SHORE

Examination Number:

Set:

2024 **YEAR 12 MID-YEAR EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using non-erasable black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7-12)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Note: Any time you have remaining should be spent reviewing your answers.

DO NOT REMOVE THIS PAPER FORM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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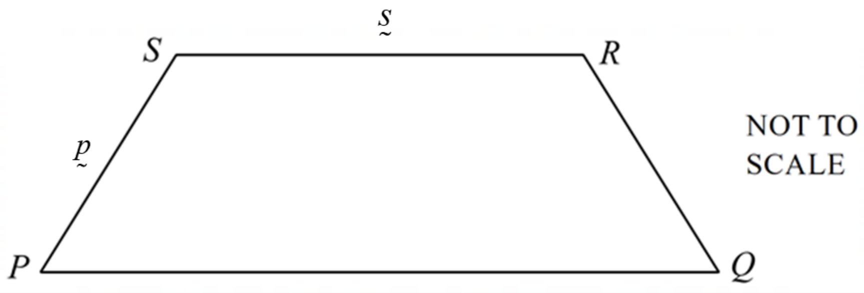
1 $PQRS$ is a trapezium where $\overrightarrow{PS} = \underset{\sim}{p}$, $\overrightarrow{SR} = \underset{\sim}{s}$, $\overrightarrow{PQ} = 2\overrightarrow{SR}$ and SR is parallel to PQ.



Which of the following is equivalent to $\overrightarrow{QS}$?

A. $2\underset{\sim}{s} + \underset{\sim}{p}$

B. $2\underset{\sim}{s} - \underset{\sim}{p}$

C. $\underset{\sim}{p} - 2\underset{\sim}{s}$

D. $-\underset{\sim}{p} - 2\underset{\sim}{s}$

2 Using the substitution $u = \tan x$, what is $\int \tan x \sec^2 x \, dx$?

A. $\frac{1}{2}\tan x + C$

B. $\frac{1}{2}\tan^2 x + C$

C. $\tan x + C$

D. $\tan^2 x + C$

3 Which of the following Cartesian equations represents the parametric equations $x = 2\cos\theta - 1$ and $y = 2 + 2\sin\theta$?

A. $\left(\frac{x-1}{2}\right)^2 + \left(\frac{y+2}{2}\right)^2 = 1$

B. $\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 4$

C. $(x+1)^2 + (y-2)^2 = 1$

D. $(x+1)^2 + (y-2)^2 = 4$

4 How many solutions for the equation $\sin 2x - \sin x = 0$ are in the domain $[0, 2\pi]$?

A. 2

B. 3

C. 4

D. 5

5 A spherical balloon is expanding so that its volume is increasing at a rate of $50\,\text{mm}^3$ per second. At what rate is the radius expanding when its radius is $10\,\text{mm}$?
Answer to 2 decimal places.

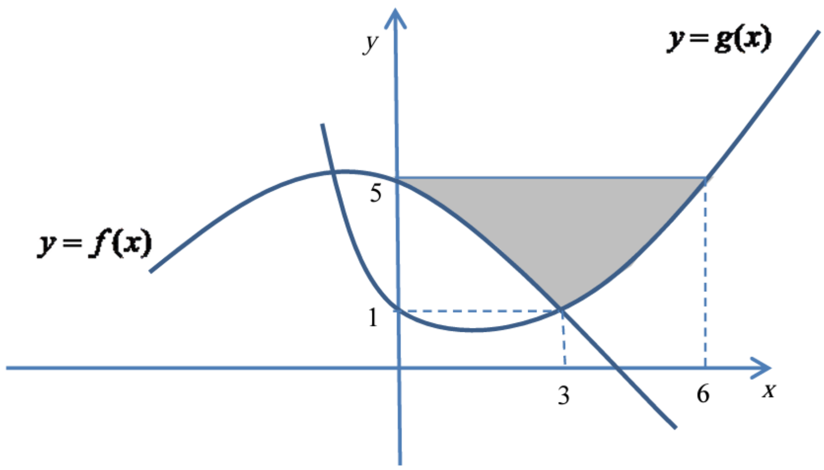
A. 0.02 mms^{-1}

B. 0.04 mms^{-1}

C. 0.08 mms^{-1}

D. 0.16 mms^{-1}

6 Consider the two functions $y = f(x)$ and $y = g(x)$ below.



Which of the following definite integrals is equal to the area of the shaded region?

A. $\int_1^5 (g(x) - f(x))dx$

B. $\int_1^5 (f(x) - g(x))dx$

C. $\int_0^3 (5 - g(x))dx + \int_3^6 (5 - f(x))dx$

D. $\int_0^3 (5 - f(x))dx + \int_3^6 (5 - g(x))dx$

7 Which expression is $\int \frac{1}{u^2 + 6u + 10} du$ equivalent to?

A. $\tan^{-1}(u+3) + C$

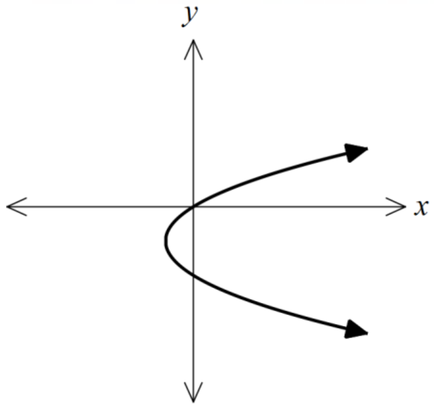
B. $\frac{1}{3}\tan^{-1}(u+3) + C$

C. $\tan^{-1}\left(\frac{u+3}{\sqrt{10}}\right) + C$

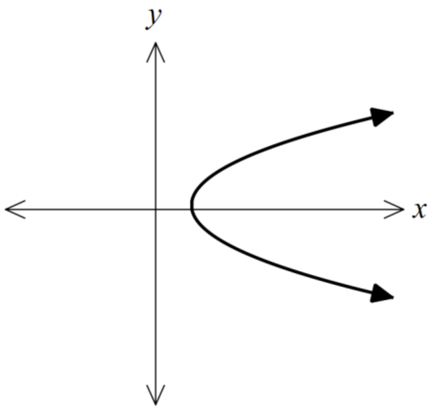
D. $\frac{1}{\sqrt{10}}\tan^{-1}(u+3) + C$

8 Which of the following is a part of the graph given by parametric equations $x = t^2 + 2t$ and $y = t$?

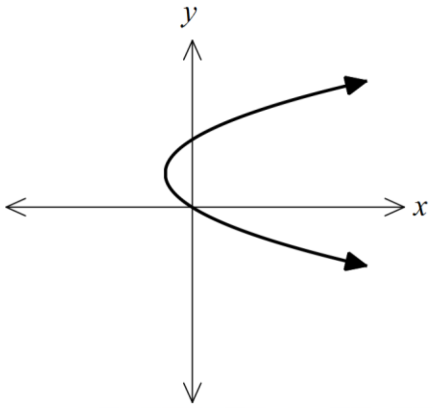
A.



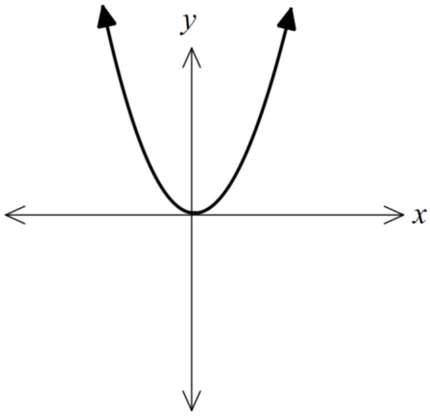
B.



C.



D.



9 What is the evaluation of $\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1} x\, dx$?

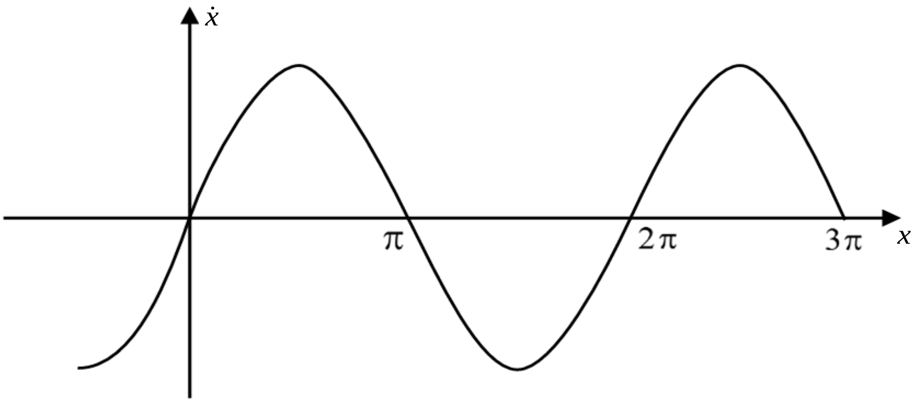
A. $\frac{3}{2}$

B. $\frac{3}{4}$

C. $\frac{3\pi}{4}$

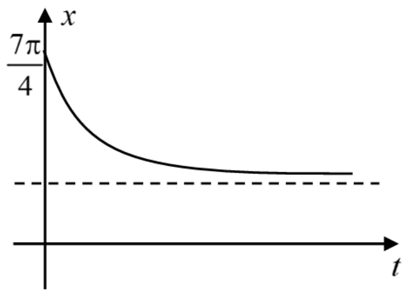
D. $\frac{3\pi}{2}$

10 The velocity of a particle moving along the x axis is given by $\dot{x} = \sin x$ and is graphed below.

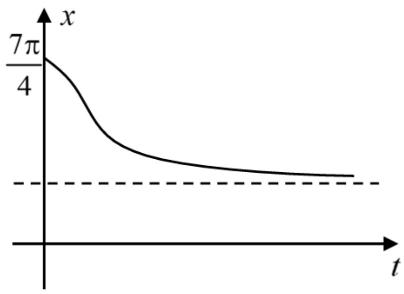


At $t = 0$, the particle is located at $x = \dfrac{7\pi}{4}$. Which of the following graphs best illustrates the motion of the particle for $t \geq 0$? (Graphs not drawn to scale.)

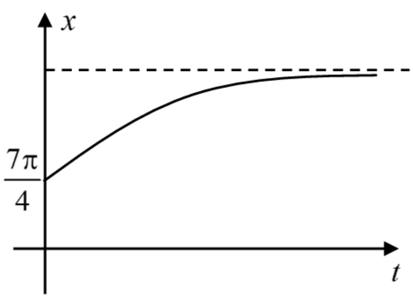
A.



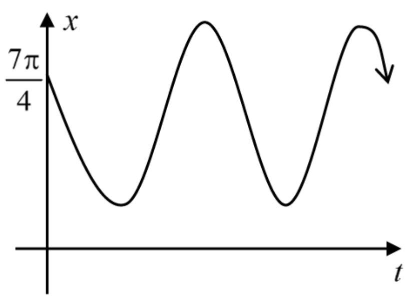
B.



C.



D.



End of Section 1

Section II

60 marks
Attempt Questions 11−14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (16 marks) Use a SEPARATE writing booklet.

(a) Given $\underset{\sim}{u} = (-10, -1)$, find the unit vector in the same direction of $\underset{\sim}{u}$. **1**

(b) Solve the inequality $4 \le \dfrac{8}{x-3}$. **3**

(c) Given $\cos A = \dfrac{7}{9}$ and $\sin B = \dfrac{1}{3}$, where A and B are acute, using an appropriate trigonometric identity, show that $\cos 2B = \cos A$. **2**

(d) Consider the function $f(x) = \dfrac{x}{x+2}$.

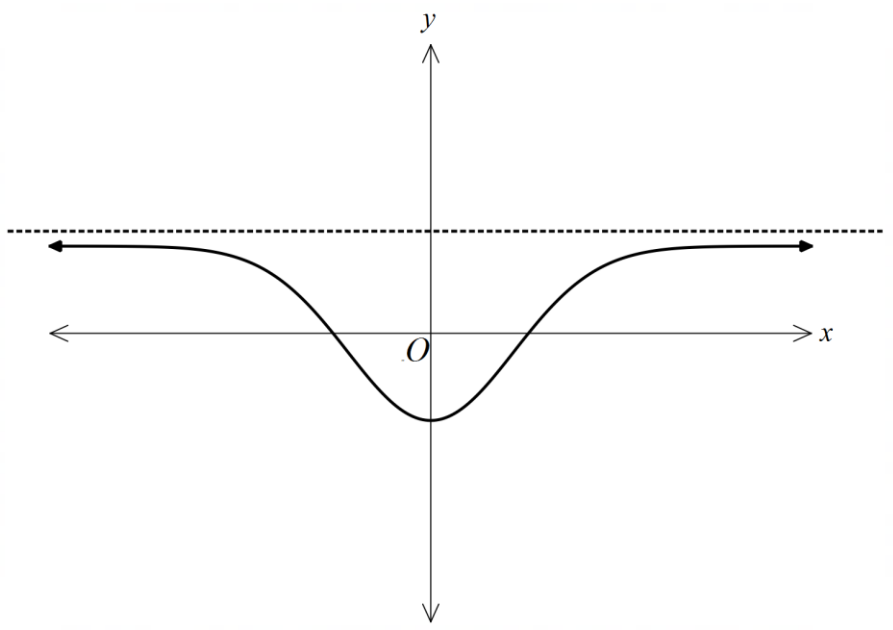
(i) Show that $f'(x) > 0$ for all x in the domain and hence explain why the function has an inverse function. **2**

(ii) Find the equation for $f^{-1}(x)$. **2**

(e) Use the substitution $u = \log_e x$ to find $\displaystyle\int_e^{e^2} \frac{1}{x(\log_e x)^2}\,dx$. **3**

For question (f), answer parts (i) and (ii) on the separate answer sheet provided. Submit the answer sheet inside the booklet for question 11.

(f) The diagram below shows the graph of $y = f(x)$ where $f(x) = 1 - 2e^{-x^2}$.



(i) Find the equation of the horizontal asymptote. **1**

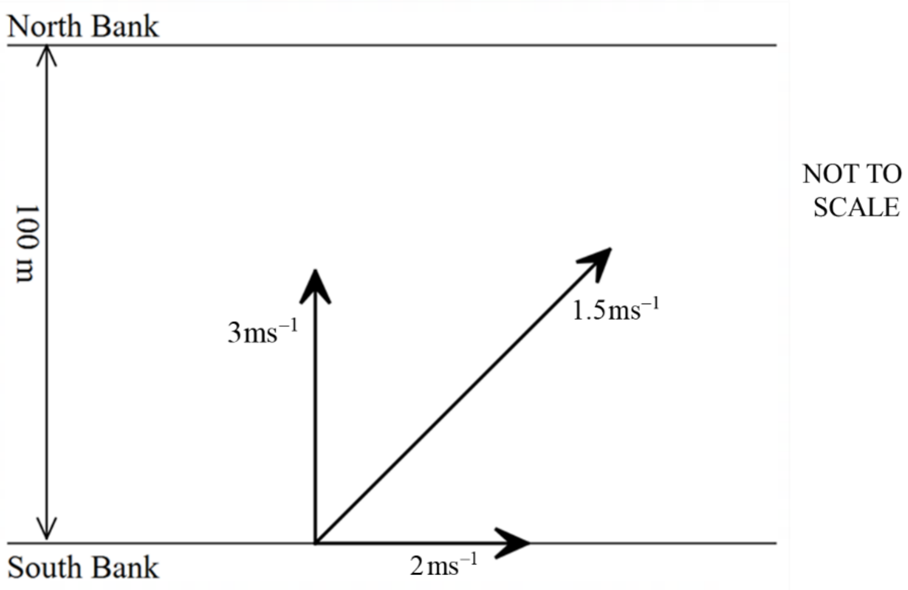
(ii) The graph of this function **is also on a separate answer sheet**. On the same set of axes **on the answer sheet**, sketch $y = \sqrt{f(x)}$ showing the asymptote and the co-ordinates of any points of intersection between $y = f(x)$ and $y = \sqrt{f(x)}$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Simon intends to row a boat from the south bank of a river to meet his friend on the north bank. The river is 100 metres wide. Simon's rowing speed is $3\,\text{ms}^{-1}$. The river is flowing east at a rate of $2\,\text{ms}^{-1}$.

The boat is also being impacted by a blowing wind from the south-west, which pushes it at $1.5\,\text{ms}^{-1}$. Simon starts rowing across the river by steering the boat such that it is perpendicular to the south bank.



(i) Show that the resultant velocity $\underset{\sim}{v}$, of Simon's boat can be expressed as the component vector **2**

$$\underset{\sim}{v} = \left(2+\sqrt{1.125}\right)\underset{\sim}{i} + \left(3+\sqrt{1.125}\right)\underset{\sim}{j}$$

(ii) Calculate the speed of the boat, correct to 1 decimal place. **1**

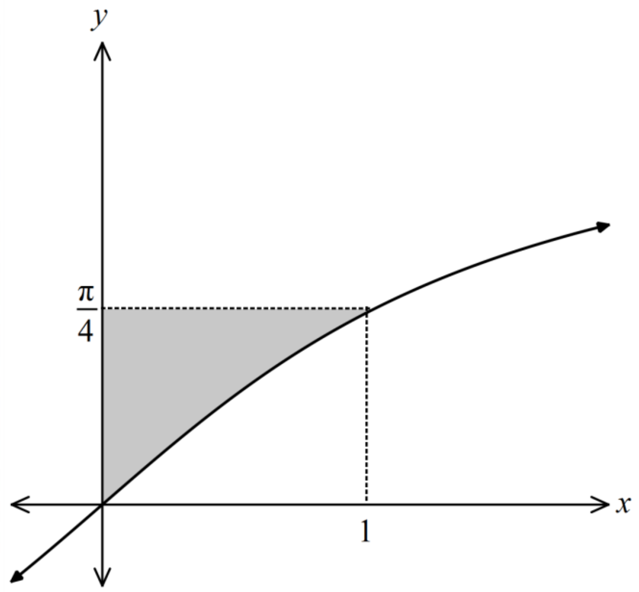
(iii) Determine the distance travelled from Simon's starting point to his landing point and how long it will take him to reach the north bank. **3**

(b) Assuming that people have, at most, 200,000 hairs on their head, prove that in a city of 8 million people, there are at least two people with exactly the same number of hairs on their head. **2**

(c) Given that the roots of $x^2 - 2x - 1 = 0$ are $\tan\alpha$ and $\tan\beta$, determine the value of $\alpha+\beta$. **2**

(d) (i) Show that $\dfrac{d}{dy}\left(\ln(\cos y)\right) = -\tan y$. **1**

(ii) The diagram below shows a portion of the graph of $y = \tan^{-1} x$. Using part (i) show that its shaded area is given by $A = \dfrac{1}{2}\ln 2$ units². **4**



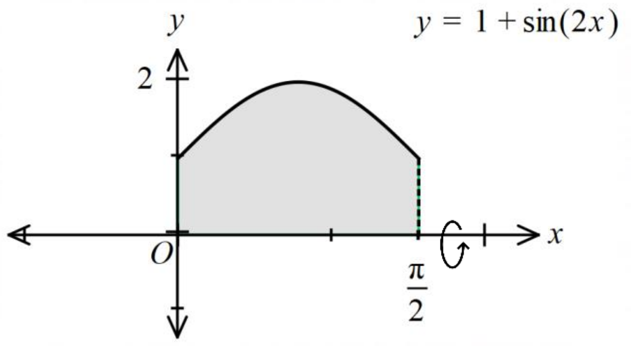
End of Question 12

Question 13 (14 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\sin(M-2N)\cos(M+2N)=\frac{1}{2}(\sin 2M-\sin 4N)$. **2**

(ii) Hence show that the exact value of $\sin(22.5^\circ)\cos(7.5^\circ)=\frac{2+\sqrt{6}-\sqrt{2}}{8}$. **4**

(b) The graph of $y=1+\sin(2x)$ is shown below for $0\le x\le\frac{\pi}{2}$.



The shaded region is rotated about the x axis to form a solid of revolution. Find the volume of the solid leaving your answer in exact form. **4**

(c) Consider the points $P(a,2a)$, $Q(-a,5a)$, $R(3a,4a)$ and $S(9a,12a)$ where a is a positive real number.

(i) Express $\overrightarrow{PQ}$ in component form. **1**

(ii) Given the length of the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$ is 12, find the value of a. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) In the expansion of $(1+x)^n$, the coefficient of x^4 is twice the coefficient of x^3. Find the value of n. **3**

(b) Find the derivative of $y=\cos^{-1}\left(\sqrt{2t-1}\right)$. Leave answer in simplified form. **2**

(c) Use the principle of mathematical induction to prove that $2^{n+1}(5+7^n)$ is divisible by 6 for all positive integers n. **3**

(d) Let each different arrangement of all the letters in PERMUTATION be called a word.

(i) Find an expression for the number of words possible. **1**

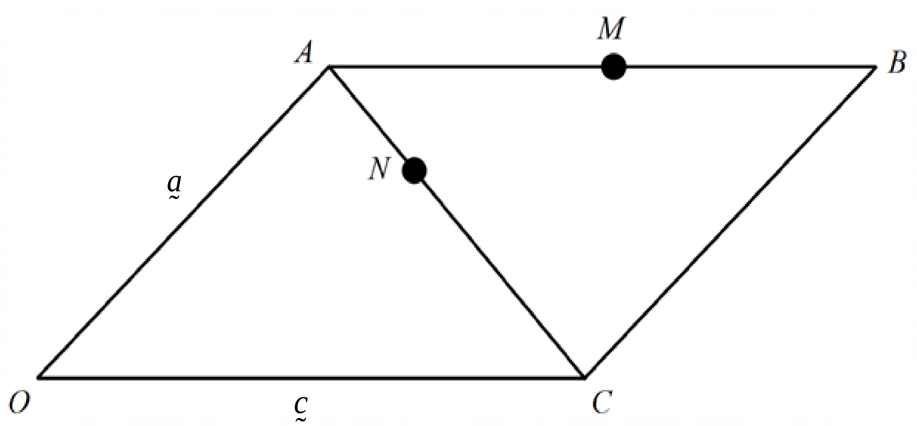
(ii) If a word is selected at random, find the probability that all the vowels are together. **2**

(iii) How many words are there in which the vowels occur in the order AEIOU from left to right though not necessarily together? **1**

(e) In the figure, $OABC$ is a parallelogram where the point M is the midpoint of AB. **3**

The point N lies on the diagonal AC so that $AN:NC=1:2$.

Let $\overrightarrow{OA}=\underset{\sim}{a}$, and $\overrightarrow{OC}=\underset{\sim}{c}$. Show that O, N and M are collinear.

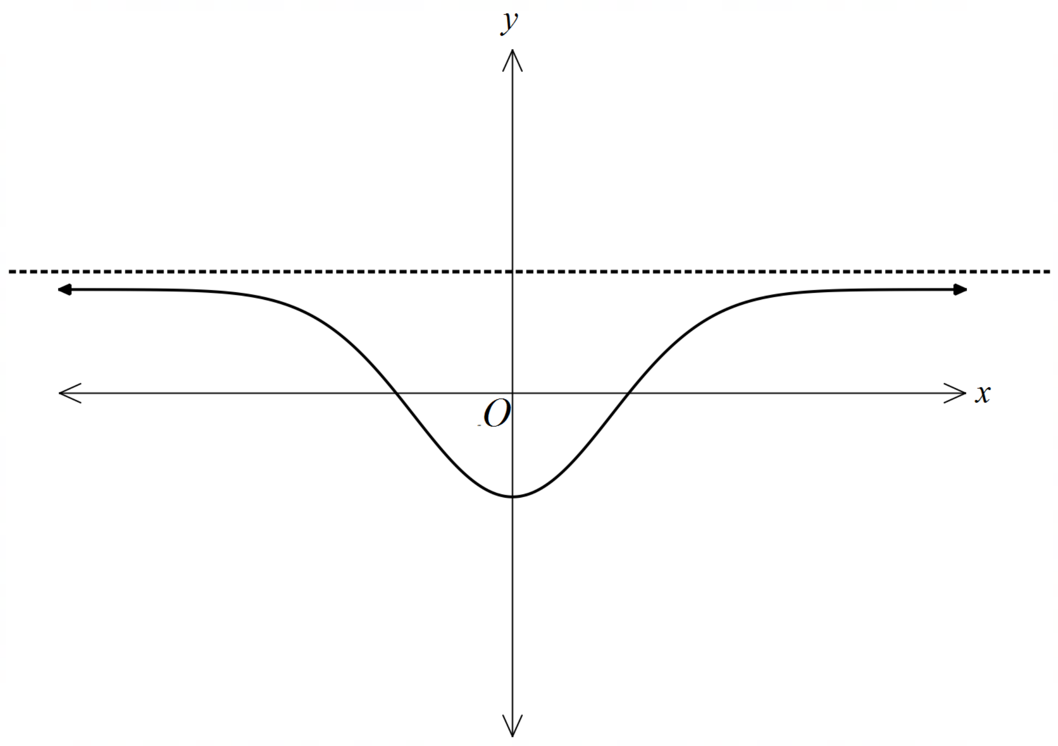


☺ END OF PAPER ☻

Examination Number:………………………………. Set:……………………..

Insert this sheet inside your answer booklet for Question 11

f) The diagram below shows the graph of $y = f\left(x\right)$ where $f\left(x\right) = 1 - 2e^{-x^2}$.



(i) Find the equation of the horizontal asymptote. **1**

……………………………………………………………………………………

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(ii) On the same set of axes above, sketch $y = \sqrt{f(x)}$ showing the asymptote and the co-ordinates of any points of intersection between $y = f(x)$ and $y = \sqrt{f(x)}$. **2**

SHORE

Set

2024

Year 12 Mid-Year Examination

Examination Number

Mathematics Extension 1

Section I Multiple Choice

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = A. 2 B. 6 C. 8 D. 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

Start → Here

1. A ○ B ○ C ○ D ○

2. A ○ B ○ C ○ D ○

3. A ○ B ○ C ○ D ○

4. A ○ B ○ C ○ D ○

5. A ○ B ○ C ○ D ○

6. A ○ B ○ C ○ D ○

7. A ○ B ○ C ○ D ○

8. A ○ B ○ C ○ D ○

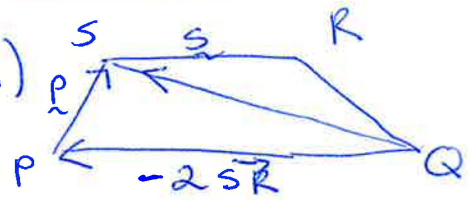
9. A ○ B ○ C ○ D ○

10. A ○ B ○ C ○ D ○

Year 12 Ext 1 Mid Year Exam 2024

Multiple Choice

MC's
C
B
D
D
B
D
A
A
C
B

1) 

$-2\vec{SR} + p$

$= p - 2\vec{SR}$

$= p - 2\underset{\sim}{s}$

C

2) $\int \tan(x)\sec^2 x\,dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\,dx$

$\int u\,du$

$= \frac{u^2}{2} + c$

$= \frac{1}{2}\tan^2 x + c$

B

3) $x = 2\cos\theta - 1$ — ①

$y = 2 + 2\sin\theta$ — ②

from ① $\cos\theta = \frac{x+1}{2}$

② $\sin\theta = \frac{y-2}{2}$

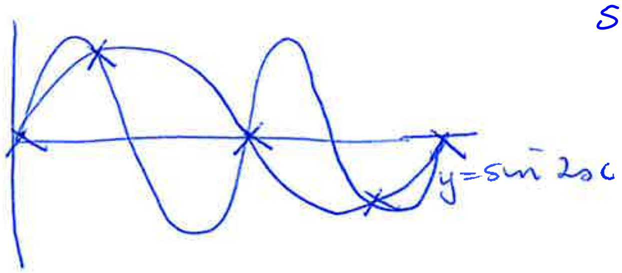
using $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$

$(x+1)^2 + (y-2)^2 = 4$

D

4) $\sin 2x - \sin x = 0$



Graphs intersect at 5 points

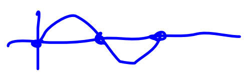
OR $2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

D

$\sin x = 0$ or $\cos x = \frac{1}{2}$

3 sol'ns 2 sol'ns



$\therefore$ 5 sol'ns

5) $\frac{dV}{dt} = 50$ $\qquad V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}.$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times 50$$

$$= \frac{50}{4\pi r^2}$$

B

when $r=10$ $\quad \frac{dr}{dt} = ?$

$$\frac{dr}{dt} = \frac{50}{4.\pi.10^2}$$

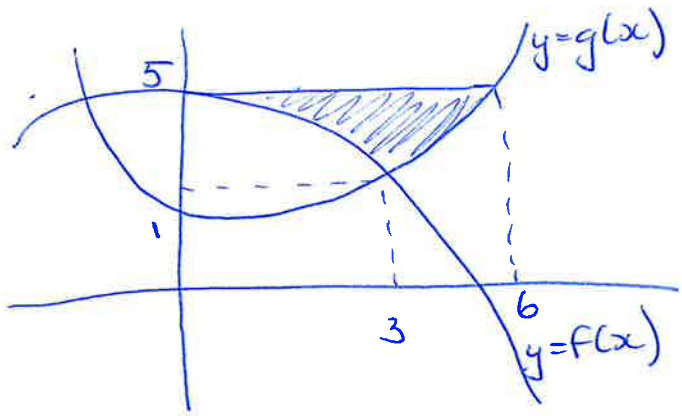
$$= \frac{50}{400\pi}$$

$$= \frac{1}{8\pi}$$

$$= 0.0397\ldots$$

$= 0.04$ mm/s (2dp)

6)



D

$$\int_0^3 (5-f(x))\,dx + \int_3^6 (5-g(x))\,dx$$

7) $\int \frac{1}{u^2+6u+10}\,du$

$u^2+6u+10 = u^2+6u+3^2-3^2+10$ (CTS)

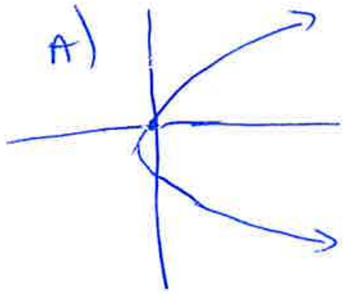
$= (u+3)^2+1$

Hence $\int \frac{1}{u^2+6u+10}\,du$ A

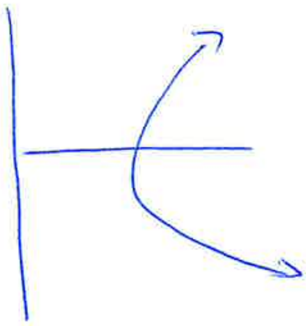
$= \int \frac{1}{(u+3)^2+1}\,du$

$= \tan^{-1}(u+3)+C$

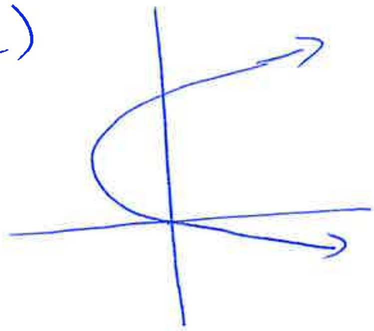
8) A)



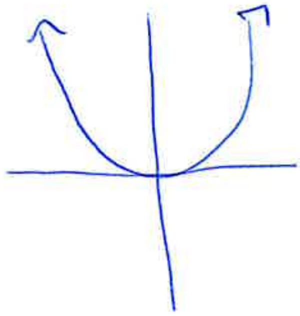
B)



C)



D)



A

$x(t)=t^2+2t$ and $y(t)=t$

when $t=0$ $x=0$ $y=0$ $(0,0)$ $\therefore$ NOT B

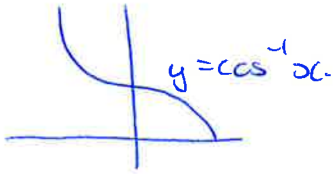
$t=1$ $x=3$ $y=1$ $(3,1)$ hard to tell

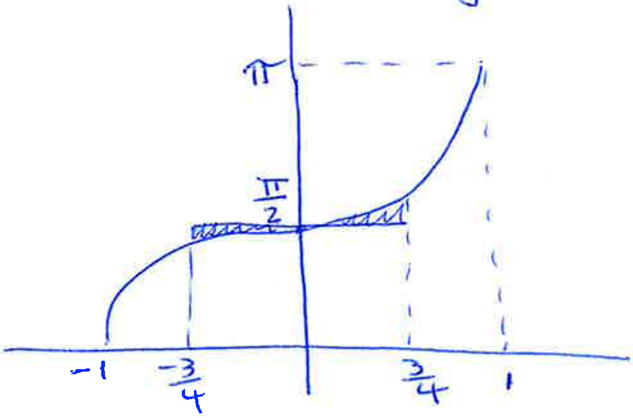
$t=-1$ $x=-1$ $y=-1$ $(-1,-1)$

only A goes through

9) $\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1}(-x)\,dx$

↑
reflect in y axis





Shading equal in size given symmetry of graph.

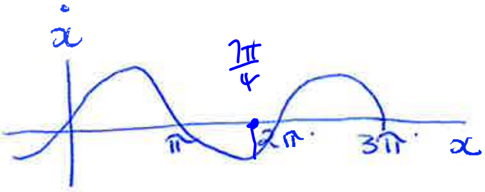
$$\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1}(-x)\,dx = \text{Area rectangle}$$

$$= \frac{6}{4} \times \frac{\pi}{2}$$

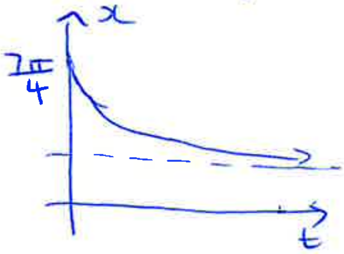
$$= \frac{3\pi}{4}$$

C

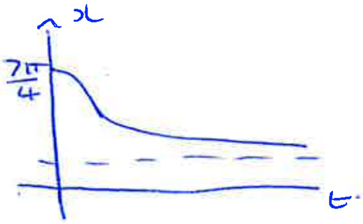
10)



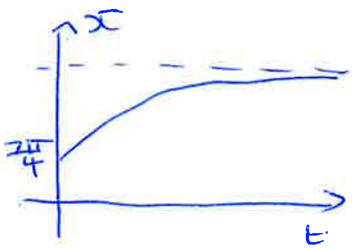
A)



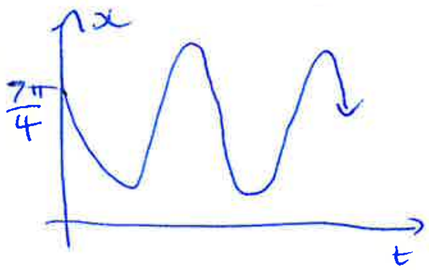
B)



C)



D)



The graph shows that at $x = \frac{7\pi}{4}$ the velocity is negative so the particle moves to the left (x decreases). As x decreases at first the velocity becomes more negative (the particle moves at a faster rate towards the left) then less negative (the particle slows down). As the particle moves towards $x = \pi$ the particle moves at an ever slower rate towards the left.

B

11a) Given $\underset{\sim}{u} = (-10, -1)$ find the unit vector in the same direction of $\underset{\sim}{u}$.

$$\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|} \qquad \underset{\sim}{u} = -10\underset{\sim}{i} - \underset{\sim}{j}$$

$$\hat{\underset{\sim}{u}} = \frac{-10\underset{\sim}{i} - \underset{\sim}{j}}{\sqrt{10^2 + 1^2}}$$

$$= \frac{-10\underset{\sim}{i} - \underset{\sim}{j}}{\sqrt{101}}$$

$$\hat{\underset{\sim}{u}} = \frac{-10}{\sqrt{101}}\underset{\sim}{i} - \frac{1}{\sqrt{101}}\underset{\sim}{j}$$

OR

$$\frac{-10\sqrt{101}}{101}\underset{\sim}{i} - \frac{\sqrt{101}}{101}\underset{\sim}{j}$$

OR

$$\left(\frac{10\sqrt{101}}{101}, \frac{\sqrt{101}}{101}\right)$$

OR

$$\frac{1}{\sqrt{101}}\begin{pmatrix} -10 \\ -1 \end{pmatrix}$$

11 b) $4 \times (x-3)^2 \leq \frac{8}{x-3} \times (x-3)^2 \qquad x \neq 3$

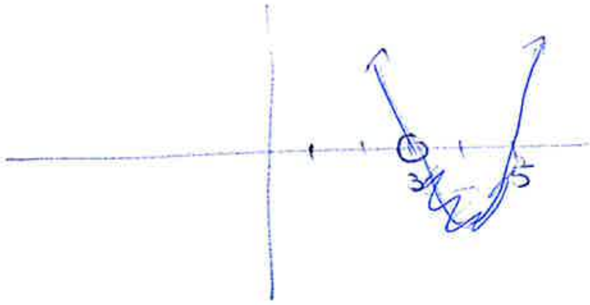
$4(x-3)^2 \leq 8(x-3)$

$4(x-3)^2 - 8(x-3) \leq 0$

$4(x-3)((x-3)-2) \leq 0$

$4(x-3)(x-5) \leq 0$

$x \neq 3$ or $x = 5$



$3 < x \leq 5$

OR

$(3, 5]$

OR

$4 \leq \frac{8}{x-3}$

Critical points

$x - 3 \neq 0$

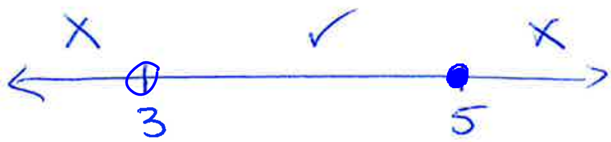
$x \neq 3$

$4x - 12 = 8$

$4x = 20$

$x = 5$

Test



$x = 0$ $\quad 4 \leq -\frac{8}{3}$ ✗

$x = 4$ $\quad 4 \leq \frac{8}{4-3}$ ✓

$x = 6$ $\quad 4 \leq \frac{8}{6-3}$ ✗

$\therefore 3 < x \leq 5$

11c) $\cos A = \frac{7}{9}$ $\sin B = \frac{1}{3}$ A, B acute, show $\cos 2B = \cos A$.

$$\begin{aligned} LHS &= \cos 2B \\ &= 1 - 2\sin^2 B \\ &= 1 - 2\left(\tfrac{1}{3}\right)^2 \\ &= 1 - \tfrac{2}{9} \\ &= \tfrac{7}{9} \\ &= \cos A \\ &= RHS \end{aligned}$$

$\therefore \cos 2B = \cos A$

$$\left[\begin{aligned} \cos 2B &= \cos^2 B - \sin^2 B \\ &= 1 - \sin^2 B - \sin^2 B \\ &= 1 - 2\sin^2 B. \end{aligned}\right]$$

11d)

$f(x) = \dfrac{x}{x+2} \quad x \neq -2$

i) $f'(x) = \dfrac{(x+2).1 - 1.x}{(x+2)^2}$

$= \dfrac{2}{(x+2)^2} > 0$ for all values of x.

As $(x+2)^2 \geq 0$ then $f'(x) > 0$ for all values of x.
$\therefore f(x)$ is monotonic increasing and hence has an inverse function.

ii) $y = \dfrac{x}{x+2}$

$x = \dfrac{y}{y+2}$

$xy + 2x = y$

$xy - y = -2x$

$y(x-1) = -2x$

$y = \dfrac{-2x}{x-1}$

$\therefore f^{-1}(x) = \dfrac{-2x}{x-1}$

OR

As $f(x)$ always has an increasing gradient, it passes the horizontal line test and hence has an inverse function.

11e) $\int_e^{e^2} \frac{1}{x(\log_e x)^2}\,dx$ let $u = \log_e x$

$u = \ln x$ when $x = e$ $u = \ln e$
$= 1$

$\frac{du}{dx} = \frac{1}{x}$ when $x = e^2$ $u = \ln e^2$
$= 2\ln e$
$= 2$

$du = \frac{dx}{x}$

Hence $\int_e^{e^2} \frac{1}{x(\log_e x)^2}\,dx$ ← All in 'x'

$= \int_1^2 \frac{du}{u^2}$ ← All in 'u'.

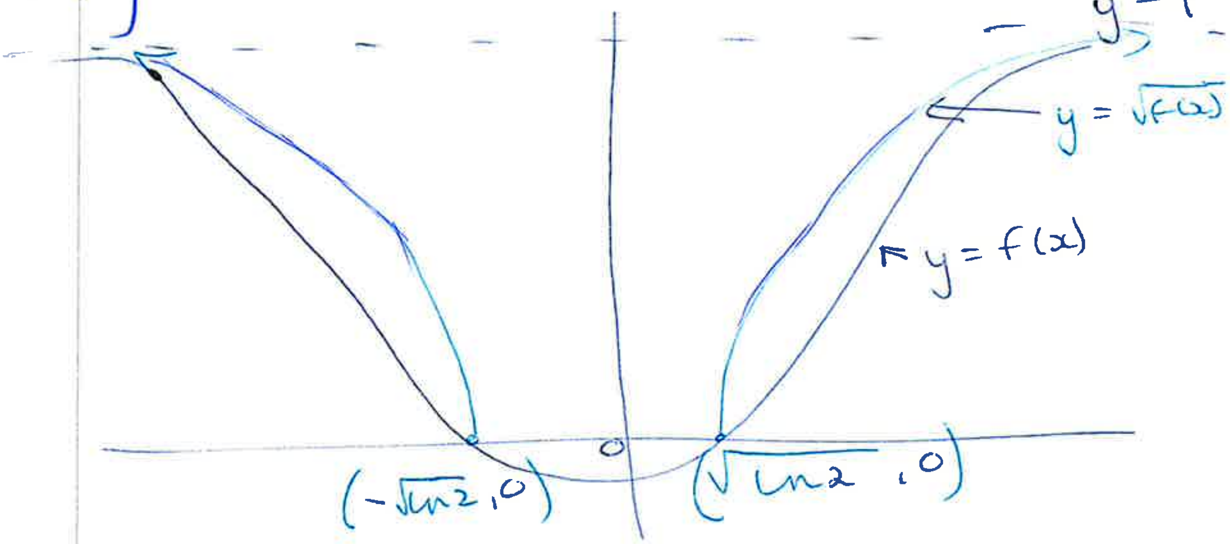
$= \left[-u^{-1}\right]_1^2$

$= -2^{-1} + 1^{-1}$

$= -\frac{1}{2} + 1$

$= \frac{1}{2}$

11f)



i) $f(x) = 1 - 2e^{-x^2}$

$x \to \infty$ from above and below $e^{x^2} \to \infty$

$\therefore \dfrac{2}{e^{x^2}} \to 0$

$\therefore f(x) \to 1$

$\therefore$ horizontal asymptote is $y=1$

ii) $0 = 1 - 2e^{-x^2}$

$\dfrac{1}{2} = \dfrac{1}{e^{x^2}}$

$e^{x^2} = 2$

$x^2 = \ln 2$

$x = \pm\sqrt{\ln 2}$

$\frac{1}{2} = e^{-x^2}$

$\ln(\frac{1}{2}) = \ln(e^{-x^2})$

$\therefore \ln(\frac{1}{2}) = -x^2$

$-\ln(\frac{1}{2}) = x^2$

Common answer: $x = \pm\sqrt{-\ln(\frac{1}{2})}$

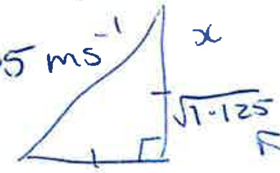
Show that $-\ln(\frac{1}{2}) > 0$.

$-\ln(2^{-1}) = \ln 2$

$\therefore x = \pm\sqrt{\ln 2}$

12a)

Components of wind vector:

i) 

$(1.5)^2 = 2x^2$

$x^2 = 1.125$

$x = \pm\sqrt{1.125}$

$x = \sqrt{1.125}$ only since $x \geq 0$

Wind vector is $\sqrt{1.125}\,\underset{\sim}{i} + \sqrt{1.125}\,\underset{\sim}{j}$

Rowing vector is $0\underset{\sim}{i} + 3\underset{\sim}{j}$

Water flow vector is $2\underset{\sim}{i} + 0\underset{\sim}{j}$

Resultant velocity $= 2\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{i} + 3\underset{\sim}{j} + \sqrt{1.125}\,\underset{\sim}{i} + \sqrt{1.125}\,\underset{\sim}{j}$

$= (2+\sqrt{1.125})\underset{\sim}{i} + (3+\sqrt{1.125})\underset{\sim}{j}$

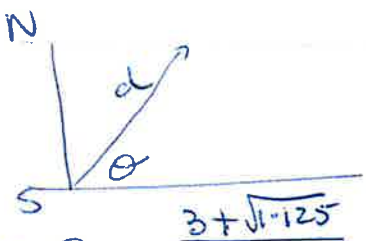
as required.

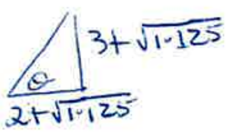
ii) Velocity is equal to magnitude of resultant vector in i)

$v = \sqrt{(2+\sqrt{1.125})^2 + (3+\sqrt{1.125})^2}$

$v = 5.0849387\ldots$

$v = 5.1\ ms^{-1}$ (1 dp)

iii) 



'd' is distance from start to finish.

$\tan\theta = \dfrac{3+\sqrt{1.125}}{2+\sqrt{1.125}}$

$\theta = \tan^{-1}\left(\dfrac{3+\sqrt{1.125}}{2+\sqrt{1.125}}\right)$

$= 52.993401\ldots$

$\doteqdot 53°$

$\sin\theta = \dfrac{100}{d}$

$d = \dfrac{100}{\sin 53}$

$d = 125.21\ldots$

$d \doteqdot 125\ m$

$\therefore$ distance from starting to landing point is 125 m.

$$\text{Time taken} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{125.21\ldots}{5.084\ldots}$$

$$= 24.6 \text{ seconds}$$

$\therefore$ time taken to reach North bank

is 24.6 seconds

OR.

$$\text{Time taken, } t_f = \frac{\text{vertical distance}}{\text{vertical component of velocity}}$$

$$= \frac{100}{3+\sqrt{1.125}}$$

$$= 24.6265\ldots$$

$$= 24.63$$

$$\text{Distance} = |v| \times t_f$$

$$= 24.63 \times 5.1$$

$$= 125.61$$

$$= 125.6\text{m}.$$

12 b)
Assume that at most, people have 200000 hairs on their head. Prove that in a city of 8 million people, there are at least two people with exactly the same number of hairs on their head.

pigeons = 8 million people

pigeons holes = 0, 1 --- 200 000 hairs

8 million people to be sorted into 200000+1 pigeon holes.

Since $\frac{800000}{200001} = 39.998$, by pigeon hole principle there are 40 people with the same number of hairs on their head.

Hence there are at least 2 people (in fact there's closer to 40) with the same number of hairs on their head.

12c)

$x^2 - 2x - 1 = 0$ $\tan\alpha$ and $\tan\beta$ are roots.

sum roots: $\tan\alpha + \tan\beta = 2$

product roots: $\tan\alpha \tan\beta = -1$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha+\beta) = \frac{2}{1+1}$$

$$\tan(\alpha+\beta) = 1$$

$$\alpha+\beta = \frac{\pi}{4}$$

$$\therefore \alpha+\beta = \frac{\pi}{4}.$$

OR.

$$x = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= 1 \pm \sqrt{2}.$$

$\tan\alpha = 1+\sqrt{2}$ $\tan\beta = 1-\sqrt{2}$

$\alpha = \tan^{-1}(1+\sqrt{2})$ $\beta = \tan^{-1}(1-\sqrt{2})$

$= 67.5$ $= -22.5$

$$\alpha + \beta = 67.5 - 22.5$$

$$= 45$$

$$= \frac{\pi}{4}$$

12d)

i) Show $\frac{d}{dy}(\ln(\cos y)) = -\tan y$

$LHS = \frac{d}{dy}(\ln(\cos y))$

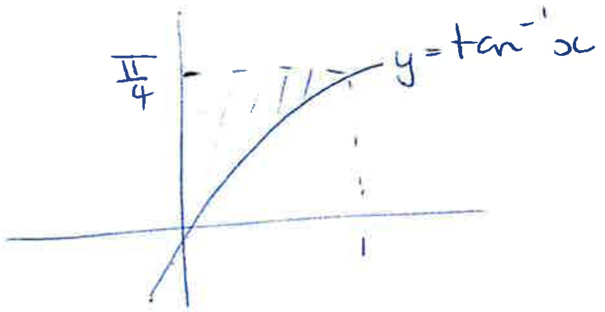
$= \frac{1}{\cos y} \cdot -\sin y$

$= \frac{-\sin y}{\cos y}$

$= -\tan y$

$= RHS$

ii)



$y = \tan^{-1} x$

$x = \tan y$

$A = \int_0^{\frac{\pi}{4}} \tan y$

$= -\int_0^{\frac{\pi}{4}} -\tan y$ (using (i))

$= -\left[\ln(\cos y)\right]_0^{\frac{\pi}{4}}$

$= -\left(\ln \cos\frac{\pi}{4}\right) - \ln(\cos 0))$

$= -\left(\ln\frac{1}{\sqrt{2}} - \ln 1\right)$

$= -\ln 2^{-\frac{1}{2}}$

$= \frac{1}{2}\ln 2$ units^2 as required

13a) Show $\sin(M-2N)\cos(M+2N) = \frac{1}{2}(\sin 2M - 4N)$

using $\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$

$LHS = \sin(M-2N)\cos(M+2N)$

$= \frac{1}{2}\left(\sin(M-2N+M+2N) + \sin(M-2N-M-2N)\right)$

$= \frac{1}{2}\left(\sin 2M + \sin(-4N)\right)$

$= \frac{1}{2}\left(\sin 2M - \sin 4N\right)$ (As sine is odd)

$=$ RHS as required.

ii) $\sin(M-2N)\cos(M+2N) = \sin(22\frac{1}{2})\cos(7\frac{1}{2})$

$M-2N = 22\frac{1}{2}$ — ① $M+2N = 7\frac{1}{2}$ — ②

① + ② $2M = 30$

$M = 15 \implies 15+2N = 7\frac{1}{2}$

$N = \frac{-15}{4}$

Hence $\sin 22\frac{1}{2}\cos 7\frac{1}{2} = \frac{1}{2}\left(\sin(2\times 15) - \sin\left(4\times\frac{-15}{4}\right)\right)$

$= \frac{1}{2}\left(\sin(30) - \sin(-15)\right)$

$= \frac{1}{2}(\sin 30 + \sin 15)$

$= \frac{1}{2}\left(\sin 30 + \sin(45-30)\right)$

$= \frac{1}{2}(\sin 30 + \sin 45\cos 30 - \sin 30\cos 45)$

$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{2}\times\frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{2}+\sqrt{3}-1}{4\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2+\sqrt{6}-\sqrt{2}}{4\cdot 2}$

$= \frac{2+\sqrt{6}-\sqrt{2}}{8}$ as required.

13b)

$$V = \pi \int_a^b y^2 \, dx$$

from graph $a=0$ $b=\frac{\pi}{2}$

$y = 1+\sin 2x$

$y^2 = (1+\sin 2x)^2$

$$V = \pi \int_0^{\frac{\pi}{2}} (1+\sin 2x)^2 \, dx$$

using

$\sin^2 x = 1-\cos^2 x$

$\sin^2 2x = 1-\cos^2 2x$

$\sin^2 2x = \frac{1}{2}(1-\cos 4x)$

$$= \pi \int_0^{\frac{\pi}{2}} (1+\sin^2 2x + 2\sin 2x) \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(1+\tfrac{1}{2}(1-\cos 4x) + 2\sin 2x\right) dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\tfrac{3}{2} - \tfrac{1}{2}\cos 4x + 2\sin 2x\right) dx$$

$$= \pi \left[\tfrac{3}{2}x - \tfrac{1}{8}\sin 4x - \cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \pi \left[\tfrac{3}{2} \times \tfrac{\pi}{2} - \tfrac{1}{8}\sin 2\pi - \cos 2\pi - 0 + 0 + \cos 0\right]$$

$$= \pi \left(\tfrac{3\pi}{4} - 0 - -1 + 1\right)$$

$$= \pi \left(\tfrac{3\pi}{4} + 2\right)$$

$$= \frac{3\pi^2 + 8\pi}{4} \text{ units}^3$$

13c)

$P(a, 2a)$ $Q(-a, 5a)$ $R(3a, 4a)$, $S(9a, 12a)$

Length of projection is $\underline{12}$ ← scalar

i) $\overrightarrow{PQ} = \begin{pmatrix} -a-a \\ 5a-2a \end{pmatrix}$ $\therefore \overrightarrow{PQ} = -2a\underset{\sim}{i} + 3a\underset{\sim}{j}$.

$= \begin{pmatrix} -2a \\ 3a \end{pmatrix}$

ii) METHOD 1: scalar projection of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$

Scalar proj $= \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{RS}}{|\overrightarrow{RS}|}$ $\overrightarrow{RS} = \begin{pmatrix} 9a-3a \\ 12a-4a \end{pmatrix}$ $= 6a\underset{\sim}{i} + 8a\underset{\sim}{j}$

$= \begin{pmatrix} 6a \\ 8a \end{pmatrix}$

Dot product $\overrightarrow{PQ} \cdot \overrightarrow{RS} = -2a \times 6a + 3a \times 8a$

$= -12a^2 + 24a^2$

$= 12a^2$

$|\overrightarrow{RS}| = \sqrt{(6a)^2 + (8a)^2}$

$= \sqrt{36a^2 + 64a^2}$

$= \sqrt{100a^2}$

$= 10a$ (as $a > 0$, given)

$\therefore \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{RS}}{|\overrightarrow{RS}|} = \dfrac{\cancel{12}^{6}\, a^{\cancel{2}}}{\cancel{10}_{5}\, \cancel{a}}$ however scalar projection of $\overrightarrow{PQ}$ onto $\overrightarrow{RS}$ is 12 (given)

$12 = \dfrac{6a}{5}$

$a = 10$

METHOD 2: using vector projection (not scalar)

Vector proj $= \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{RS}}{|\overrightarrow{RS}|^2} \cdot \overrightarrow{RS}$

$$= \frac{\begin{pmatrix} -2a \\ 3a \end{pmatrix} \begin{pmatrix} 6a \\ 8a \end{pmatrix} (6a\underset{\sim}{i} + 8a\underset{\sim}{j})}{(6a)^2 + (8a)^2}$$ as per method 1

$$= \frac{-12a^2 + 24a^2}{36a^2 + 64a^2} (6a\underset{\sim}{i} + 8a\underset{\sim}{j})$$

$$= \frac{12a^2}{100a^2} (6a\underset{\sim}{i} + 8a\underset{\sim}{j})$$

$$= \frac{3}{25} (6a\underset{\sim}{i} + 8a\underset{\sim}{j})$$

$$= \frac{18a\underset{\sim}{i}}{25} + \frac{24a}{25}\underset{\sim}{j}$$

Given projection length is 12

$$12 = \sqrt{\left(\frac{18a}{25}\right)^2 + \left(\frac{24a}{25}\right)^2}$$

$$12 = \sqrt{\frac{900a^2}{25^2}}$$

$$12 = \frac{30a}{25}$$

$$\frac{6a}{5} = 12$$

$$6a = 60$$

$$a = 10.$$

14 a)

$$(1+x)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n x^n$$

$$\text{coeff } x^4 = 2 \times \text{coeff } x^3$$

$$ {}^nC_4 = 2\, {}^nC_3$$

$$\frac{n!}{4!(n-4)!} = 2\frac{n!}{3!(n-3)!}$$

$$\frac{3!(n-3)!}{4!(n-4)!} = 2\frac{\cancel{n!}}{\cancel{n!}}$$

$$\frac{\cancel{3!}(n-3)\cancel{(n-4)!}}{4\cdot\cancel{3!}\cancel{(n-4)!}} = 2$$

$$\frac{n-3}{4} = 2$$

$$n-3 = 8$$

$$n = 11$$

14b)

$$y = \cos^{-1}\left(\sqrt{2t-1}\right)$$

$$f(x) = (2t-1)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{\cancel{2}}(2t-1)^{-\frac{1}{2}}.\cancel{2}$$

$$= (2t-1)^{-\frac{1}{2}}$$

$$(f(x))^2 = \left((2t-1)^{\frac{1}{2}}\right)^2$$

$$= 2t-1$$

$$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$= \frac{-(2t-1)^{-\frac{1}{2}}}{\sqrt{1-(2t-1)}}$$

$$= -\frac{1}{(2t-1)^{\frac{1}{2}}\sqrt{2-2t}}$$

$$= -\frac{1}{\left(\sqrt{2t-1}\right)\left(\sqrt{2-2t}\right)}$$

14c)

Prove $2^{n+1}(5+7^n)$ is divisible by 6 for all n, positive integers.

1) Prove statement true for $n=1$

$LHS = 2^{1+1}(5+7^1)$

$= 2^2(12)$

$= 4\times 2\times 6$ which is divisible by 6

$\therefore$ true for $n=1$

2) Assume statement true for $n=K$

$2^{K+1}(5+7^K) = 6M$ where M & K are positive integers

$2^{K+1}\times 5 = 6M - 7^K(2^{K+1})$

3) Prove statement true for $n=k+1$

$2^{K+1+1}(5+7^{K+1}) = 6Q$ where Q is a positive integer

$LHS = 2^{K+1+1}(5+7^{K+1})$

$= 2^{K+2}(5+7^{K+1})$

$= 2.2^{K+1}(5+7^1.7^K)$

$= 2.5.2^{K+1} + 2.7.2^{K+1}.7^K$

$= 2(6M-7^K(2^{K+1})) + 14.2^{K+1}.7^K$ by assumption.

$= 12M - 2.7^K.2^{K+1} + 14.2^{K+1}.7^K$

$= 12M + 12.7^K.2^{K+1}$

$= 12(M+7^K.2^{K+1})$

$= 6\times 2(M+7^K.2^{K+1})$

$= 6Q$ where $Q = 2(M+7^K.2^{K+1})$

$= RHS$

Statement is divisible by 6 $\therefore$ is true for $n=k+1$

4) Therefore statement is true for all positive integers n.

14 d)

i) PERMUTATION

$\# \text{ words} = \frac{11!}{2!} = 19958400$ words

ii) p(all vowels together)

$= \frac{\# \text{ ways vowels together}}{\text{total } \# \text{ ways}}$

$= \frac{\frac{7!}{2!} \times 5!}{\frac{11!}{2!}}$

$= \frac{7! \times 5!}{11!}$

$= \frac{1}{66}$

all vowels together E, U, A, I, O

PRMTTN

$\frac{7!}{2!}$ ways to arrange group (2 Ts)

$\times 5!$ ways to arrange vowels

iii) _ _ _ _ _ _ _ _ _ _ _

11 positions

- $\binom{11}{5}$ ways to choose 5 positions for vowels A, E, I, O, U and only 1 way to arrange vowels.
- $\frac{6!}{2!}$ ways to arrange constants

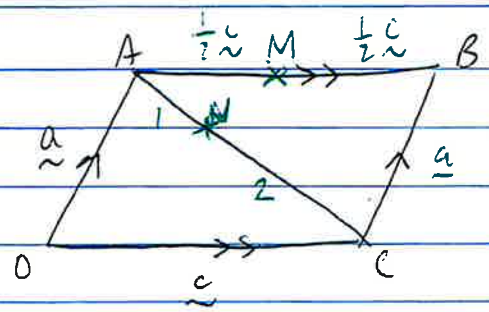
$\therefore \# \text{ words} = \binom{11}{5} \times \frac{6!}{2!} = 166320$

OR

$\frac{1}{5!}$ of the arrangements will have vowels in order $\therefore \# \text{ words} = \frac{1}{5!} \times \frac{11!}{2!} = 166320$

14e) Vector Question

OABC is a parallelogram and the point M is the midpoint of AB.
The point N lies on the diagonal AC so that AN : NC = 1:2.
Let $\vec{OA} = \underset{\sim}{a}$ and $\vec{OC} = \underset{\sim}{c}$.
Deduce, showing reasoning, that O, N & M are collinear.



$$\vec{AC} = \vec{AO} + \vec{OC}$$
$$= -\underset{\sim}{a} + \underset{\sim}{c}$$

$$\vec{AN} = \frac{1}{3}\vec{AC}$$
$$= \frac{1}{3}(-\underset{\sim}{a} + \underset{\sim}{c})$$

$$\therefore \vec{ON} = \vec{OA} + \vec{AN}$$
$$= \underset{\sim}{a} + \frac{1}{3}(-\underset{\sim}{a} + \underset{\sim}{c})$$
$$= \underset{\sim}{a} - \frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{c}$$
$$= \frac{2}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{c}$$
$$= \frac{1}{3}[2\underset{\sim}{a} + \underset{\sim}{c}]$$

$$\vec{NM} = \vec{NA} + \vec{AM}$$
$$= -\frac{1}{3}(-\underset{\sim}{a} + \underset{\sim}{c}) + \frac{1}{2}\underset{\sim}{c}$$
$$= \frac{1}{3}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{c}$$
$$= \frac{1}{3}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{c}$$
$$= \frac{1}{6}[2\underset{\sim}{a} + \underset{\sim}{c}]$$

$\therefore \vec{ON} = \vec{NM}$, ~~and~~ are in the same direction & share the point N.

$\therefore$ O, N & M are collinear.