Student's Name:___________________________ Teacher's Code : ________________________

Saint Ignatius' College, Riverview

Mathematics Assessment Task

2022

Year 12
Mathematics Extension 1
Task 1
In–Class Assessment
Date: 30 November 2021

General Instructions:

- **Time Allowed: 45 minutes**
- Write using blue or black pen only
- Board approved calculators may be used
- Attempt all questions in the space provided on the paper
- Write **your name** and **your teacher's code** in the positions indicated
- Marks may be deducted for missing or carelessly arranged working

Teachers :

- Mr R Maxwell **REM**
- Mr D Reidy **DPR**
- Mr N Mushan **NHM**
- Mr J Newey **JPN**

Topics Examined:

Question 1 **10 marks**
Polynomials

Question 2 **10 marks**
Polynomials

Question 3 **10 marks**
Mathematical Induction

Total **30 Marks**

Question 1 (10 marks). Use a SEPARATE writing booklet. **Marks**

(a) The quadratic equation $4x^2 - px - 5 = 0$ has two roots opposite in sign but equal in magnitude. Find the value of p. **2**

(b) Find the values of m and n if $(x-1)$ and $(x+2)$ are factors of

$$P(x) = x^4 + 4x^3 + mx^2 - n$$ **2**

(c) i) Show that $P(x) = 2x^3 + x^2 - 13x + 6$ is divisible by $(2x-1)$ and $(x+3)$ **2**

ii) Hence, find all the **zeros** of $P(x)$ **1**

(d) Given that the polynomial $x^2 + x + 2$ is a factor of $x^4 + 4x^2 + x + a$.

i) Using the process of Long Division of Polynomials, or otherwise, find the other factor. **2**

ii) Hence, find the value of a. **1**

Question 2 (10 marks). Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $4x^3 + 32x^2 + 79x + 60 = 0$ given that one root is the sum of the other two roots. **3**

2

(b) Consider the function $f(x) = x^3 - x^2 - 8x + 12$.

i) Show that the graph of $y = f(x)$ cuts the x-axis at one point and touches it at another. **2**

ii) Sketch the graph of $y = f(x)$ clearly identifying all intercepts. **2**

(c) The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β and γ

i) Show that $\beta + \gamma = 1 - \alpha$ **1**

ii) Using part (i), and similar results, evaluate $\dfrac{\beta+\gamma}{\alpha} + \dfrac{\gamma+\alpha}{\beta} + \dfrac{\alpha+\beta}{\gamma}$ **2**

Question 3 (10 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove by Mathematical Induction that for all positive integers $n \geq 1$ **3**

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\ldots \qquad \cdot 1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$$

(b) Prove by Mathematical Induction that for all positive integers $n \geq 1$

$7^{2n}+7^{n}+4$ is divisible by 6. **3**

(c) (i) Use Mathematical Induction to prove for all integers $n \geq 1$, that **3**

$$\left(\frac{1\times 5}{3^2}\right)\left(\frac{3\times 7}{5^2}\right)\left(\frac{5\times 9}{7^2}\right)\ldots\left(\frac{\quad \cdot 1)(2n+3)}{(2n+1)^2}\right)=\frac{2n+3}{6n+3}$$

(ii) As n approaches infinity (becomes very large), what does the value of the following expression approach **1**

$$\left(\frac{1\times 5}{3^2}\right)\left(\frac{3\times 7}{5^2}\right)\left(\frac{5\times 9}{7^2}\right)\ldots$$

END OF ASSESSMENT TASK

Student's Name:____________________________ Teacher's Code : _________________________

SUGGESTED SOLUTIONS

Saint Ignatius' College, Riverview

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Topics Examined:

Question 1 Polynomials	**10 marks**
Question 2 Polynomials	**10 marks**
Question 3 Mathematical Induction	**10 marks**
Total	**30 Marks**

(a) $4x^2 - px - 5 = 0$

$(2x - \sqrt{5})(2x + \sqrt{5}) = 0$

$\therefore \quad p = 0.$

(b) $P(x) = x^4 + 4x^3 + mx^2 - n$

$P(1) = 1 + 4 + m - n = 0$

$5 + m - n = 0 \quad \text{..(1)}$

$P(-2) = 16 - 32 + 4m - n = 0$

$-16 + 4m - n = 0 \quad \text{..(2)}$

(2) − 1

$-21 + 3m = 0$

$\underline{m = 7}$

Sub into (1) $\quad 5 + 7 - n = 0$

$\underline{n = 12}$

(c) $P(x) = 2x^3 + x^2 - 13x + 6$

$2x - 1 = 0$

$x = \frac{1}{2}$

$P(\frac{1}{2}) = 2(\frac{1}{2})^3 + (\frac{1}{2})^2 - 13(\frac{1}{2}) + 6$

$= \frac{1}{4} + \frac{1}{4} - \frac{1}{2}$

$= 0 \quad \therefore (2x - 1)$ is a factor

$P(-3) = 2(-3)^3 + (-3)^2 - 13(-3) + 6$

$= -54 + 9 + 39 + 6$

$= 0 \quad \therefore (x+3)$ is a factor

let $(x + a)$ be the third factor

$\therefore -1 \times 3 \times a = 6$

$a = -2$, zeros: $\frac{1}{2}, -3, 2.$

(d) (i)

$$\begin{array}{r} x^2 - x + 3 \\ x^2+x+2 \overline{) x^4 + \quad + 4x^2 + x + a} \\ \underline{x^4 + x^3 + 2x^2} - \\ -x^3 + 2x^2 + x \quad - \\ \underline{-x^3 - x^2 - 2x} \\ 3x^2 + 3x + a \quad - \\ \underline{3x^2 + 3x + 6} \end{array}$$

(ii) $r = a - 6 = 0$

$\therefore \underline{\underline{a = 6}}$

Q2. (a) $4x^3 + 32x^2 + 79x + 60 = 0$

let the roots be α, β and $(\alpha+\beta)$

$\alpha + \beta + (\alpha + \beta) = \frac{-b}{a}$

$2\alpha + 2\beta = -\frac{32}{4}$

$2(\alpha+\beta) = -8$

$\alpha + \beta = -4$

$\alpha \cdot \beta \cdot (\alpha+\beta) = \frac{-d}{a}$

$= \frac{-60}{4}$

$= -15$

$\therefore \alpha\beta(-4) = -15$

$\alpha\beta = \frac{15}{4}$

$\alpha + \beta = -4$

$\therefore \beta = -4 - \alpha$

(a) CONTINUED

$\alpha\beta = \frac{15}{4}$

$\alpha(-4-\alpha) = \frac{15}{4}$

$-4\alpha - \alpha^2 = \frac{15}{4}$

$\therefore \alpha^2 + 4\alpha + \frac{15}{4} = 0$

$\therefore \alpha = \dfrac{-4 \pm \sqrt{16 - 4.1.\frac{15}{4}}}{2}$

$= \dfrac{-4 \pm 1}{2}$

$= -\frac{3}{2}, \ -\frac{5}{2}$

$\therefore \alpha + \beta = -4$

if $\alpha = -\frac{3}{2}$,

$\beta = -4 + \frac{3}{2}$

$= -\frac{5}{2}$

roots are: $-4, -\frac{5}{2}, -\frac{3}{2}$

(b) $f(x) = x^3 - x^2 - 8x + 12$

By inspection $(x+3)(x-2)(x-2)$

check:

$f(2) = 2^3 - 2^2 - 8(2) + 12$

$= 8 - 4 - 16 + 12$

$= 0$

$f(-3) = (-3)^3 - (-3)^2 - 8(-3) + 12$

$= -27 - 9 + 24 + 12$

$= 0$

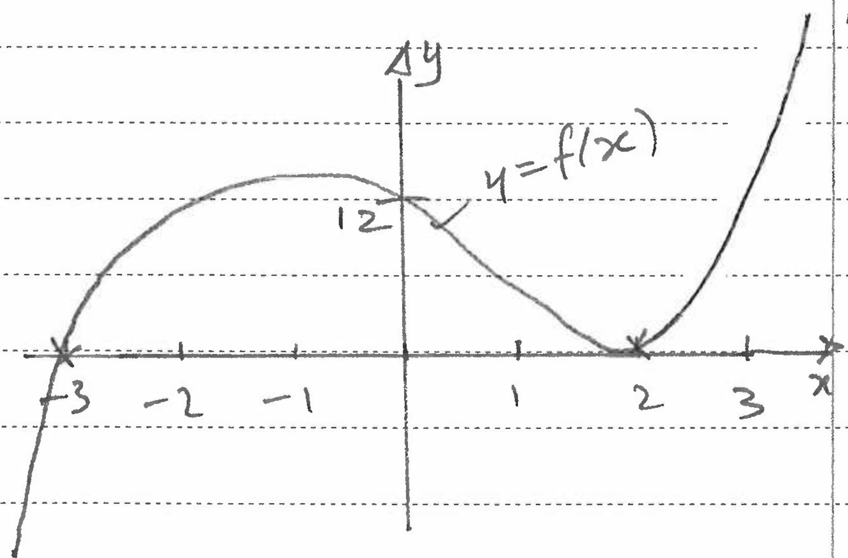
$f(x) = (x-2)(x+3)(x+a)$

$-2 \times 3 \times a = 12$

$a = -2$

$\therefore (x-2)$ is a repeated factor.

(b) Continued



Mark 1: Correct shape and position

Mark 2: Three intercepts shown

(c) $x^3 - x^2 - 5x + 2 = 0$

(i) $\alpha + \beta + \gamma = \frac{-(-1)}{1}$

$= 1$

$\therefore \alpha + \beta = 1 - \gamma$

(ii) Similarly $\alpha + \gamma = 1 - \beta$, $\beta + \gamma = 1 - \alpha$

$\therefore \frac{\beta+\gamma}{\alpha} + \frac{\alpha+\gamma}{\beta} + \frac{\alpha+\beta}{\gamma} = \frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} + \frac{1-\gamma}{\gamma}$

$= \frac{\beta\gamma(1-\alpha) + \alpha\gamma(1-\beta) + \alpha\beta(1-\gamma)}{\alpha\beta\gamma}$

$= \frac{\beta\gamma - \alpha\beta\gamma + \alpha\gamma - \alpha\beta\gamma + \alpha\beta - \alpha\beta\gamma}{\alpha\beta\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} - 3\frac{\cancel{\alpha\beta\gamma}}{\cancel{\alpha\beta\gamma}}$

$= \frac{\frac{c}{a}}{\frac{-d}{a}} - 3$

$= \frac{-5}{-2} - 3$

$= -\frac{1}{2}$

(a) Prove $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \dots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

1. Prove true for $n=1$

i.e. $n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

$LHS = 1 \times 2 \times 3$
$= 6$

$RHS = \frac{1}{4}\cdot 1(2)(3)(4)$
$= 6$

$\therefore$ true for $n=1$.

2. Assume true for $n=K$

i.e. $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \dots + K(K+1)(K+2) = \frac{1}{4}K(K+1)(K+2)(K+3)$

3. Prove true for $n=K+1$

i.e. $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \dots + K(K+1)(K+2) + (K+1)(K+2)(K+3)$
$= \frac{1}{4}(K+1)(K+2)(K+3)(K+4)$

$LHS = 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \dots + K(K+1)(K+2) + (K+1)(K+2)(K+3)$
$= \frac{1}{4}K(K+1)(K+2)(K+3) + (K+1)(K+2)(K+3)$
$= \left(\frac{1}{4}K + 1\right)(K+1)(K+2)(K+3)$
$= \frac{1}{4}(K+4)(K+1)(K+2)(K+3$
$= RHS$

$\therefore$ true for $K+1$ if true for K. As true for $n=1$, true for $n=2$ and so on. True for all $n \geq 1$.

(b) Prove $7^{2n} + 7^{n} + 4$ is divisible × 6, $n \geq 1$

1. Prove true for $n = 1$

$7^2 + 7 + 4 = 60$

$= 6 \times 10$ ∴ true for $n=1$

2. Assume true for $n = K$

i.e. $7^{2K} + 7^{K} + 4 = 6M$, M is an integer

3. Prove true for $n = K+1$

i.e. $7^{2(K+1)} + 7^{K+1} + 4 = 6N$, N is an integer.

$LHS = 7^2 . 7^{2K} + 7 . 7^{K} + 4$

$= 7^2 \left[6M - 7^K - 4 \right] + 7.7^K + 4$

$= 6 \times 49M - 42.7^K - 48.4$

$= 6 \left[49M - 7.7^K - 32 \right]$

$= 6N$ [as $49M$, 7.7^K and 32 are all integers]

∴ True for $n = K+1$ if true for $n = K$

as true for $n=1$, true for $n=2$, and so on... true for all $n \geq 1$.

(e) RTP $\left(\frac{1\times5}{3^2}\right)\left(\frac{3\times7}{5^2}\right)\left(\frac{5\times9}{7^2}\right)\cdots\left[\frac{(2n-1)(2n+3)}{(2n+1)^2}\right]=\frac{2n+3}{6n+3}$

1. Prove true for $n=1$.

$LHS = \frac{(2-1)(2+3)}{(2+1)^2}$ $\quad$ $RHS = \frac{2+3}{6+3}$

$= \frac{5}{9}$ $\quad$ $= \frac{5}{9}$

$\therefore$ true for $n=1$.

2. Assume true for $n=k$.

i.e. $\left(\frac{1\times5}{3^2}\right)\left(\frac{3\times7}{5^2}\right)\left(\frac{5\times7}{7^2}\right)\cdots\left[\frac{(2k-1)(2k+3)}{(2k+1)^2}\right]=\frac{2k+3}{6k+3}$

3. Prove true for $n=k+1$

i.e. $\left(\frac{1\times5}{3^2}\right)\left(\frac{3\times7}{5^2}\right)\left(\frac{5\times7}{7^2}\right)\cdots\left[\frac{(2k-1)(2k+3)}{(2k+1)^2}\right]\left[\frac{(2k+1)(2k+5)}{(2k+3)^2}\right]=\frac{2k+5}{6k+9}$

$LHS = \left(\frac{1\times5}{3^2}\right)\left(\frac{3\times7}{5^2}\right)\cdots\left[\frac{(2k-1)(2k+3)}{(2k+1)^2}\right]\left[\frac{(2k+1)(2k+5)}{(2k+3)^2}\right]$

$= \left[\frac{\cancel{2k+3}}{6k+3}\right]\left[\frac{(2k+1)(2k+5)}{(2k+3)^{\cancel{2}}}\right]$

$= \frac{\cancel{(2k+1)}(2k+5)}{3\cancel{(2k+1)}(2k+3)}$

$= \frac{2k+5}{6k+9}$

$= RHS$

$\therefore$ true for $n=k+1$ if true for $n=k$.

as true for $n=1$, true for $n=2$ and so on ... $\therefore$ true for all $n\geq1$.

Q3 (c) (ii) [Formal limits not covered]

$$\left(\frac{1\times5}{3^2}\right)\left(\frac{3\times7}{5^2}\right)\left(\frac{5\times9}{7^2}\right)\ldots = \frac{2n+3}{6n+3}$$

let $n = 10\,000$ say,

$$\therefore \frac{2(10\,000)+3}{6(10000)+3} = 0.33336\ldots$$

$$\doteqdot \frac{1}{3}.$$

Seems to be heading to $\frac{1}{3}$.