NORTH SYDNEY GIRLS HIGH SCHOOL

2022

HSC Assessment Task 3

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations

Total marks – 41

Section I – 5 marks (pages 2 – 4)

- Attempt Questions 1 – 5
- Allow about 7 minutes for this section

Section II – 36 marks (pages 5 – 7)

- Attempt Questions 6 – 8
- Allow about 53 minutes for this section

NAME:______________________________ TEACHER:___________________

STUDENT NUMBER: | | | | | | | | | | |

Question	1-5	6	7	8	Total
Inverse Functions	/4	/7	/7	/7	**/25**
Further Integration	/1	/5	/2	/5	**/13**
Further Trigonometry			/3		**/3**
	/5	/12	/12	/12	**/41**

Section I

5 marks
Attempt Questions 1-5
Allow about 7 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

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1 What is the domain of the function $y = \arcsin\left(\frac{x}{5}\right)$?

A. $x \in \left[-\frac{1}{5}, \frac{1}{5}\right]$

B. $x \in [-5, 5]$

C. $x \in \left[-\frac{\pi}{10}, \frac{\pi}{10}\right]$

D. $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$

2 Which expression is an antiderivative of $\sin^2 3x$?

A. $-\frac{1}{3}\cos^2 3x$

B. $-\frac{\sin^3 3x}{\cos 3x}$

C. $\frac{x}{2}+\frac{1}{12}\sin 6x$

D. $\frac{x}{2}-\frac{1}{12}\sin 6x$

3 Which expression is equivalent to $\int \frac{dx}{5+4x^2}$?

A. $\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2x}{\sqrt{5}} + c$

B. $\frac{1}{\sqrt{5}}\tan^{-1}\frac{2x}{\sqrt{5}} + c$

C. $\frac{2}{\sqrt{5}}\tan^{-1}\frac{2x}{\sqrt{5}} + c$

D. $\frac{1}{4}\tan^{-1}\frac{2x}{\sqrt{5}} + c$

4 Which of the following is a general solution for the equation $\tan 3x = \sqrt{3}$?

A. $x = \frac{\pi}{6} + \frac{2k\pi}{3}, \quad x = \frac{7\pi}{6} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z}$

B. $x = \frac{\pi}{6} + 2k\pi, \quad x = \frac{7\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$

C. $x = \frac{\pi}{9} + \frac{2k\pi}{3}, \quad x = \frac{4\pi}{9} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z}$

D. $x = \frac{\pi}{9} + 2k\pi, \quad x = \frac{4\pi}{9} + 2k\pi, \quad k \in \mathbb{Z}$

5. Consider the function $f(x) = \cos x + \sin x$, $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

Which of the following is a correct equation for $f^{-1}(x)$?

A. $y = \sin^{-1}\frac{x}{\sqrt{2}} - \frac{\pi}{4}$

B. $y = \sin^{-1}\frac{x}{\sqrt{2}} + \frac{\pi}{4}$

C. $y = \cos^{-1}\frac{x}{\sqrt{2}} - \frac{\pi}{4}$

D. $y = \cos^{-1}\frac{x}{\sqrt{2}} + \frac{\pi}{4}$

Section II

36 marks
Attempt Questions 6 - 8
Allow about 53 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (12 marks) Use a SEPARATE writing booklet

(a) Find the exact value of $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$. **1**

(b) Differentiate $\sin^{-1}\left(x^2\right)$. **2**

(c) Evaluate $\int_0^{\frac{4}{3}} \frac{dx}{\sqrt{16-9x^2}}$. **2**

(d) Find $\int \cos 2x \sin 5x \, dx$. **2**

(e) (i) Sketch the graph of $y = 12\sin^{-1}\frac{x}{2}$, showing all important features. **2**

(ii) The region in the first quadrant bounded by $y = 12\sin^{-1}\frac{x}{2}$ and the line $y = 3\pi x$ is rotated around the y-axis to form a solid of revolution. **3**

Find the volume of this solid.

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet

(a) Use the substitution $u = \sqrt{x}$ to find $\displaystyle\int \frac{dx}{\sqrt{x}\left(x+1\right)}$. **2**

(b) Find the exact value of $\sin\left(\sin^{-1}\left(\tfrac{3}{7}\right) + \cos^{-1}\left(\tfrac{1}{5}\right)\right)$. **3**

(c) Given that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ find all solutions of the polynomial equation $8x^3 - 6x - 1 = 0$ in exact trigonometric form. **3**

(d) Consider the function $f\left(x\right) = 2\sin^{-1}\sqrt{x} - \sin^{-1}\left(2x-1\right)$ for $0 \le x \le 1$.

(i) Show that $f'\left(x\right) = 0$ for $0 < x < 1$. **2**

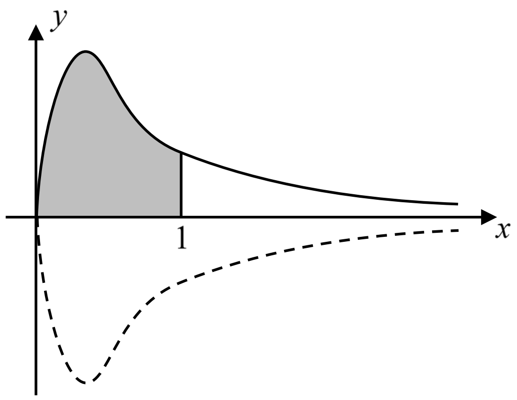
(ii) Hence sketch the graph of $y = f\left(x\right)$. **2**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet

(a) (i) Use the substitution $u = 3x + 1$ to show that $\int_0^1 \frac{x}{(3x+1)^2}\,dx = \frac{2}{9}\ln 2 - \frac{1}{12}$. **3**

(ii) The diagram shows the shaded region bounded by the curve $y = \frac{6\sqrt{x}}{3x+1}$, the x-axis, and the line $x = 1$. **2**



This region is to be rotated about the x-axis to form a solid of revolution.

Use part (i) to find the volume of this solid in exact form.

(b) Use a trigonometric identity or otherwise to evaluate $\int_0^{\frac{1}{2}} \sin\left(2\cos^{-1} x\right) dx$. **3**

(c) A runner and a cyclist compete in a race along a long straight running track. **4**

They start at the same time at point S at the beginning of the track, each moving at their own constant speed, where the cyclist is three times as fast as the runner.

An observer stands at P, one metre from S, where SP is perpendicular to the track.

What is the greatest angle between the runner and the cyclist, as seen by the observer, as the race proceeds?

End of Paper

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2022 HSC Ext 1 Task 3 Solutions

(I) Multiple Choice

Summary: 1. B 2. D 3. A 4. C 5. D

1. $-1 \leq \frac{x}{5} \leq 1$

$-5 \leq x \leq 5$ (B)

2. $\int \sin^2 3x \, dx = \frac{1}{2}\int (1 - \cos 6x)\,dx$

$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right)$

$= \frac{x}{2} - \frac{\cos 6x}{12}$ (D)

3. $\int \frac{dx}{5+4x^2} = \frac{1}{4}\int \frac{dx}{\frac{5}{4}+x^2}$

$= \frac{1}{4} \cdot \frac{2}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}/2} + c$

$= \frac{1}{2\sqrt{5}} \tan^{-1} \frac{2x}{\sqrt{5}} + c$ (A)

4. $\tan 3x = \sqrt{3}$

$3x = \frac{\pi}{3} + 2k\pi, \ \frac{4\pi}{3} + 2k\pi, \ k \in \mathbb{Z}$

$x = \frac{\pi}{9} + \frac{2k\pi}{3}, \ \frac{4\pi}{9} + \frac{2k\pi}{3}$ (C)

5. $y = \cos x + \sin x, \quad 0 \leq x - \frac{\pi}{4} \leq \pi, \quad \frac{\pi}{2} \leq x + \frac{\pi}{4} \leq \frac{3\pi}{2}$

$y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$ OR $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ [aux. angle method]

Inverse: $x = \sqrt{2}\cos\left(y - \frac{\pi}{4}\right)$ OR $x = \sqrt{2}\sin\left(y + \frac{\pi}{4}\right)$

$\frac{x}{\sqrt{2}} = \cos\left(y - \frac{\pi}{4}\right)$ OR $\frac{x}{\sqrt{2}} = \sin\left(y + \frac{\pi}{4}\right)$

$y - \frac{\pi}{4} = \cos^{-1}\frac{x}{\sqrt{2}}$ OR $y + \frac{\pi}{4} = \pi - \sin^{-1}\frac{x}{\sqrt{2}}$ [match domain]

$y = \cos^{-1}\frac{x}{\sqrt{2}} + \frac{\pi}{4}$ OR $y = -\sin^{-1}\frac{x}{\sqrt{2}} + \frac{3\pi}{4}$ [D]

Question 6

(a) $\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$\sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= -\sin^{-1}\left(\sqrt{3}/2\right)$

$= \boxed{-\frac{\pi}{3}}$ [1]

(b) $\frac{d}{dx}\sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \times 2x$

$= \boxed{\frac{2x}{\sqrt{1-x^4}}}$ [2]

(c) $\int_0^{4/3} \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\int_0^{4/3} \frac{dx}{\sqrt{16/9 - x^2}}$

$= \frac{1}{3}\left[\sin^{-1}\frac{3x}{4}\right]_0^{4/3}$

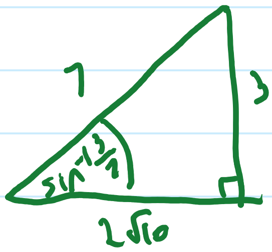
$= \frac{1}{3}\left(\sin^{-1}1 - \sin^{-1}0\right)$

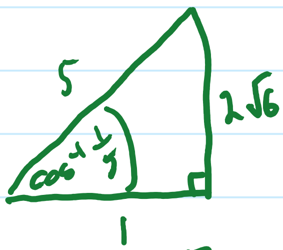
$= \boxed{\frac{\pi}{6}}$ [2]

OR $\int_0^{4/3} \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\int_0^{4/3} \frac{3}{\sqrt{16-(3x)^2}}\,dx$

$= \frac{1}{3}\left[\sin^{-1}\frac{3x}{4}\right]_0^{4/3}$

(d) $\sin\left(\sin^{-1}\frac{3}{7} + \cos^{-1}\frac{1}{5}\right) = \sin\left(\sin^{-1}\frac{3}{7}\right)\cos\left(\cos^{-1}\frac{1}{5}\right) + \cos\left(\sin^{-1}\frac{3}{7}\right)\sin\left(\cos^{-1}\frac{1}{5}\right)$





$= \frac{3}{7} \times \frac{1}{5} + \frac{2\sqrt{10}}{7} \times \frac{2\sqrt{6}}{5}$

$= \frac{3}{35} + \frac{4\sqrt{60}}{35}$

$= \boxed{\frac{3+8\sqrt{15}}{35}}$ [3]

(e) $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$$\cos 2x \sin 5x = \frac{1}{2}[\sin(2x+5x) - \sin(2x-5x)]$$
$$= \frac{1}{2}(\sin 7x + \sin 3x)$$

$$\int \cos 2x \sin 5x \, dx = \frac{1}{2}\int (\sin 7x + \sin 3x)\, dx$$
$$= \boxed{-\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + c}$$

[2]

(f) $$\int \frac{dx}{\sqrt{x}(x+1)} = 2\int \frac{1}{(\sqrt{x})^2+1} \cdot \frac{dx}{2\sqrt{x}}$$
$$= 2\int \frac{du}{u^2+1}$$
$$= 2\tan^{-1} u + c$$
$$= \boxed{2\tan^{-1}(\sqrt{x}) + c}$$

[3]

$u = x^{1/2}$

$du = \frac{1}{2} x^{-1/2}\, dx$

$= \frac{1}{2\sqrt{x}}\, dx$

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Question 7

(a) (i) $f(x) = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$

$$f'(x) = \frac{\cancel{2}}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{\cancel{2}\sqrt{x}} - \frac{1}{\sqrt{1-(2x-1)^2}} \times 2$$

$$= \frac{1}{\sqrt{x}\cdot\sqrt{1-x}} - \frac{2}{\sqrt{1-(4x^2-4x+1)}}$$

$$= \frac{1}{\sqrt{x-x^2}} - \frac{2}{\sqrt{4x-4x^2}}$$

$$= \frac{1}{\sqrt{x-x^2}} - \frac{1}{\sqrt{x-x^2}}$$

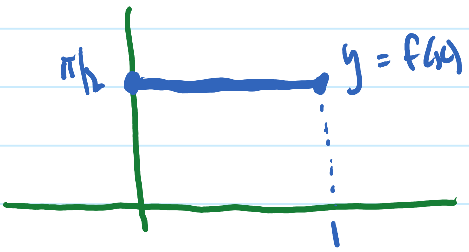
$$= 0$$ [2]

(ii) There are no discontinuities in $[0,1]$.

$f(0) = 2\sin^{-1}0 - \sin^{-1}(-1)$

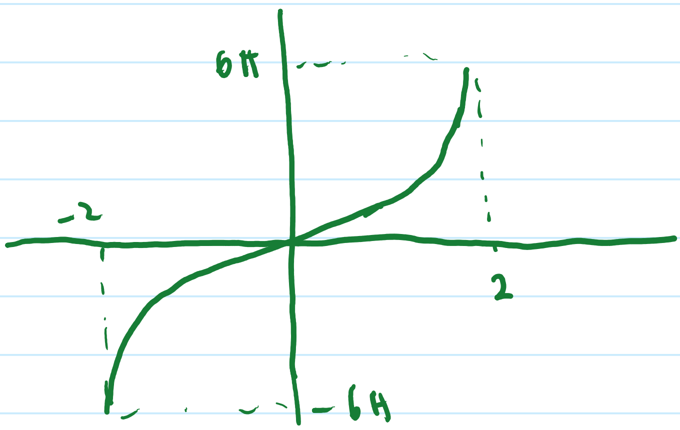
$= \frac{\pi}{2}$

As $f'(x) = 0$, and no discontinuities, $f(x)$ is a constant function



[2]

(b) (i)



[2]

(ii) $y = 3\pi x \Rightarrow x = 2,\ y = 6\pi$

$y = 12\sin^{-1}\frac{x}{2} \Longrightarrow \frac{y}{12} = \sin^{-1}\frac{x}{2}$

$\frac{x}{2} = \sin\frac{y}{12}$

$x = 2\sin\frac{y}{12}$

$V = \pi\int_0^{6\pi}\left(2\sin\frac{y}{12}\right)^2 dy - \frac{1}{3}\pi\times 2^2\times 6\pi$ (Vol. of cone: $\frac{1}{3}\pi r^2 h$)

$= 4\pi\int_0^{6\pi}\sin^2\frac{y}{12} - 8\pi^2$

$= 2\pi\int_0^{6\pi}\left(1-\cos\frac{y}{6}\right)dy - 8\pi^2$

$= 2\pi\left[y - 6\sin\frac{y}{6}\right]_0^{6\pi} - 8\pi^2$

$= 2\pi\left[6\pi - 0\right] - 8\pi^2$

$= \boxed{4\pi^2 \text{ units}^3}$ [3]

(c) let $x = \sin\theta$ $\quad 8x^3 - 6x - 1 = 0 \Rightarrow 8\sin^3\theta - 6\sin\theta - 1 = 0$

$-2(3\sin\theta - 4\sin^3\theta) = 1$

$\sin 3\theta = -\frac{1}{2}$

$3\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}, \frac{35\pi}{6}, \quad 3\theta\in[0,6\pi]$

$\theta = \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}, \quad \theta\in[0,2\pi]$

$x = \sin\frac{7\pi}{18}, \sin\frac{11\pi}{18}, \sin\frac{19\pi}{18}, \sin\frac{23\pi}{18}, \sin\frac{31\pi}{18}, \sin\frac{35\pi}{18}$

but $\sin\frac{7\pi}{18} = \sin\frac{11\pi}{18}$, $\sin\frac{19\pi}{18} = \sin\frac{35\pi}{18}$, $\sin\frac{23\pi}{18} = \sin\frac{31\pi}{18}$

$\therefore \boxed{x = \sin\frac{7\pi}{18}, \sin\frac{19\pi}{18}, \sin\frac{23\pi}{18}}$ [3]

alternatively, $x = \sin\frac{7\pi}{18}, -\sin\frac{\pi}{18}, -\sin\frac{5\pi}{18}$

OR $\sin 70^\circ, \sin 190^\circ, \sin 230^\circ$

OR $\sin 70^\circ, -\sin 10^\circ, -\sin 50^\circ$

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Question 8

(a) (i) $\displaystyle\int_0^1 \frac{x\,dx}{(3x+1)^2} = \int_1^4 \frac{1}{u^2}\cdot\frac{1}{3}(u-1)\cdot\frac{1}{3}\,du$

let $u = 3x+1$

$du = 3dx$

$x=0,\ u=1$

$x=1,\ u=4$

$x = \frac{1}{3}(u-1)$

$\displaystyle = \frac{1}{9}\int_1^4 \frac{u-1}{u^2}\,du$

$\displaystyle = \frac{1}{9}\int_1^4 \left(\frac{1}{u} - \frac{1}{u^2}\right)du$

$\displaystyle = \frac{1}{9}\left[\ln u + \frac{1}{u}\right]_1^4$

$\displaystyle = \frac{1}{9}\left\{\left(\ln 4 + \frac{1}{4}\right) - (0+1)\right\}$

$\displaystyle = \frac{1}{9}\left(2\ln 2 - \frac{3}{4}\right)$

$\displaystyle = \frac{2}{9}\ln 2 - \frac{1}{12}$ [3]

(ii) $\displaystyle V = \pi\int_0^1 \left(\frac{6\sqrt{x}}{3x+1}\right)^2 dx$

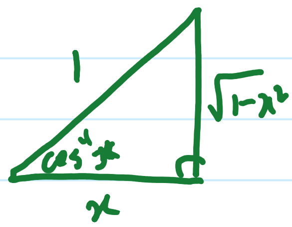
$\displaystyle = 36\pi\int_0^1 \frac{x}{(3x+1)^2}\,dx$

$\displaystyle = 36\pi \times \left(\frac{2}{9}\ln 2 - \frac{1}{12}\right)$

$= \boxed{\pi(8\ln 2 - 3)\ \text{units}^3}$ [2]

(b) $\sin(2\cos^{-1}x) = 2\sin(\cos^{-1}x)\cos(\cos^{-1}x)$

$= 2\cdot\sqrt{1-x^2}\cdot x$

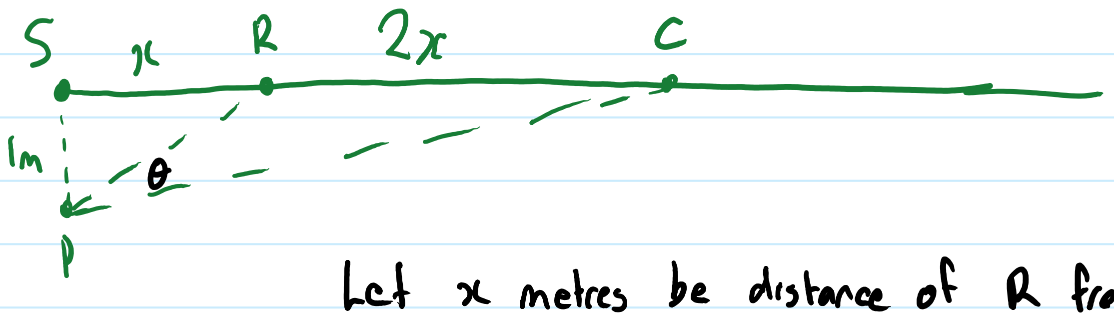


$\displaystyle\int_0^{1/2} \sin(2\cos^{-1}x)\,dx = -\int_0^{1/2} (1-x^2)^{1/2}\cdot(-2x)\,dx$

$\displaystyle = -\frac{2}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^{1/2}$

$\displaystyle = -\frac{2}{3}\left(\frac{3\sqrt{3}}{8} - 1\right) = \boxed{\frac{1}{12}\left(8 - 3\sqrt{3}\right)}$ [3]

(c)



Let x metres be distance of R from S at same time after starting
$\therefore$ $3x$ metres is distance of C from S at that same time.
Let θ be angle between R and C as seen from P
Let $\alpha = S\hat{P}C$, $\beta = S\hat{P}R$

$\tan\alpha = \frac{3x}{1} = 3x$, $\tan\beta = x$

$\alpha = \tan^{-1} 3x$ $\quad\quad$ $\beta = \tan^{-1} x$

$$\theta = \alpha - \beta$$
$$= \tan^{-1} 3x - \tan^{-1} x$$

$$\frac{d\theta}{dx} = \frac{3}{1+9x^2} - \frac{1}{1+x^2}$$
$$= \frac{3(1+x^2) - (1+9x^2)}{(1+9x^2)(1+x^2)}$$
$$= \frac{2-6x^2}{(1+9x^2)(1+x^2)}$$

For max/min, $\frac{d\theta}{dx} = 0 \Rightarrow 6x^2 = 2$
$x = \frac{1}{\sqrt{3}}$ $(x > 0)$

As $(1+9x^2)(1+x^2) > 0 \ \forall x$, sign of $\frac{d\theta}{dx}$ is sign of $2-6x^2$

if $x < \frac{1}{\sqrt{3}}$, $x^2 < \frac{1}{3}$, $6x^2 < 2 \Rightarrow 2-6x^2 > 0 \Rightarrow \frac{d\theta}{dx} > 0$

if $x > \frac{1}{\sqrt{3}}$, $x^2 > \frac{1}{3}$, $6x^2 > 2 \Rightarrow 2-6x^2 < 0 \Rightarrow \frac{d\theta}{dx} < 0$

$\therefore$ max θ when $x = \frac{1}{\sqrt{3}}$

$$\theta_{max} = \tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}$$
$$= \frac{\pi}{3} - \frac{\pi}{6}$$
$$= \boxed{\frac{\pi}{6}}$$

(4)

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