NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2023 | **HSC Assessment Task 3**

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 8, show relevant mathematical reasoning and/or calculations

Total marks – 39

Section I – 5 marks (pages 2 – 5)

- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 34 marks (pages 6 – 8)

- Attempt Questions 6 – 8
- Allow about 52 minutes for this section

NAME:____________________________ **TEACHER:**__________________

STUDENT NUMBER: | | | | | | | | | | |

Question	1-5	6	7	8	Total
Inverse Functions	/3	/9		/5	/17
Further Integration	/2	/3	/8	/2	/15
Related Rates			/3	/5	/8
	/5	/12	/11	/12	/40

Section I

5 marks
Attempt Questions 1-5
Allow about 8 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

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1 Which of the following is the domain of $y = \sin^{-1} 3x$?

A. $-3 \le x \le 3$

B. $-\frac{1}{3} \le x \le \frac{1}{3}$

C. $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$

D. $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$

2 Which of the following is an antiderivative of $\cos 3x \sin x$?

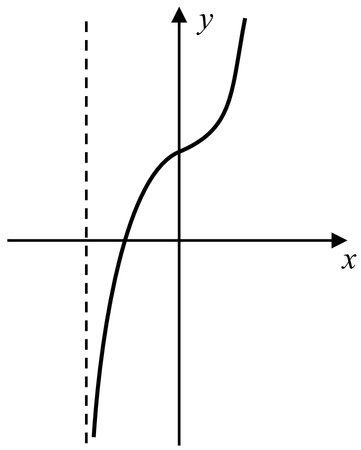
A. $\frac{1}{2}(\sin 4x - \sin 2x)$

B. $\frac{1}{2}(\sin 4x + \sin 2x)$

C. $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x$

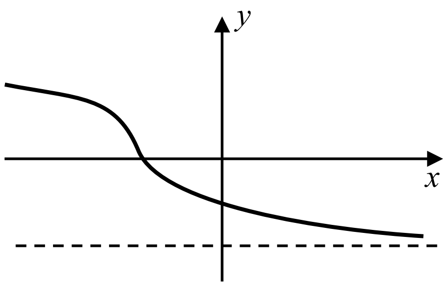
D. $\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x$

3 The graph of the function $y = f(x)$ is shown below.

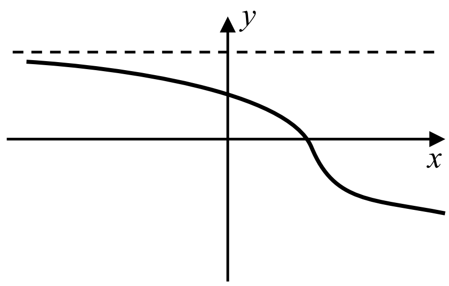


Which of the following could be the graph of $f^{-1}(x)$?

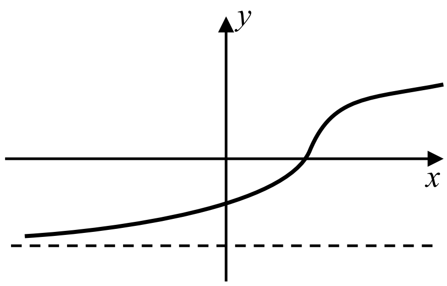
A.



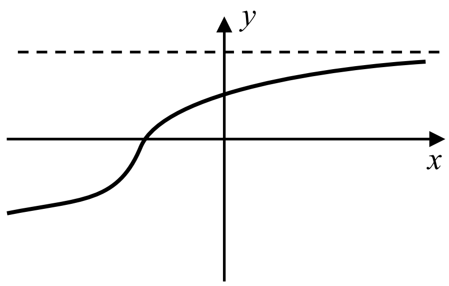
B.



C.



D.



4 A region on the Cartesian plane is bounded by the curve $y = x^4$, the y-axis, and the line $y = 16$.

A solid of revolution is formed by rotating this region about the y-axis.

Which of the following is the correct expression for the volume of this solid?

A. $V = \pi\int_0^2 x^8 \, dx$

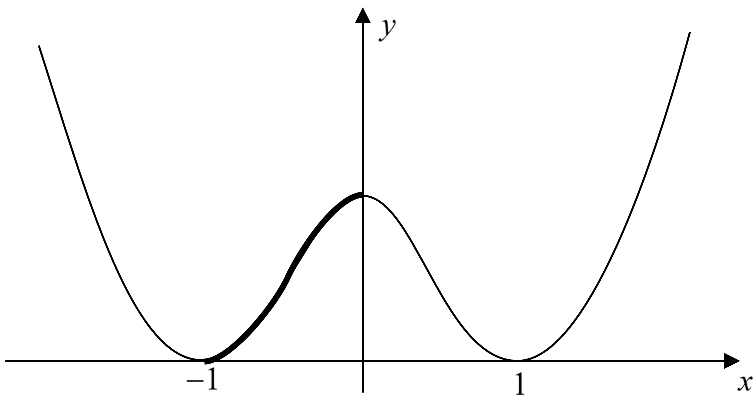
B. $V = \pi\int_0^{16} x^8 \, dx$

C. $V = \pi\int_0^2 y^{\frac{1}{2}} \, dy$

D. $V = \pi\int_0^{16} y^{\frac{1}{2}} \, dy$

5 The graph of $y = f(x)$ where $f(x) = \left(x^2 - 1\right)^2$ is shown below.

The part of the curve with domain $-1 \le x \le 0$ is highlighted.



What is the equation of the inverse function $f^{-1}(x)$ if the domain of $f(x)$ is restricted to $-1 \le x \le 0$?

A. $y = \sqrt{1 + \sqrt{x}}$

B. $y = -\sqrt{1 + \sqrt{x}}$

C. $y = \sqrt{1 - \sqrt{x}}$

D. $y = -\sqrt{1 - \sqrt{x}}$

Section II

35 marks
Attempt Questions 6 - 8
Allow about 52 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 **(**12 marks**)** Use a SEPARATE writing booklet

(a) Differentiate $\cos^{-1}\left(x^2\right)$. **2**

(b) Find an antiderivative of $\dfrac{1}{\sqrt{9-4x^2}}$. **2**

(c) Use the substitution $u = x^4 + 4x$ to find $\displaystyle\int_0^2 \frac{x^3+1}{\sqrt{x^4+4x+1}}\,dx$. **3**

(d) (i) Write down the x-intercept of $y = 1 - \tan^{-1} x$. **1**

(ii) Sketch the graph of $y = 1 - \tan^{-1} x$ showing all important features. **2**

(e) Find the exact value of $\cos^{-1}\left(\cos\left(\dfrac{7\pi}{5}\right)\right)$. **2**

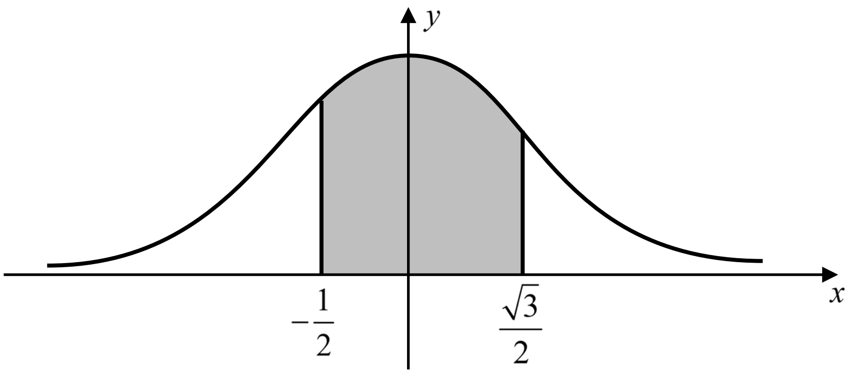
Show all necessary working.

End of Question 6

Question 7 (11 marks) Use a SEPARATE writing booklet

(a) Find $\int \sin^2 2x \, dx$. **2**

(b) The graph of $y = \dfrac{1}{\sqrt{1+4x^2}}$ is shown below. **3**



The shaded region in the diagram is bounded by the curve $y = \dfrac{1}{\sqrt{1+4x^2}}$, the x-axis and the lines $x = -\dfrac{1}{2}$ and $x = \dfrac{\sqrt{3}}{2}$. Find the exact volume of the solid of revolution formed when the shaded region is rotated about the x-axis.

(c) Use the substitution $u = e^{-x} + 1$ to find the indefinite integral $\displaystyle\int \frac{e^{-2x}}{e^{-x}+1} dx$. **3**

(d) The area A of an equilateral triangle is increasing at a rate of 4 cm^2/s. **3**
The side length of the triangle is x cm.

Find the exact rate of increase of the side length of the triangle when its area is $\sqrt{3}$ cm.

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet

(a) (i) Given $a > 1$, use the substitution $x = \dfrac{1}{u}$ to rewrite the integral **2**

$$\int_1^a \frac{dx}{1+x^2} \text{ as an integral in } u.$$

(ii) Hence, deduce the value of $\tan^{-1}(a) + \tan^{-1}\left(\dfrac{1}{a}\right)$. **2**

(b) Consider the curve $y = \log_e x$ for $x \geq 1$. A variable point P lies on the curve and the interval joining P to the origin O makes an angle of θ with the positive x-axis.

(i) Explain with the aid of a diagram why $\theta = \tan^{-1}\left(\dfrac{\log_e x}{x}\right)$. **1**

(ii) Show that $\dfrac{d\theta}{dx} = \dfrac{1 - \log_e x}{x^2 + (\log_e x)^2}$. **2**

(iii) Find the coordinates of the point on the curve where $\dfrac{d\theta}{dx} = 0$, and explain the geometrical significance of this result. **2**

(iv) P moves on the curve so that its y-coordinate increases at a constant rate of 0.5 units/s. **3**

Find, in degrees per second, the rate at which OP is rotating when P is 2 units above the x-axis.

End of Paper

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2023 Extension 1 Assessment 3 Solutions and Markers Feedback

Section 1

1. $-1 \le 3x \le 1$ **B**

$-\frac{1}{3} \le x \le \frac{1}{3}$

2. $\cos 3x \sin x = \frac{1}{2}\left[\sin(3x+x) - \sin(3x-x)\right]$ (products to sums) **C**

$= \frac{1}{2}(\sin 4x - \sin 2x)$

$$\int \cos 3x \sin x \; dx = \frac{1}{2}\int (\sin 4x - \sin 2x)\,dx$$
$$= \frac{1}{2}\left[-\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x\right] + c$$
$$= \frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + c$$

3. −ve vert asymptote become −ve horz asymptote **C**

Eliminate B and D.

+ve y-intercept become +ve x-intercept

Eliminate A

4. $y = x^4 \;\Rightarrow\; x^2 = y^{\frac{1}{2}}$ **D**

y-values already given − don't convert

5. −ve x-values become −ve y-values **D**

Can only be achieved with B or D

$-1 \le x \le 0$ becomes $-1 \le y \le 0$

But B gives y-values less than −1.

Question 6

(a) Differentiate $\cos^{-1}(x^2)$. **2**

$$\frac{d}{dx}\left(\cos^{-1}(x^2)\right) = \frac{-1}{\sqrt{1-(x^2)^2}} \times 2x$$

$$= \frac{-2x}{\sqrt{1-x^4}}$$

Generally well done. Some minor errors with not correctly finding $f'(x)$.

(b) Find an antiderivative of $\frac{1}{\sqrt{9-4x^2}}$. **2**

$$\int \frac{1}{\sqrt{9-4x^2}}dx = \frac{1}{2}\int \frac{2}{\sqrt{3^2-(2x)^2}}dx$$

$$= \frac{1}{2}\sin\left(\frac{2x}{3}\right)$$

Generally, well done. $+C$ was not required here, as you only needed to find a possible antiderivative. This is a standard integral from the reference sheet. Some students didn't recognise the form.

(c) Use the substitution $u = x^4 + 4x$ to find $\int_0^2 \frac{x^3+1}{\sqrt{x^4+4x+1}}\,dx$. **3**

$$u = x^4 + 4x$$

$$du = (4x^3+4)\,dx$$

$$= 4(x^3+1)\,dx$$

$$x = 0,\ u = 0$$

$$x = 2,\ u = 24$$

$$\int_0^2 \frac{x^3+1}{\sqrt{x^4+4x+1}}dx = \frac{1}{4}\int_0^{24}\frac{du}{\sqrt{u+1}} = \frac{1}{4}\int_0^{24}(u+1)^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\left[\frac{(u+1)^{\frac{1}{2}}}{1/2}\right]_0^{24} = \frac{1}{2}\left[\sqrt{u+1}\right]_0^{24}$$

$$= \frac{1}{2}(\sqrt{25}-\sqrt{1}) = \frac{4}{2} = 2$$

Mostly well done. Several students didn't recognise the form of the integral and could not get subsequent marks.

(d) (i) Write down the x-intercept of $y = 1 - \tan^{-1} x$. **1**

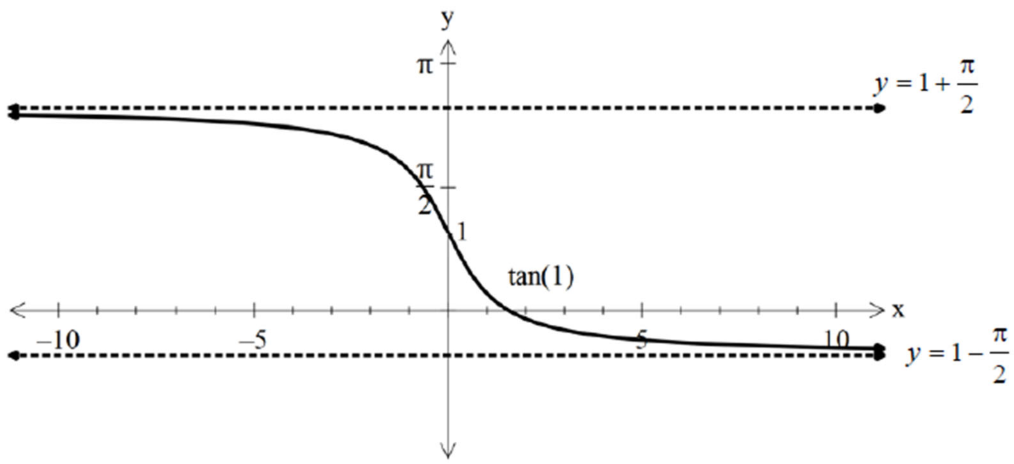
For x-intercept, set $y = 0$

$1 - \tan^{-1} x = 0 \Rightarrow \tan^{-1} x = 1$

$x = \tan 1$

Poorly done. Most common mistake: $x = \frac{\pi}{4}$.

(ii) Sketch the graph of $y = 1 - \tan^{-1} x$ showing all important features. **2**



Responses were mixed, with most students losing part of the marks from careless construction of the graphs. Asymptotes and intercepts should be calculated and provided whenever you do a sketch. For this graph, the y-intercept is the midpoint of the graph and should be halfway between the asymptotes. Scales also need to be not obviously wrong. Don't just use lines because they are there on the page. $1 - \frac{\pi}{2} \approx -0.6$ and $1 + \frac{\pi}{2} \approx 2.6$.

(e) Find the exact value of $\cos^{-1}\left(\cos\left(\frac{7\pi}{5}\right)\right)$. **2**

Show all necessary working.

$$\begin{aligned}\cos^{-1}\left(\cos\left(\frac{7\pi}{5}\right)\right) &= \cos^{-1}\left(\cos\left(\pi + \frac{2\pi}{5}\right)\right)\\ &= \cos^{-1}\left(\cos\left(\pi - \frac{2\pi}{5}\right)\right)\\ &= \cos^{-1}\left(\cos\left(\frac{3\pi}{5}\right)\right)\\ &= \frac{3\pi}{5}\end{aligned}$$

Again, responses were mixed. Many students were unaware of how to find the related angle in the relevant quadrant.

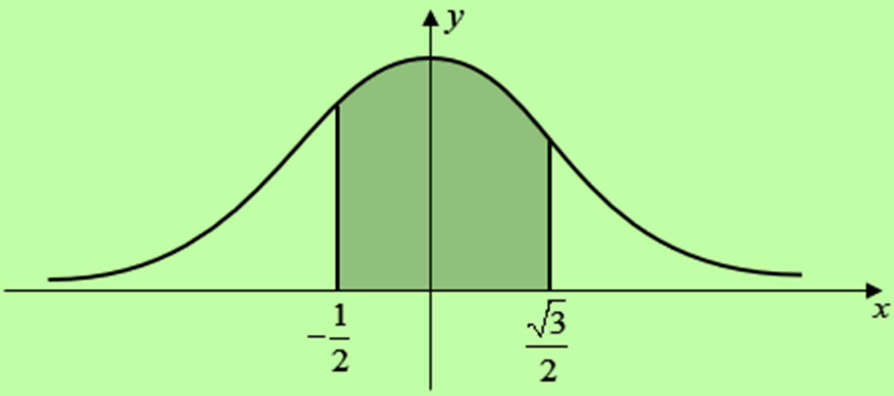
Question 7

(a) Find $\int \sin^2 2x\,dx$. **2**

$$\int \sin^2 2x\ dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

Generally well done.

(b) The graph of $y = \dfrac{1}{\sqrt{1+4x^2}}$ is shown below. **3**



The shaded region in the diagram is bounded by the curve $y = \dfrac{1}{\sqrt{1+4x^2}}$, the x-axis and the lines $x = -\dfrac{1}{2}$ and $x = \dfrac{\sqrt{3}}{2}$. Find the exact volume of the solid of revolution formed when the shaded region is rotated about the x-axis.

$$V = \pi \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{1}{\sqrt{1+4x^2}}\right)^2 dx$$
$$= \pi \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{1+4x^2}$$
$$= \frac{\pi}{2} \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2dx}{1+4x^2}$$
$$= \frac{\pi}{2}\left[\tan^{-1} 2x\right]_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$
$$= \frac{\pi}{2}\left[\tan^{-1}\sqrt{3} - \tan^{-1}(-1)\right]$$
$$= \frac{\pi}{2}\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \frac{\pi}{2}\cdot\frac{7\pi}{12}$$
$$= \frac{7\pi}{24}\ \text{units}^2$$

Many students wasted time by splitting the integral into two.
Others could not get the correct exact value for $\tan^{-1}\sqrt{3}$.
Too many students could not multiply π by π to get π^2.

(c) Use the substitution $u = e^{-x} + 1$ to find the indefinite integral $\int \frac{e^{-2x}}{e^{-x}+1}\,dx$. **3**

$$u = e^{-x} + 1$$
$$du = -e^{-x}dx$$

$$\begin{aligned}\int \frac{e^{-2x}}{e^{-x}+1}dx &= -\int \frac{e^{-x}}{e^{-x}+1}\left(-e^{-x}dx\right)\\ &= -\int \frac{u-1}{u}du\\ &= -\int\left(1-\frac{1}{u}\right)du\\ &= \ln|u| - u + c\\ &= \ln\left(e^{-x}+1\right) - e^{-x} + c \quad \left[\text{including the } -1 \text{ with the } c\right]\end{aligned}$$

Quite well done, but many errors arose with sign.

(d) The area A of an equilateral triangle is increasing at a rate of 4 cm²/s. **3**

The side length of the triangle is x cm.

Find the exact rate of increase of the side length of the triangle when its area is $\sqrt{3}$ cm.

$$A = \frac{1}{2}x^2 \sin 60° = \frac{\sqrt{3}}{4}x^2$$

$$\frac{dA}{dx} = \frac{\sqrt{3}}{2}x$$

When $A = \sqrt{3}$, $\frac{\sqrt{3}}{4}x^2 = \sqrt{3}$

$$x^2 = 4$$
$$x = 2$$

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{dA}\cdot\frac{dA}{dt}\\ &= \frac{2}{x\sqrt{3}}\times 4\end{aligned}$$

$$\begin{aligned}\left.\frac{dx}{dt}\right|_{A=\sqrt{3}} &= \frac{2}{2\sqrt{3}}\times 4\\ &= \frac{4}{\sqrt{3}}\text{ cm/s}\end{aligned}$$

Generally whole marks only, with a few esceptions.

Many couldn't get the correct area formula, and many couldn't simplify it.

Many couldn't correctly reciprocate $\frac{\sqrt{3}}{2}x$.

You should be showing which derivatives you are multiplying, not just the calculation
A number didn't convert from an area of $\sqrt{3}$ to a side length of 2.

Question 8

(a) (i) Given $a > 1$, use the substitution $x = \dfrac{1}{u}$ to rewrite the integral **2**

$$\int_1^a \frac{dx}{1+x^2} \text{ as an integral in } u.$$

$$\int_1^a \frac{dx}{1+x^2} \text{ as an integral in } u.$$

$$\int_1^a \frac{dx}{1+x^2} = \int_1^{\frac{1}{a}} \frac{1}{1+\left(\frac{1}{u}\right)^2} \cdot \left(-\frac{1}{u^2}\,du\right)$$

$$= \int_1^{\frac{1}{a}} \frac{\cancel{u^2}}{u^2+1} \cdot \left(-\frac{1}{\cancel{u^2}}\,du\right)$$

$$= \int_{\frac{1}{a}}^{1} \frac{du}{u^2+1}$$

Let $x = \dfrac{1}{u}$

$\dfrac{dx}{du} = -\dfrac{1}{u^2}$

$dx = -\dfrac{1}{u^2}\,du$

When $x = 1, u = 1$

When $x = a, u = \dfrac{1}{a}$

This was straightforward and generally well done. However, too many students obtained $x = \dfrac{1}{u}$ correctly but then went on to get $\dfrac{dx}{du} = \ln|u|$, mixing up the derivative and the primitive. Some students forgot to change the limits to be in terms of u.

The question asks for an integral in terms of u, it was not necessary to evaluate this integral but many students did.

(ii) Hence, deduce the value of $\tan^{-1}(a) + \tan^{-1}\left(\dfrac{1}{a}\right)$. **2**

From (i) $\displaystyle\int_1^a \frac{dx}{1+x^2} = \int_{\frac{1}{a}}^{1} \frac{du}{u^2+1}$

$$\tan^{-1} a - \tan^{-1} 1 = \tan^{-1} 1 - \tan^{-1}\left(\frac{1}{a}\right)$$

$$\tan^{-1} a + \tan^{-1}\left(\frac{1}{a}\right) = 2\tan^{-1} 1 = 2\left(\frac{\pi}{4}\right)$$

$$\tan^{-1} a + \tan^{-1}\left(\frac{1}{a}\right) = \frac{\pi}{2}$$

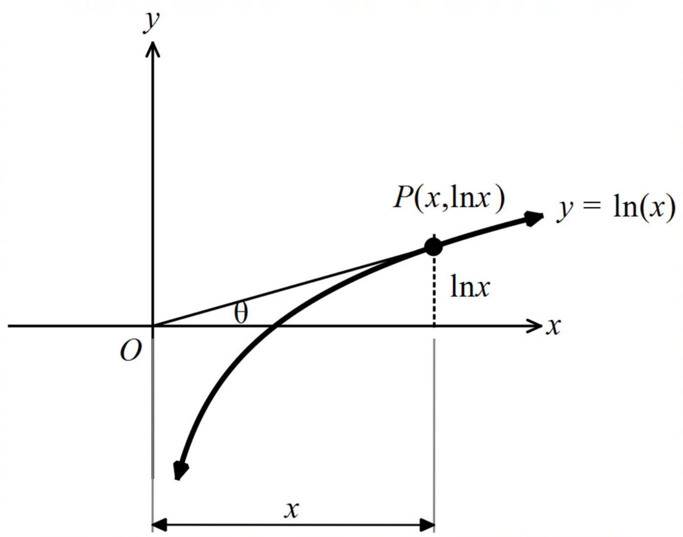
This is a "Hence" question. Students should look to make a link to the previous part. As a general guide, somewhere in your response the words "using part (i)" should appear.

In part (i) we have shown one integral to be the same as another. Both are able to be evaluated using the inverse tan primitive. Equating these yields the solution.

Some students did not use the "hence" and attempted to prove the result using the compound angle result for tan or by differentiating the expression, with varying degrees of success. These responses could not be awarded credit.

(b) Consider the curve $y = \log_e x$ for $x \geq 1$. A variable point P lies on the curve and the interval joining P to the origin O makes an angle of θ with the positive x-axis.

(i) Explain with the aid of a diagram why $\theta = \tan^{-1}\left(\frac{\log_e x}{x}\right)$. **1**



Using right angle trigonometry, $\tan\theta = \dfrac{\ln x}{x}$

So, $\theta = \tan^{-1}\left(\dfrac{\ln x}{x}\right)$

Generally well done. Almost all students drew the correct diagram and were able to explain why $\tan\theta = \dfrac{\ln x}{x}$. Students are encouraged to draw clear diagrams and annotate their diagram.

(ii) Show that $\dfrac{d\theta}{dx} = \dfrac{1-\log_e x}{x^2+(\log_e x)^2}$. **2**

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{\log_e x}{x}\right)^2} \times \frac{\frac{1}{x}.x - \log_e x.1}{x^2}$$

$$= \frac{\cancel{x^2}}{x^2+(\log_e x)^2} \times \frac{1-\log_e x}{\cancel{x^2}}$$

$$= \frac{1-\log_e x}{x^2+(\log_e x)^2}$$

Pretty well done. Students are encouraged to break up a complex question into smaller components, and it was pleasing to see that a large number of students successfully used this technique in this question. These students first used the quotient rule to find the derivative of $\dfrac{\ln x}{x}$ before differentiating $\tan^{-1}\left(\dfrac{\ln x}{x}\right)$. This is a "show" question – students are advised to show all steps of working and not take shortcuts when simplifying.

(iii) Find the coordinates of the point on the curve where $\dfrac{d\theta}{dx} = 0$, and explain the geometrical significance of this result. **2**

$$\frac{d\theta}{dx} = 0 \Rightarrow 1-\log_e(x) = 0$$

$\log_e(x) = 1$

$x = e;$ so $y = 1$

Coordinates are $(e,1)$

Testing nature of stationary point: the sign of $\dfrac{d\theta}{dx}$ is the same as the sign of $1-\log_e x$ as $x^2+(\log_e x)^2 > 0$

x	$\frac{1}{e}$	e	e^2
$1-\log_e x$	2	0	-1
	+	0	–

Therefore, there is a maximum turning point at $(e,1)$ ie θ is maximum at $(e,1)$.

The geometrical implication of this is that OP is a tangent to the curve $y = \log_e(x)$ at the point $(e,1)$.

Almost all students successfully found the x value which makes the derivative zero. However, the question asks for coordinates of the point and several students omitted to find the y-coordinate.
The question asks for the geometric significance. Several students simply stated that this was a stationary point, others claimed it was a maximum but without justification. Some students did establish the nature of the stationary point before stating it was a maximum.
A number of students who claimed the point $(e,1)$ was a maximum turning point on the curve $y=\ln x$ indicating confusion regarding what derivative they were working with.

Geometrically, on the diagram drawn in part (i), a maximum value of θ translates to when OP is a tangent to the curve $y=\ln x$. Only a couple of students realised this.
Full credit was ony awarded to those students who said θ was maximum at this point (with some justification either using calculus or by using diagrams) <u>and</u> also stated this was a point of tangency.

(iv) P moves on the curve so that its y-coordinate increases at a constant rate of 0.5 units/s. **3**

Find, in degrees per second, the rate at which OP is rotating when P is 2 units above the x-axis.

We are given that $\frac{dy}{dt}=0.5$ and we need to find $\frac{d\theta}{dt}$ when $y=2$

$$\frac{d\theta}{dt}=\frac{d\theta}{dx}\times\frac{dx}{dy}\times\frac{dy}{dt} \text{ (Chain Rule)}$$

Now $\frac{d\theta}{dx}=\frac{1-\log_e(x)}{x^2+\left[\log_e(x)\right]^2}$ (from (ii)) and $y=\log_e(x)$ so $\frac{dy}{dx}=\frac{1}{x}$ and $\frac{dx}{dy}=x$

$$\frac{d\theta}{dt}=\frac{1-\log_e(x)}{x^2+\left[\log_e(x)\right]^2}\times x\times 0.5$$

When $y=2$, $\log_e(x)=2$ so $x=e^2$

$$\begin{aligned}\frac{d\theta}{dt}&=\frac{1-2}{e^4+4}\times e^2\times 0.5\\ &=\frac{-e^2}{2e^4+8} \text{ radians/second}\\ &=\frac{-90e^2}{\pi\left(e^4+4\right)} \text{ degrees/second}\\ &\approx -3.6 \text{ deg/sec}\end{aligned}$$

OP is rotating clockwise at a rate of 3.6 degrees per second.

Reasonably well attempted for a last question.
Several students had difficulty interpreting the question – both in understanding what was given and what was asked for.
Some incorrectly assumed the question was asking for the rate at which the length of *OP* was changing. A lot of tedious working followed but could not be awarded much credit.
Others assumed that $\frac{dx}{dt} = 0.5$ and the question was asking for $\frac{d\theta}{dt}$ when $x = 2$, which simplifies the question considerably.

Many who did have the correct approach to the question did not realise that their answer was in radians per second and would need to be converted to degrees per second, which incurred a ½ mark penalty. On a side note, if the question asks for answer in degrees per second, it does not require an answer in whole number of degrees per second.

A cautionary note for the future on some conceptual errors:

- some students substituted $x = e^y$ into the expression for $\frac{d\theta}{dx}$ from part (ii) and assumed this gives them $\frac{d\theta}{dy}$. It doesn't. $\frac{d\theta}{dy} = \frac{d\theta}{dx} \times \frac{dx}{dy}$.
- some students substituted a value before differentiating. Once you substitute a value you have a constant - differentiating will yield zero.
- Some students obtained an expression for *OP* in terms of two variables eg x and y or x and θ and then differentiated with respect to one of the variables say x, treating the other variable as if it were a constant. The variables are not independent of each other. Try to express everything in terms of one variable before differentiating.