NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2021 | HSC Assessment Task 3

Mathematics Extension 1

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations

Total marks – 38

Section I – 5 marks (pages 2 – 3)

- Attempt Questions 1 – 5
- Allow about 7 minutes for this section

Section II – 33 marks (pages 5 – 7)

- Attempt Questions 6 – 8
- Allow about 48 minutes for this section

NAME:_______________________________ TEACHER:____________________

STUDENT NUMBER: | | | | | | | | | | |

Question	1-5	6	7	8	Total
Inverse Functions	/3	/5	/5	/8	**/21**
Further Integration	/2	/3	/6		**/11**
Induction		/3		/3	**/6**
	/5	/11	/11	/11	**/38**

Section I

5 marks
Attempt Questions 1-5
Allow about 7 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

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1 Which of the following is the derivative of $\cos^{-1}\left(e^{-x}\right)$?

A. $\dfrac{e^{x}}{\sqrt{1-e^{2x}}}$

B. $\dfrac{-e^{x}}{\sqrt{1-e^{2x}}}$

C. $\dfrac{e^{-x}}{\sqrt{1-e^{-2x}}}$

D. $\dfrac{-e^{-x}}{\sqrt{1-e^{-2x}}}$

2 What is the exact value of $\tan^{-1}\left(\tan\dfrac{5\pi}{6}\right)$?

A. $-\dfrac{1}{\sqrt{3}}$

B. $\sqrt{3}$

C. $\dfrac{5\pi}{6}$

D. $-\dfrac{\pi}{6}$

3 What is an antiderivative of $\sin^2 2x$?

A. $\frac{x}{2}+\frac{1}{8}\sin 2x$

B. $\frac{x}{2}-\frac{1}{8}\sin 2x$

C. $\frac{x}{2}+\frac{1}{8}\sin 4x$

D. $\frac{x}{2}-\frac{1}{8}\sin 4x$

4 Using the trigonometric product-to-sum results, what is $\int \sin 3x \sin x \, dx$?

A. $\frac{1}{4}\sin 2x-\frac{1}{8}\sin 4x$

B. $\frac{1}{8}\sin 4x-\frac{1}{4}\sin 2x$

C. $\frac{1}{4}\cos 2x-\frac{1}{8}\cos 4x$

D. $\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x$

5 By considering symmetry, what is the value of $\int_{-a}^{a} \cos^{-1} x \, dx$ where $-1 \le a \le 1$?

A. 0

B. $\frac{a\pi}{2}$

C. $a\pi$

D. $2a\pi$

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Section II

33 marks
Attempt Questions 6 - 8
Allow about 48 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks**)** Use a SEPARATE writing booklet

(a) Evaluate $\int_0^4 \frac{1}{16+x^2}\,dx$. **2**

(b) (i) What is the domain of $y=\cos^{-1}(3x+2)$? **1**

(ii) Sketch the graph of $y=\cos^{-1}(3x+2)$. **2**

(c) Use the substitution $u=\cos^2 x$ to find $\int \frac{\sin 2x}{1+\cos^4 x}\,dx$. **3**

(d) Using the Principle of Mathematical Induction, prove that $3^{2n}+7$ is divisible by 8 for every integer $n \geq 1$. **3**

End of Question 6

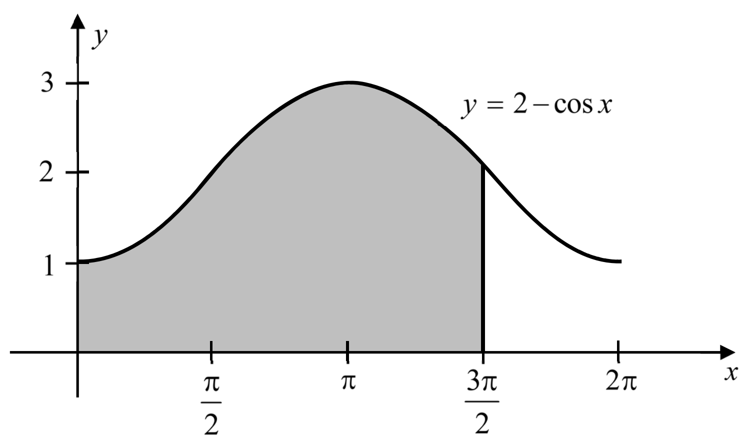
Question 7 (11 marks) Use a SEPARATE writing booklet

(a) (i) If $\theta = \tan^{-1} A + \tan^{-1} B$ then show that $\tan\theta = \dfrac{A+B}{1-AB}$. **2**

(ii) Hence solve the equation $\tan^{-1} 3x + \tan^{-1} 2x = \dfrac{\pi}{4}$. **3**

(b) Use the substitution $u^2 = 5 - x$ to evaluate $\displaystyle\int_1^4 (x-1)\sqrt{5-x}\,dx$. **3**

(c) The region bounded by the axes, the curve $y = 2 - \cos x$, and the lines $x = 0$ and $x = \dfrac{3\pi}{2}$ is rotated about the x axis. **3**



Find the exact volume of the resulting solid of revolution.

End of Question 7

Question 8 (11 marks) Use a SEPARATE writing booklet

(a) Prove by Mathematical Induction that for positive integer values of n, **3**

$$\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!} = \frac{n!-1}{n!}$$

(b) Consider the function $f(x)=\arcsin\left(\dfrac{x^2-1}{x^2+1}\right)$.

(i) Show that $f'(x)=\dfrac{2x}{|x|\left(x^2+1\right)}$. **2**

(ii) Restrict the domain of $f(x)$ to the largest interval for which $f(x)$ is decreasing, and find the equation of $f^{-1}(x)$ for this restricted function. **3**

(iii) Without differentiating $f^{-1}(x)$, find the gradient of $y=f^{-1}(x)$ at $x=\dfrac{\pi}{6}$. **3**

End of Paper

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Year 12 Extension 1 Assessment 3 Solutions

Multiple Choice [5 marks]

Summary:
1. C 2. D 3. D
4. A 5. C

1. $\frac{d}{dx}\left(\cos^{-1}e^{-x}\right) = -\frac{1}{\sqrt{1-(e^{-x})^2}} \times -e^{-x}$

$= \frac{e^{-x}}{\sqrt{1-e^{-2x}}}$ [C]

2. $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= -\frac{\pi}{6}$ [D]

3. $\cos 4x = 1 - 2\sin^2 2x \Rightarrow \sin^2 2x = \frac{1}{2}(1-\cos 4x)$

$\int \sin^2 2x \, dx = \frac{1}{2}\int(1-\cos 4x)\,dx$

$= \frac{1}{2}\left(x - \frac{1}{4}\cos 4x\right) + c$

$= \frac{x}{2} - \frac{1}{8}\sin 4x + c$ [D]

4. $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\sin 3x \sin x = \frac{1}{2}\left[\cos(3x-x) - \cos(3x+x)\right]$

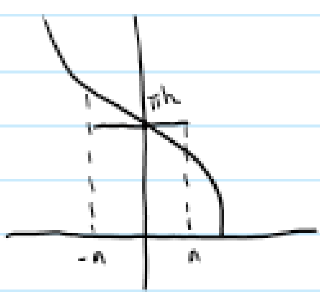
$= \frac{1}{2}(\cos 2x - \cos 4x)$

$\int \sin 3x \sin x \, dx = \frac{1}{2}\int(\cos 2x - \cos 4x)\,dx$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right) + c$

$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + c$ [A]

5.



By symmetry, area under $\cos^{-1}x$ is equal to area of rectangle.

$\int_{-a}^{a} \cos^{-1}x \, dx = 2a \times \frac{\pi}{2}$

$= a\pi$ [C]

Question 6

(a) $\int_0^4 \frac{dx}{16+x^2} = \frac{1}{4}\left[\tan^{-1}\frac{x}{4}\right]_0^4$

$= \frac{1}{4}\left(\tan^{-1}1 - \tan^{-1}0\right)$

$= \frac{1}{4}\left(\frac{\pi}{4} - 0\right)$

$= \boxed{\frac{\pi}{16}}$

[2]

(b)(i) $y = \cos^{-1}(3x+2)$

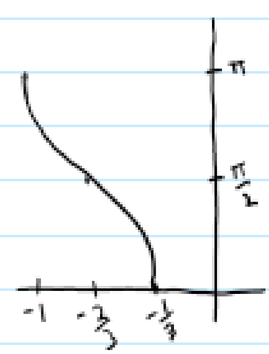
$-1 \le 3x+2 \le 1$

$-3 \le 3x \le -1$

$\boxed{-1 \le x \le -\frac{1}{3}}$

[1]

(ii)



[2]

(c) $\int \frac{\sin 2x}{1+\cos^4 x}\,dx = -\int \frac{-\sin 2x\,dx}{1+\cos^4 x}$

$u = \cos^2 x$

$du = 2\cos x \cdot (-\sin x)\,dx$

$= -\sin 2x\,dx$

$u^2 = \cos^4 x$

$= -\int \frac{du}{1+u^2}$

$= -\tan^{-1} u + c$

$= \boxed{-\tan^{-1}(\cos^2 x) + c}$

[3]

(d) RTP: $8 \mid (3^{2n}+7)$, $n \in \mathbb{N}$

Test $n=1$: $3^{2(1)}+7 = 16$

$= 8(2)$

$\therefore$ true for $n=1$

Assume true for $n=k$: ie. $3^{2k}+7 = 8M$, $M \in \mathbb{N}$

Prove true for $n=k+1$: $3^{2(k+1)}+7 = 3^{2k+2}+7$

$= 3^{2k}\cdot 3^2 + 7$

$= 9(8M-7)+7$ (by ass.)

$= 72M - 56$

$= 8(9M-7)$

$= 8P$

where $P = 9M-7 \in \mathbb{N}$ as $M \in \mathbb{N}$

$\therefore$ true for $n=k+1$ when true for $n=k$

$\therefore$ by mathematical induction, $8 \mid (3^{2n}+7)$, $n \in \mathbb{N}$

[3]

Question 7

(a)(i) $\theta = \tan^{-1}A + \tan^{-1}B$

$\tan\theta = \tan(\tan^{-1}A + \tan^{-1}B)$

$= \dfrac{\tan(\tan^{-1}A) + \tan(\tan^{-1}B)}{1 - \tan(\tan^{-1}A)\cdot\tan(\tan^{-1}B)}$

$= \dfrac{A+B}{1-AB}$ [2]

Well done by most students.

(ii) $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$

$\tan^{-1}\left(\dfrac{3x+2x}{1-3x\cdot 2x}\right) = \dfrac{\pi}{4}$

$\dfrac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$

$5x = 1 - 6x^2$

$6x^2 + 5x - 1 = 0$

$(6x-1)(x+1) = 0$

$x = \frac{1}{6}, -1$

but $\tan^{-1}(-3) + \tan^{-1}(-2) < 0$

$\therefore \boxed{x = \frac{1}{6}}$ [3]

The most common error was failing to check that the answers satisfied the original equation. A number of students left $\frac{\pi}{4}$ in the equation instead of $\tan\frac{\pi}{4} = 1$, which led to an ugly quadratic equation. As usual, there were a number of careless errors when rearranging terms, particularly with signs.

1b) Let $u^2 = 5 - x$

Then $x = 5 - u^2$

$$\frac{dx}{du} = -2u \quad \text{so} \quad dx = -2u\,du$$

$$x - 1 = 5 - u^2 - 1$$
$$= 4 - u^2$$

x	1	4
u	2	1

using $u = \sqrt{5-x}$

$$\int_1^4 (x-1)\sqrt{5-x}\,dx = \int_2^1 (4-u^2).u.(-2u\,du)$$

$$= -2\int_2^1 (4-u^2)\,u^2\,du$$

$$= 2\int_1^2 (4u^2 - u^4)\,du$$

$$= 2\left[4\frac{u^3}{3} - \frac{u^5}{5}\right]_1^2$$

$$= 2\left[\left(\frac{4\times 8}{3} - \frac{32}{5}\right) - \left(\frac{4}{3} - \frac{1}{5}\right)\right]$$

$$= \frac{94}{15}$$

(b) $\int_1^4 (x-1)\sqrt{5-x}\,dx$

$u = 5 - x \qquad \Rightarrow (x-1) = (5-u) - 1$

$du = -dx \qquad\qquad = 4 - u$

$x = 1 \rightarrow u = 4$

$x = 4 \rightarrow u = 1$

$= \int_4^1 (4-u)\sqrt{u}\,(-du)$

$= \int_1^4 \left(4u^{1/2} - u^{3/2}\right) du$

$= \left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_1^4$

$= \left(\frac{64}{3} - \frac{64}{5}\right) - \left(\frac{8}{3} - \frac{2}{5}\right)$

$= \boxed{\frac{94}{15}}$

[3]

SOLUTION DOES NOT MATCH QN.

Given the large number of potential sources for error, it was pleasing to see so many students get the correct answer. The biggest issue was an inability to find a correct substitution for dx, with many students confusing dx and du. Even students who made the correct substitution often did an overly complicated derivation. There is no need to differentiate $\sqrt{5-x}$ – see the solutions. A few students did not change the limits of integration.

(c) $V = \pi \int_0^{\frac{3\pi}{2}} (2 - \cos x)^2\,dx$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$= \pi \int_0^{\frac{3\pi}{2}} \left(4 - 4\cos x + \cos^2 x\right) dx$

$= \pi \int_0^{\frac{3\pi}{2}} \left(4 - 4\cos x + \frac{1}{2}(1 + \cos 2x)\right) dx$

$= \frac{\pi}{2} \int_0^{\frac{3\pi}{2}} \left(9 - 8\cos x + \cos 2x\right) dx$

$= \frac{\pi}{2} \left[9x - 8\sin x + \frac{1}{2}\sin 2x\right]_0^{3\pi/2}$

$= \frac{\pi}{2}\left(\left(\frac{27\pi}{2} + 8 + 0\right) - 0\right)$

$= \boxed{\frac{\pi}{4}(27\pi + 16) \text{ units}^3}$

[3]

Well done up to the point of performing the calculation. Too many errors crept in at the end. A few students left off the π. Most knew the method for integrating $\cos^2 x$. It was disappointing to see how many students could not correctly expand $(2 - \cos x)^2$.

Question 8

(a) RTP: $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots+\frac{n-1}{n!}=\frac{n!-1}{n!}$

Test $n=1$: LHS $=\frac{0}{1!}=0$

RHS $=\frac{1!-1}{1!}=0=$ LHS

$\therefore$ true for $n=1$

Assume true for $n=k$: ie. $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\dots+\frac{k-1}{k!}=\frac{k!-1}{k!}$

Prove true for $n=k+1$: ie. RTP:

$\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\dots+\frac{k-1}{k!}+\frac{k}{(k+1)!}=\frac{(k+1)!-1}{(k+1)!}$

$$\begin{aligned}
\text{LHS} &= \frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\dots+\frac{k-1}{k!}+\frac{k}{(k+1)!} \\
&= \frac{k!-1}{k!}+\frac{k}{(k+1)!} \quad \text{(by assumption)} \\
&= \frac{(k+1)(k!-1)}{(k+1)!}+\frac{k}{(k+1)!} \\
&= \frac{(k+1)k! - \cancel{k} - 1 + \cancel{k}}{(k+1)!} \\
&= \frac{(k+1)!-1}{(k+1)!} \\
&= \text{RHS}
\end{aligned}$$

$\therefore$ true for $n=k+1$ when true for $n=k$

$\therefore$ by M.I., $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\dots+\frac{n-1}{n!}=\frac{n!-1}{n!}$, $n\in\mathbb{N}$

[3]

(b) (i) $f(x) = \sin^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}} \times \frac{(x^2+1)\cdot 2x - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{\cancel{x^2+1}}{\sqrt{(x^2+1)^2-(x^2-1)^2}} \times \frac{4x}{(x^2+1)^{\cancel{2}}}$$

$$= \frac{4x}{\sqrt{4x}\cdot(x^2+1)}$$

$$= \frac{2x}{|x|(x^2+1)}$$

[2]

(ii) as $|x| > 0$ and $x^2+1 > 0$, $f'(x) < 0 \Rightarrow x < 0$

$$y = \sin^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

Inverse: $x = \sin^{-1}\left(\frac{y^2-1}{y^2+1}\right)$

$$\sin x = \frac{y^2-1}{y^2+1}$$

$$y^2\sin x + \sin x = y^2 - 1$$

$$1 + \sin x = y^2(1-\sin x)$$

$$y^2 = \frac{1+\sin x}{1-\sin x}$$

$$y = \pm\sqrt{\frac{1+\sin x}{1-\sin x}}$$

$x < 0$ for $f(x) \Rightarrow y < 0$ for $f^{-1}(x)$

$$\therefore \boxed{f^{-1}(x) = -\sqrt{\frac{1+\sin x}{1-\sin x}}}$$

[3]

(iii) If $y = f^{-1}(x)$, $m = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = \frac{1}{\left.\frac{dx}{dy}\right|_{x=\frac{\pi}{6}}}$

$$= \frac{1}{\left.\frac{dy}{dx}\right|_{y=\frac{\pi}{6}}}$$ for $y = f(x)$

$$f^{-1}\left(\frac{\pi}{6}\right) = -\sqrt{\frac{1+\sin\frac{\pi}{6}}{1-\sin\frac{\pi}{6}}}$$

$$= -\sqrt{\frac{1+\frac{1}{2}}{1-\frac{1}{2}}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= -\sqrt{\frac{2+1}{2-1}}$$

$$= -\sqrt{3}$$

$\therefore m = \frac{1}{\left.\frac{dy}{dx}\right|_{x=-\sqrt{3}}}$ where $y = f(x)$

$$= \frac{|-\sqrt{3}|\left((-\sqrt{3})^2+1\right)}{2(-\sqrt{3})}$$

$$= \frac{4\sqrt{3}}{-2\sqrt{3}}$$

$$= \boxed{-2}$$

[3]