The Scots College

Year 12

HSC Extension 1 Mathematics

Task 1

February 2013

Time Allowed: 45 minutes

Weighting: 10%

Instructions to Students:

- Show all working
- Board approved calculators may be used
- Table of Standard Integrals attached
- Start a new booklet for each Question.

Question/ Learning Intention	Polynomials	Applications of Calculus
1	/9	
2	/7	

3		**/10**
4		**/11**
Total **/37**		

Question 3: 10 Marks (start a new booklet) **Marks**

a) Find $\int x\sqrt{9-x^2}\ \ dx$, using the substitution $u = 9 - x^2$. **3**

b) Find $\int \frac{1-x}{(1+x)^3}\ dx$, using the substitution $u = 1 + x$. **3**

c) At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = 2cos^2x + 1$. If $y = \pi$ when $x = \pi$, find the value of y when $x = 2\pi$. **4**

Question 4: 11 Marks (start a new booklet)

2

a) Evaluate $\int_0^{\pi} 2sinxcos^2 x\ \ dx$

b) The area bounded by the curve $y = sinx$ between $x = 0$ and $\frac{\pi}{2}$ and the x-axis is rotated about the x-axis. Find the volume of the solid of revolution. **3**

c) i) By expanding $\cos(2\theta + 2\theta)$ or otherwise, show that : **3**

$$cos4\theta = 1 - 8sin^2\theta + 8sin^4\theta$$

ii) Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (sin^2 \theta - sin^4\theta)\ d\theta$ **3**

END OF EXAM

Question 1 9 Marks (start a new booklet) Marks

a)i) Given the polynomial $P(x) = 2x^3 - 9x^2 + kx + 6$, find the value of k if $(x - 3)$ is a factor of $P(x)$. **2**

ii) Hence, or otherwise, determine all the roots of the equation $P(x) = 0$. **3**

b) If the equation $5x^3 - 6x^2 - 29x + 6 = 0$ has roots α, β, γ find the value of the following (leave your answer in fraction form) :

i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

Question 2 7 Marks (start a new booklet) — Marks

a)i)	Show that the equation $f(x) = x^3 - 8x + 8$ has a zero between $x = -3$ and $x = -4$.	**2**
ii)	Taking $x = -3.5$ as the first approximation of the solution of the equation $f(x) = 0$, use Newton's method once to find a closer approximation, giving your answer to 2 decimal places.	**2**
b)	The polynomial $P(x) = 6x^3 - 7x^2 + ax + b$ has a zero at $x = -1$ and the remaining zeros are reciprocals $(\alpha, \frac{1}{\alpha})$. Determine the value of $a\ and\ b$.	**3**

⊛ Record Results on top of each section

① Peter
② Leonard
③ Dave
④ Arun

2.25pm

Yr 12 Mathematics Extension 1

SOLUTIONS: TASK 1. Feb 2013

Q1

(a) (i) $P(x) = 2x^3 - 9x^2 + kx + 6$.

$P(3) = 0$

$2(3)^3 - 9(3)^2 + 3k + 6 = 0$ ✓

$54 - 81 + 3k + 6 = 0$

$3k = 21$

$\underline{k = 7}$ ✓

2

(ii)
$$\begin{array}{r|l} & 2x^2 - 3x - 2 \\ \hline x-3 & 2x^3 - 9x^2 + 7x + 6 \\ & 2x^3 - 6x^2 \\ \hline & -3x^2 + 7x \\ & -3x^2 + 9x \\ \hline & -2x + 6 \\ & -2x + 6 \\ \hline & 0 \end{array}$$ ✓

3

$2x^2 - 3x - 2 = (2x+1)(x-2)$ ✓

$\therefore P(x) = (x-3)(x-2)(2x+1) = 0$

$\therefore x = \underline{3, 2 \text{ or } -\tfrac{1}{2}}$ ✓

2:28

Q 1
(b) $5x^3 - 6x^2 - 29x + 6 = 0$.

α, β, γ are roots.

$\alpha+\beta+\gamma = 6/5$

$\alpha\beta+\beta\gamma+\gamma\alpha = -29/5$

$\alpha\beta\gamma = -6/5$

(i) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$ ✓

$= \frac{-29/5}{-6/5}$ $\frac{1}{2}$

$= \underline{\underline{29/6}}$ $\frac{1}{2}$ ✓

2

(ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ✓

$= \left(\frac{6}{5}\right)^2 - 2(-29/5)$

$= \frac{36}{25} + \frac{58}{5}$

2

$= \frac{36+290}{25}$

$= \underline{\underline{\frac{326}{25}}}$ ✓ (note: question asked for fraction form)

2.30

Question 2

(a) (i) $f(x) = x^3 - 8x + 8$

$f(-3) = (-3)^3 - 8(-3) + 8$

$= -27 + 24 + 8$ $\frac{1}{2}$

$= 5 > 0$

$\frac{1}{2}$ each sub.

1 for explanation

$f(-4) = (-4)^3 - 8(-4) + 8$

total. 2

$= -64 + 32 + 8$ $\frac{1}{2}$

$= -24 < 0$ 1 ← reasoning

$\therefore$ A ~~root~~ lies between $x = -3$ and $x = -4$.

(ii) $x = -3.5$ $f(x) = x^3 - 8x + 8$

$f(-3.5) = -6.875$ $f'(x) = 3x^2 - 8$.

$f'(-3.5) = 28.75$

$$x_1 = x - \frac{f(x)}{f'(x)}$$

$$= -3.5 - \frac{-6.875}{28.75}$$ ✓

$= -3.2608\ldots$

$\simeq \underline{\underline{-3.26}}$ ✓

2.34.

$\boxed{Q2(b)}$ $P(x) = 6x^3 - 7x^2 + ax + b$

$P(-1) = 0.$

The roots are $-1, \alpha, \frac{1}{\alpha}.$

Prod of roots $= -1 \times \alpha \times \frac{1}{\alpha} = -1 = -b/6.$

$\therefore \boxed{b = 6}$ ✓

$$\left.\begin{aligned} -1 + \alpha + \frac{1}{\alpha} &= 7/6. \\ \text{or} \quad \alpha + \frac{1}{\alpha} &= \frac{7}{6} + 1 \\ \alpha + \frac{1}{\alpha} &= 13/6 \quad — (1) \end{aligned}\right\} \text{sum roots}$$

✓

~~Prod~~ sum roots 2 at a time.

$-1 \times \alpha + -1 \times \frac{1}{\alpha} + \alpha \times \frac{1}{\alpha} = a/6$

or $-\alpha - \frac{1}{\alpha} + 1 = a/6,$

or $-(\alpha + \frac{1}{\alpha}) + 1 = a/6$

or $-\frac{13}{6} + 1 = a/6$ (from (1) $\alpha + \frac{1}{\alpha} = 13/6$)

or $a/6 = -7/6.$

$\therefore \boxed{a = -7.}$ ✓

$\therefore a = -7, b = 6$

2.37

Q3

(a) $I = \int x\sqrt{9-x^2}\,dx$ $\qquad u = 9-x^2$

$$\frac{du}{dx} = -2x$$

$$\frac{du}{-2} = x\,dx$$

$$I = \int u^{\frac{1}{2}}\left(\frac{du}{-2}\right)$$

$$= -\frac{1}{2}\int u^{\frac{1}{2}}du$$

$$= -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C$$

$$= -\frac{1}{3}(9-x^2)^{3/2} + C$$

(b) $\int \frac{1-x}{(1+x)^3}\,dx$ $\qquad u = 1+x$, $du = dx$

$x = u-1$

$1-x = 1-u+1$
$= 2-u.$

$$= \int \frac{2-u}{u^3}\,du$$

$$= \int (2u^{-3} - u^{-2})\,du$$

$$= 2\frac{u^{-2}}{-2} - \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{(1+x)^2} + \frac{1}{1+x} + C$$

2.40

Q4

(a) $$\int_0^{\pi} 2\sin x \cos^2 x \, dx.$$

$\cos x = u$

$-\sin x \, dx = du.$

$\sin x \, dx = -du.$

$x = 0, \quad u = 1$

$x = \pi, \quad u = -1$

$$= 2\int_1^{-1} u^2(-du)$$

$$= -2\int_1^{-1} u^2 \, du$$

$$= -2\left[\frac{u^3}{3}\right]_1^{-1}$$

$$= -2\left[\frac{-1}{3} - \frac{1}{3}\right]$$

$$= -2\left(-\frac{2}{3}\right)$$

$$= \frac{4}{3}$$

2.42

Question 3

(c) $\frac{dy}{dx} = 2\cos^2 x + 1$ (π, π) is a point.

$$y = \int (2\cos^2 x + 1)\, dx.$$

$$= \int ((1 + \cos 2x + 1)\, dx$$

$$= \left[2x + \frac{1}{2}\sin 2x\right] + C$$

Sub (π, π)

2.44. $\pi = 2\pi + \frac{1}{2}\sin 2\pi + C$ $\quad C = -\pi.$

$y = 2x + \frac{1}{2}\sin 2x - \pi.$ When $x = 2\pi$

$y = 4\pi + \frac{1}{2}\sin 4\pi - \pi = 3\pi$

Question 4 b)

$$V = \pi \int_0^{\pi/2} y^2 \, dx$$

$$= \pi \int_0^{\pi/2} \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$$ ✓

$$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi/2}$$ ✓

$$= \frac{\pi}{2} \left[\left(\frac{\pi}{2} - \frac{1}{2} \sin \pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{4} \ u^3.$$ ✓

2.45

Question 4

$\cos(2\theta + 2\theta)$

c) (i)

$= \cos^2 2\theta - \sin^2 2\theta$

$= (1 - 2\sin^2\theta)^2 - 4\sin^2\theta\cos^2\theta$

$= 1 - 4\sin^2\theta + 4\sin^4\theta - 4\sin^2\theta(1 - \sin^2\theta)$

$= 1 - 4\sin^2\theta + 4\sin^4\theta - 4\sin^2\theta + 4\sin^4\theta$

$\cos 4\theta = 1 - 8\sin^2\theta + 8\sin^4\theta$

(ii) $1 - 8\sin^2\theta + 8\sin^4\theta = \cos 4\theta$

$8\sin^2\theta - 8\sin^4\theta = 1 - \cos 4\theta$

$\therefore \sin^2\theta - \sin^4\theta = \frac{1}{8}(1 - \cos 4\theta)$

$$\int_{\pi/4}^{\pi/2} (\sin^2\theta - \sin^4\theta)\,d\theta = \frac{1}{8}\int_{\pi/4}^{\pi/2} (1 - \cos 4\theta)\,d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \left(\frac{\pi}{4} - \frac{1}{4}\sin \pi\right)\right]$$

$$= \frac{1}{8}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{32}$$

2.49