NORTH SYDNEY GIRLS HIGH SCHOOL

2024 | **HSC Assessment Task 3**

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 60 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Section II, show relevant mathematical reasoning and/or calculations

Total marks – 40

Section I – 4 marks (pages 2 – 3)

- Attempt Questions 1 – 4
- Allow about 7 minutes for this section

Section II – 36 marks (pages 5 – 7)

- Attempt Questions 5-7
- Allow about 53 minutes for this section

NAME:______________________________ **TEACHER:**____________________

STUDENT NUMBER:

Question	1 – 4	5	6	7	Total
Inverse Functions	Q1,4 /2	/3	/4	/5	**/14**
Further Integration Techniques	Q2,3 /2	/3	/8		**/13**
Further Trigonometry and Rates of Change		/6		/7	**/13**
	/4	**/12**	**/12**	**/12**	**/40**

Section I

4 marks
Attempt Questions 1-4
Allow about 7 minutes for this section

Use the multiple choice answer sheet for Questions 1-4.

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1 What is the domain of the function $\sin^{-1}(3x)$?

A. $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$

B. $-\frac{1}{3} \le x \le \frac{1}{3}$

C. $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$

D. $-3 \le x \le 3$

2 Which of the following is a primitive of $\cos^2\left(\frac{2x}{3}\right)$?

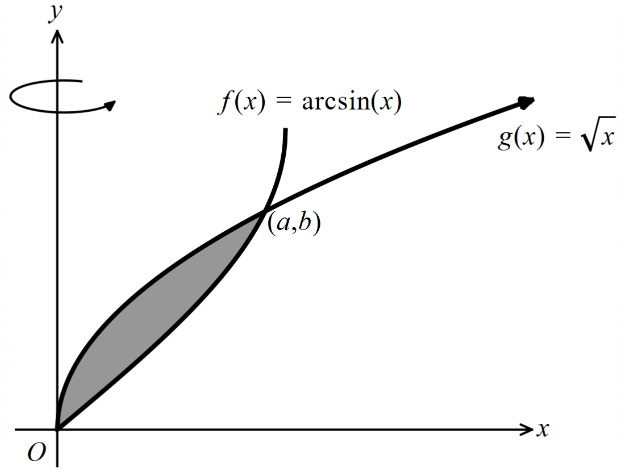
A. $\frac{x}{2}+\frac{3}{4}\sin\left(\frac{4x}{3}\right)$

B. $\frac{x}{2}-\frac{3}{4}\sin\left(\frac{4x}{3}\right)$

C. $\frac{x}{2}+\frac{3}{8}\sin\left(\frac{4x}{3}\right)$

D. $\frac{x}{2}-\frac{3}{8}\sin\left(\frac{4x}{3}\right)$

3 The curves $f(x) = \arcsin x$ and $g(x) = \sqrt{x}$ intersect at $(0,0)$ and (a,b) in the first quadrant as shown. The region enclosed by the curves is rotated about the y-axis.



Which of the following gives the volume of the solid of revolution formed?

A. $\pi \int_0^a x - \left(\sin^{-1} x\right)^2 \, dx$

B. $\pi \int_0^b \sin^2 y - y^2 \, dy$

C. $\pi \int_0^b \left(\sin y - y^2\right)^2 \, dy$

D. $\pi \int_0^b \sin^2 y - y^4 \, dy$

4 What is the exact value of $\sin^{-1}\left(\cos(a\pi)\right)$ where a can be any value in the interval $(1, 1.5)$?

A. $a\pi$

B. $\pi - a\pi$

C. $a\pi - \dfrac{3\pi}{2}$

D. $a\pi + \dfrac{\pi}{2}$

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Section II

36 marks
Attempt Questions 5-7
Allow about 53 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 5-7, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (12 marks) Use a SEPARATE writing booklet

(a) Differentiate $\cos^{-1}(3x)$. **1**

(b) Find $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2}$. **2**

(c) Use the substitution $u = 3x+1$ to evaluate $\int_0^8 \frac{x}{\sqrt{3x+1}}dx$ **3**

(d) The radius of a circular oil spill is increasing at a rate of 3 metres per minute. Find the rate at which the area covered by the oil spill is increasing, when the radius is 20 metres? **2**

(e) By writing $\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, solve $\cos x = \sin x + 1$ in the domain $0 \le x \le 2\pi$. **4**

Question 6 (12 marks) Use a SEPARATE writing booklet

(a) Find $\int \cos 3\theta \sin 5\theta \, d\theta$. **2**

(b) Let $f(x) = x^3 + 3x + 1$.

(i) Explain why $f^{-1}(x)$ exists. **1**

(ii) Find the point(s) on the graph of $y = f^{-1}(x)$ where the gradient is $\frac{1}{6}$. **3**

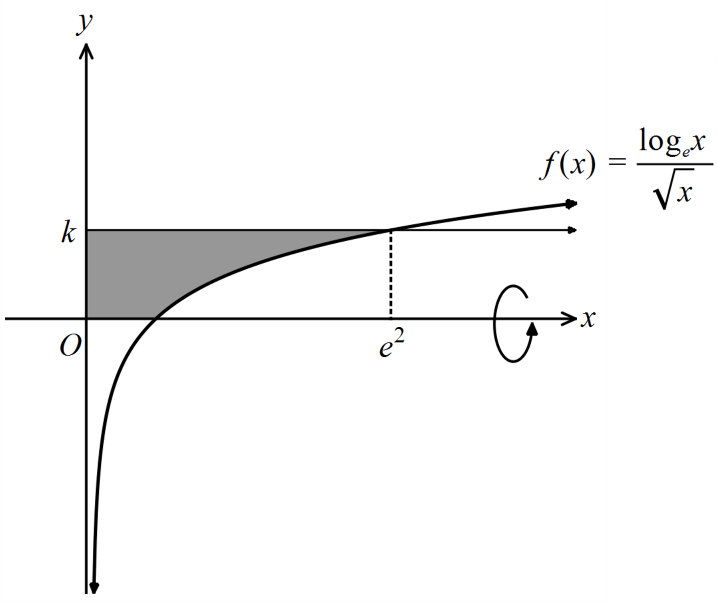
(c) (i) Use the substitution $u = \log_e x$ to find $\int \frac{(\log_e x)^2}{x} dx$. **2**

(ii) Consider the function $f(x) = \frac{\log_e x}{\sqrt{x}}$. **4**

The line $y = k$ intersects the graph of $y = f(x)$ at $x = e^2$.

A student designs a glass ornament which can be modelled by rotating the region bounded by the curve, the coordinate axes and the line $y = k$ about the x-axis.

By first finding the value of k, find the exact volume of glass used in the ornament.



Question 7 (12 marks) Use a SEPARATE writing booklet

(a) You are given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. DO NOT PROVE THIS.

(i) By substituting $x = \cos\theta$ in the polynomial equation $8x^3 - 6x + 1 = 0$, show that $\cos 3\theta = -\frac{1}{2}$. **2**

(ii) Using the result of part (i), prove that $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{1}{8}$. **2**

(b) Consider $f(x) = \tan^{-1}\left(\frac{\sin x}{\cos x - 1}\right)$, $0 < x < \pi$.

(i) Show that $f'(x)$ is constant. **3**

(ii) Hence sketch the graph of $f(x)$. **2**

(c) A curve C is defined by the following set of parametric equations: **3**

$$\left.\begin{aligned} x &= \sin\theta \\ y &= e^{-\cos\theta} \end{aligned}\right\}, \text{ where } 0 \le \theta \le 2\pi$$

A point P traces the curve C as the value of θ changes at a constant rate of 0.2 radians per second.

Find the exact rate at which the gradient m of the curve at P is changing when $\theta = \frac{2\pi}{3}$.

NOTE: The gradient m of the curve C at any point P on the curve is given by $\frac{dy}{dx}$.

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Multiple Choice

1. $-1 \leq 3x \leq 1$

$-\frac{1}{3} \leq x \leq \frac{1}{3}$ $\quad\therefore$ B

2. $\int \cos^2 \frac{2x}{3}\, dx = \frac{1}{2}\int \left(1 + \cos\frac{4x}{3}\right) dx$

$= \frac{1}{2}\left(x + \frac{3}{4}\sin\frac{4x}{3}\right)$

$= \frac{x}{2} + \frac{3}{8}\sin\frac{4x}{3}$ $\quad\therefore$ C

3. For outer radius

$y = \arcsin x$

$x = \sin y$

$V_{outer} = \pi\int_0^b \sin^2 y\, dy$

For inner radius

$y = \sqrt{x}$

$x = y^2$

$V_{inner} = \pi\int_0^b y^4\, dy$

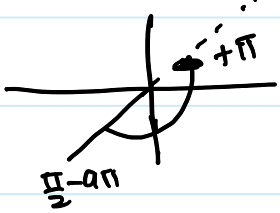
$V = \pi\int_0^b (\sin^2 y - y^4)\, dy$ $\quad\therefore$ D

4. $\sin^{-1}(\cos a\pi) = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - a\pi\right)\right)$

Now $\quad \pi < a\pi < \frac{3\pi}{2}$ $\quad\because 1 < a < \frac{3}{2}$

$\therefore$

$-\pi < \frac{\pi}{2} - a\pi < -\frac{\pi}{2}$



Related angle in 1st Quad $= \left(\frac{\pi}{2} - a\pi\right) + \pi$

$= \frac{3\pi}{2} - a\pi$

$\sin^{-1}\left(\sin\left(\frac{\pi}{2} - a\pi\right)\right) = -\left(\frac{3\pi}{2} - a\pi\right) = a\pi - \frac{3\pi}{2}$

$\therefore$ C

Question 5

a) $\frac{d}{dx}\cos^{-1}(3x) = \frac{-1}{\sqrt{1-(3x)^2}} \times 3 = \frac{-3}{\sqrt{1-9x^2}}$

b) $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{9+4x^2} = \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \frac{2\,dx}{3^2+(2x)^2}$

$= \frac{1}{2} \times \frac{1}{3}\left[\tan^{-1}\frac{2x}{3}\right]_0^{\frac{\sqrt{3}}{2}}$

$= \frac{1}{6}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1} 0\right)$

$= \frac{1}{6} \times \left(\frac{\pi}{6} - 0\right)$

$= \frac{\pi}{36}$

(c) $\int_0^8 \frac{x}{\sqrt{3x+1}}\,dx$

Let $u = 3x+1$

$\frac{du}{dx} = 3$ so $\frac{du}{3} = dx$

Also, $x = \frac{u-1}{3}$

x	0	8
u	1	25

$= \int_1^{25} \frac{u-1}{3\sqrt{u}} \times \frac{du}{3}$

$= \frac{1}{9}\int (u-1)u^{-\frac{1}{2}}\,du$

$= \frac{1}{9}\int u^{\frac{1}{2}} - u^{-\frac{1}{2}}.\,du$

$= \frac{1}{9}\left[\frac{2}{3}u^{3/2} - 2u^{\frac{1}{2}}\right]_1^{25} = \frac{1}{9}\left[\left(\frac{2}{3}\times 125 - 10\right) - \left(\frac{2}{3} - 2\right)\right]$

$= \frac{224}{27}$

Question 5 (contd)

(d) $\frac{dr}{dt} = 3$ $\frac{dA}{dt} = ?$ when $r = 20$

$A = \pi r^2$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ (Chain Rule)

$= (2\pi r)\, 3$

$= 6\pi r$

When $r = 20$, $\frac{dA}{dt} = 6\pi \times 20$

$= 120\pi$

The area covered by the oil spill is increasing at a rate of 120π m^2/min.

(e) Consider $\cos x - \sin x = R\cos(x+\alpha)$

$= R\cos\alpha\cos x - R\sin\alpha\sin x$

Equating coefficients of $\cos x$ and $\sin x$

$R\cos\alpha = 1$ ①

$R\sin\alpha = 1$ ②

①^2 + ②^2 : $R^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + 1^2$

$R^2 = 2$

$R = \sqrt{2}$, $R > 0$

from ① $\cos\alpha = \frac{1}{R} = \frac{1}{\sqrt{2}}$

$\therefore \alpha = \frac{\pi}{4}$ as $0 \leq \alpha \leq \frac{\pi}{2}$

Question 5 contd

(e) contd.

$\therefore \quad \cos x - \sin x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

Now $\quad \cos x = \sin x + 1$

so $\quad \cos x - \sin x = 1$

$\therefore \quad \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = 1$

$\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad 0 \leqslant x \leqslant 2\pi$

$\frac{\pi}{4} \leqslant x + \frac{\pi}{4} \leqslant 2\pi + \frac{\pi}{4}$

$x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$

$x = 0, \frac{3\pi}{2}, 2\pi$

Question 6

a) $\int \cos 3\theta \sin 5\theta \, d\theta = \int \frac{\sin(5\theta + 3\theta) + \sin(5\theta - 3\theta)}{2} \, d\theta$

using products to sums $A = 5\theta$ $B = 3\theta$

$= \frac{1}{2} \int (\sin 8\theta + \sin 2\theta) \, d\theta$

$= \frac{1}{2} \left(-\frac{\cos 8\theta}{8} - \frac{\cos 2\theta}{2} \right) + C$

$= -\frac{\cos 8\theta}{16} - \frac{\cos 2\theta}{4} + C$

$= -\frac{1}{16} (\cos 8\theta + 4\cos 2\theta) + C$

(b) $f(x) = x^3 + 3x + 1$

(i) $f'(x) = 3x^2 + 3$

> 0 for all x as $x^2 \geq 0$

$\therefore$ $f(x)$ is monotonically increasing and

$\therefore$ $f^{-1}(x)$ exists

(ii) corresponding points on $f(x)$ have a gradient of 6.

$f'(x) = 6 \Rightarrow 3x^2 + 3 = 6$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

Pts on $f(x)$ with gradient of 6 are $(1, 5)$ and $(-1, -3)$

Corresponding pts on $f(x)$ $(5, 1)$ and $(-3, -1)$

have a gradient of $\frac{1}{6}$.

Question 6 contd

(c) (i) $\int \frac{(\ln x)^2}{x}\,dx$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ so $du = \frac{1}{x}dx$

$= \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \frac{(\ln x)^3}{3} + C$

(ii) $k = \frac{\ln(e^2)}{\sqrt{e^2}} = \frac{2}{e}$

$\text{Volume} = \text{Volume}_{\text{cylinder}} - \text{Volume}_{\text{solid of revolution}}$

$= \pi\left(\frac{2}{e}\right)^2 \times e^2 - \pi\int_1^{e^2}\left(\frac{\ln x}{\sqrt{x}}\right)^2 dx$

$= \pi \times \frac{4}{\cancel{e^2}} \times \cancel{e^2} - \pi\int_1^{e^2}\frac{(\ln x)^2}{x}\,dx$

$= 4\pi - \pi\left[\frac{(\ln x)^3}{3}\right]_1^{e^2}$ using (i)

$= 4\pi - \pi\left(\frac{8}{3} - 0\right)$

$= 4\pi - \frac{8\pi}{3}$

$= \frac{4\pi}{3}$

$\therefore$ volume of glass used is $\frac{4\pi}{3}$ cm^3.

Question 7

(a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (given)

(i) $8x^3 - 6x + 1 = 0$ Sub $x = \cos\theta$

$8\cos^3\theta - 6\cos\theta + 1 = 0$

$2(4\cos^3\theta - 3\cos\theta) = -1$

$2\cos 3\theta = -1$

$\cos 3\theta = -\frac{1}{2}$

(ii) $\cos 3\theta = -\frac{1}{2}$

$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{12\pi}{3}, \ldots$

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{12\pi}{9}, \ldots$

As $x = \cos\theta$ then $x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$ are the solutions to $8x^3 - 6x + 1 = 0$

Using product of roots:

$\cos\frac{2\pi}{9} \cdot \cos\frac{4\pi}{9} \cdot \cos\frac{8\pi}{9} = -\frac{1}{8}$ $\left(-\frac{d}{a}\right)$

(b) $f(x) = \tan^{-1}\left(\frac{\sin x}{\cos x - 1}\right)$

$f'(x) = \frac{1}{1 + \left(\frac{\sin x}{\cos x - 1}\right)^2} \times \frac{\cos x(\cos x - 1) - \sin x(-\sin x)}{(\cos x - 1)^2}$

$= \frac{\cancel{(\cos x - 1)^2}}{(\cos x - 1)^2 + \sin^2 x} \times \frac{\cos^2 x - \cos x + \sin^2 x}{\cancel{(\cos x - 1)^2}}$

Question 7 (contd)

(b) contd.

$$f'(x) = \frac{1-\cos x}{\cos^2 x + 1 - 2\cos x + \sin^2 x}$$

$$= \frac{1-\cos x}{2 - 2\cos x}$$

$$= \frac{1-\cancel{\cos x}}{2(1-\cancel{\cos x})}$$

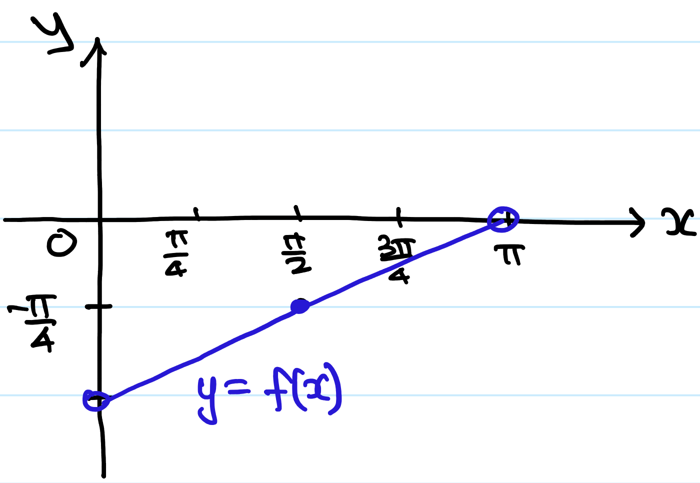
$$= \frac{1}{2}$$

(ii) $f'(x) = \frac{1}{2}$ so graph of $f(x)$ is a straight line with gradient $\frac{1}{2}$.

Now $f\left(\frac{\pi}{2}\right) = \tan^{-1}\left(\frac{1}{0-1}\right)$

$= \tan^{-1}(-1)$

$= -\frac{\pi}{4}$



Question 7 (contd)

(c) $x = \sin\theta$ $\qquad \frac{d\theta}{dt} = 0.2$ (given)

$y = e^{-\cos\theta}$

$\frac{dm}{dt} = ?$ when $\theta = \frac{2\pi}{3}$

$$m = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} \quad \text{(Chain Rule)}$$

$$= \frac{dy}{d\theta} \times \frac{1}{\frac{dx}{d\theta}}$$

$$= \sin\theta\, e^{-\cos\theta} \times \frac{1}{\cos\theta}$$

$$= \tan\theta\, e^{-\cos\theta}$$

So, $\frac{dm}{dt} = \frac{dm}{d\theta} \times \frac{d\theta}{dt}$ (Chain Rule)

$$= \left[\sec^2\theta \left(e^{-\cos\theta}\right) + \tan\theta . \sin\theta \left(e^{-\cos\theta}\right)\right] \times 0.2$$

At $\theta = \frac{2\pi}{3}$, $\frac{dm}{dt} = \left((-2)^2 e^{+\frac{1}{2}} + (-\sqrt{3}) \frac{\sqrt{3}}{2} e^{+\frac{1}{2}}\right) \times \frac{1}{5}$

$$= \sqrt{e} \times \left(4 - \frac{3}{2}\right) \times \frac{1}{5}$$

$$= \frac{1}{2}\sqrt{e}$$

the gradient of the curve at P is changing at a rate of $\frac{\sqrt{e}}{2}$ per second.