SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2003
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 3

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections. Section A (Questions 1 and 2), Section B (Questions 3 and 4) and Section C (Questions 5 and 6).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 84 Marks

- All questions are of equal value.

Examiner: *B. Opferkuch*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

QUESTION 1. Use a *separate* Writing Booklet **Marks**

(a) Find a primitive of $\dfrac{1}{\sqrt{1-x^2}}$. **1**

(b) Evaluate $\displaystyle\int_0^{\ln 5} e^{-x}\, dx$ **2**

(c) Find $\displaystyle\lim_{h\to 0} \frac{\sin\frac{h}{2}}{h}$ **1**

(d) A box contains 5 red marbles and 4 white marbles. **2**
Three marbles are drawn in succession without replacement,
What is the probability that any two marbles are red and one is white?

(e) The sector OAB of a circle centre O and radius r has an area of $\dfrac{3\pi}{4}\ \text{cm}^2$. **2**

If the arc AB subtends an angle of $\dfrac{\pi}{6}$ at O, find the length of the arc AB.

(f) If $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \tan^{-1}(1) = k\pi$ (where k is rational) **2**

Find the value of k.

(g) Find $\displaystyle\int \frac{\ln 3x}{x}\, dx$, using the substitution $u = \ln 3x$. **4**

QUESTION 2. Use a *separate* Writing Booklet. **Marks**

(a) Differentiate the following with respect to x. **8**

(i) $x \cos x$

(ii) $\log_e(\cos 5x)$

(iii) $\tan^{-1}\left(\dfrac{x}{3}\right)$

(iv) $e^{\tan x}$

(v) $\left\{(1 + \cos^{-1}(3x)\right\}^3$

(b) Find a primitive of: **6**

(i) $x^2 e^{2x^3+1}$

(ii) $\dfrac{5}{4 + x^2}$

(iii) $\dfrac{x^2}{x^3 + 7}$

(iv) $-\sin(\pi - x)$

QUESTION 3. Use a *separate* Writing Booklet. **Marks**

(a) In choosing three letters from the word PROBING, and assuming each choice is equally likely, what is the probability of choosing just one vowel? **2**

(b) (i) In how many ways can the numbers 1, 2, 3, 4, 5, 6 be arranged around a circle? **3**

(ii) How many of these arrangements have at least two even numbers together?

(c) (i) Show that the function $f(x) = x^2 + e^{-\frac{1}{2}x} - 5$ has a root between -2 and -1. **4**

(ii) Taking $x = -2$ as a first approximation, apply Newton's method once, to show that the root of $f(x) = 0$ is approximately $-\dfrac{18}{e+8}$.

(d) (i) Show that $\dfrac{1+x}{1-x} = -1 + \dfrac{2}{1-x}$. **3**

(ii) Hence find $\displaystyle\int \frac{1+x}{1-x}\,dx$

(e) Write down the general solution of $\sqrt{3}\tan\theta - 1 = 0$. Leave answer in exact radian form. **2**

QUESTION 4. Use a *separate* Writing Booklet **Marks**

(a) Consider the curves $y = \sin x$ and $y = \cos 2x$. **5**

(i) Sketch the graphs of the two curves on the same axes, in the domain $\frac{-\pi}{2} \le x \le \frac{\pi}{6}$.

(ii) Show that the curves intersect at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

(iii) Hence find the exact area bounded by the two curves.

(b) Consider the function $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$. **3**

(i) Evaluate $f(2)$.

(ii) State the domain and range of $y = f(x)$.

(iii) Sketch the graph of $y = f(x)$.

(c) Find the exact value of $\tan\left[2\tan^{-1}\left(-\frac{1}{2}\right)\right]$. **3**

(d) The function $g(x)$ is given by $g(x) = \cos^{-1} x + \sin^{-1} x,\ \ 0 \le x \le 1$. **3**

(i) Find $g'(x)$.

(ii) Sketch the graph of $y = g(x)$.

QUESTION 5. Use a *separate* Writing Booklet **Marks**

(a) Consider the function $f(x) = x\sin^{-1}(x^2)$. **8**

(i) State the domain and range of $f(x)$.

(ii) Find $f'(x)$.

(iii) Show that there is a horizontal inflexion at the origin.

(iv) What is the slope of the tangent at $x = 1$.

(v) Sketch the curve which represents $f(x)$.

(b) (i) Show that $\displaystyle\int \frac{x}{\sqrt{x^2 + 2x - 3}}\,dx = \int \frac{x}{\sqrt{(x+1)^2 - 4}}\,dx$. **3**

(ii) Using the substitution $2\sec u = x + 1$, find $\displaystyle\int \frac{x}{\sqrt{x^2 + 2x - 3}}\,dx$

(c) The curve $y = \dfrac{1}{\sqrt{1+x^2}}$ is rotated about the x-axis. **3**

Find the volume of the solid enclosed between $x = \dfrac{1}{\sqrt{3}}$ and $x = \sqrt{3}$.

QUESTION 6. Use a *separate* Writing Booklet **Marks**

(a) Evaluate $\int_0^{\sqrt{2}} x\sqrt[3]{x^2+1}\,dx -$ using the substitution $u = x^2 + 1$. **3**

Answer in exact form.

(b) Consider the equation $\cos 3x = \sin x$ **3**

(i) Show that $\cos\left(\frac{\pi}{2} - A\right) = \sin A$

(ii) Hence, or otherwise, find the general solution of the equation $\cos 3x = \sin x$.

(c) (i) Show that $\frac{d}{dx}\tan^3 x = 3\sec^4 x - 3\sec^2 x$. **4**

(ii) Using (i) or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \sec^4 x\,dx -$

(d) Show that the exact value of $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \frac{2\pi - 3\sqrt{3}}{24}$ **4**

THIS IS THE END OF THE PAPER

2003 Ext 1 #3

(1) (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

(b) $\int_0^{\ln 5} e^{-x}\,dx = -e^{-x}\Big]_0^{\ln 5}$ $\qquad e^{\ln x} = x$

$= -\left[e^{-\ln 5} - e^0\right]$

$= -\left[e^{\ln\frac{1}{5}} - 1\right]$

$= 1 - \frac{1}{5} = \frac{4}{5}$

(c) $\lim\limits_{h\to 0} \frac{\sin\frac{h}{2}}{h} = \frac{1}{2}\lim\limits_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$ $\qquad \left[\lim\limits_{u\to 0}\frac{\sin u}{u} = 1\right]$

$= \frac{1}{2}\times 1$

$= \frac{1}{2}$

(d) 5R 4W

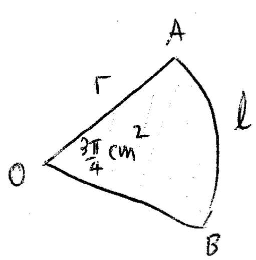
<u>method 1:</u>

$\frac{\binom{5}{2}\times\binom{4}{1}}{\binom{9}{3}} = \frac{10}{21}$

method 2: RRW

$\left(\frac{5}{9}\times\frac{4}{8}\times\frac{4}{7}\right) \times 3$

$= \frac{10}{21}$

(e) 

$\therefore \frac{1}{2}r^2\theta^c = \frac{3\pi}{4}$

$\therefore \frac{1}{2}r^2\times\frac{\pi}{6} = \frac{3\pi}{4}$

$\therefore r^2 = \frac{3}{4}\times 12 = 9$

$\therefore r = 3$

$\therefore l = r\theta$

$= 3\times\frac{\pi}{6}$

$= \frac{\pi}{2}$ cm

(f) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \tan^{-1}(1) = \frac{\pi}{6} + \frac{\pi}{4} = \frac{10\pi}{24} = \frac{5\pi}{12}$

$\therefore \boxed{k = \frac{5}{12}}$

(g) $\int \frac{\ln 3x}{x}\,dx$ $\quad u = \ln 3x$

$\therefore du = \frac{3}{3x}\,dx = \frac{dx}{x}$

$= \int \ln 3x \times \frac{dx}{x}$

$= \int u\,du = \frac{1}{2}u^2 + c$

$= \frac{1}{2}\ln^2 3x + c$

(2) (a) (i) $\frac{d(x\cos x)}{dx} = x(-\sin x) + (1)\cos x$
$= \cos x - x\sin x$

(ii) $\frac{d(\ln(\cos 5x))}{dx} = \frac{1}{\cos 5x} \times -5\sin 5x$

$= -\frac{5\sin 5x}{\cos 5x} = -5\tan 5x$

(iii) $\frac{d\left(\tan^{-1}\left(\frac{x}{3}\right)\right)}{dx} = \frac{1}{3} \times \frac{1}{1+\left(\frac{x}{3}\right)^2} = \frac{1}{3} \times \frac{9}{9+x^2}$

$= \frac{3}{9+x^2}$

(iv) $\frac{d(e^{\tan x})}{dx} = \sec^2 x\, e^{\tan x}$

(v) $\frac{d\left(\left[1+\cos^{-1}(3x)\right]^3\right)}{dx} = 3\left[1+\cos^{-1}(3x)\right]^2 \times \frac{-3}{\sqrt{1-9x^2}} = \frac{-9\left(1+\cos^{-1}(3x)\right)^2}{\sqrt{1-9x^2}}$

2(b) (i) $\int x^2 e^{2x^3+1}\,dx$

$= \frac{1}{6}\int (6x^2) e^{2x^3+1}\,dx$ $\qquad \left[N.B. \ \frac{d(2x^3+1)}{dx} = 6x^2 \right]$

$= \frac{1}{6} e^{2x^3+1} + C$

(ii) $\int \frac{5}{4+x^2}\,dx = \frac{5}{2}\int \frac{2}{4+x^2}\,dx$

$= \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

(iii) $\int \frac{x^2}{x^3+7}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3+7}\,dx = \frac{1}{3}\ln|x^3+7| + C$

(iv) $-\sin(\pi - x) = -\sin x$

$\therefore \int -\sin x\,dx = \cos x + C$

(3) (a) PRBN IG

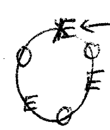
3 letters $= \binom{6}{3} = 20$ $\qquad \therefore \frac{12}{20} = \frac{3}{5}$

1 vowel $= \binom{4}{2}\times\binom{2}{1} = 6\times 2$

(b) (i) 6 numbers $\Rightarrow 5! = 120$

(ii) 1 3 5 2 4 6

No even numbers together means alternating



$\leftarrow$ place an even number first

$\therefore 2! \times 3! = 2\times 3 = 12$ ways

$\therefore$ Prob $= \frac{12}{120} = \frac{1}{10}$

$\therefore$ At least 2 $= 1 - \frac{1}{10} = \frac{9}{10}$

(c) (i) $f(x) = x^2 + e^{-\frac{1}{2}x} - 5$ is continuous

$f(-2) = 4 + e - 5 = e - 1 > 0$ ($\because e > 3$)

$f(-1) = 1 + e^{\frac{1}{2}} - 5 = e^{\frac{1}{2}} - 4 < 0$

$\therefore$ $f(-2) \cdot f(-1) < 0$ and with f continuous

$\exists c$ s.t. $f(c) = 0$, $-2 < c < -1$

(ii) $x_0 = -2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\Big[$ $f(-2) = e - 1$

$f'(x) = 2x - \frac{1}{2}e^{-\frac{1}{2}x}$

$f'(-2) = -4 - \frac{1}{2}e$

$= \frac{-8 - e}{2} = -\frac{e+8}{2}$

$$\therefore x_1 = -2 - \frac{e-1}{-\left(\frac{e+8}{2}\right)}$$

$$= -2 + \frac{2(e-1)}{e+8} = \frac{-2(e+8) + 2(e-1)}{e+8}$$

$$= \frac{-2e - 16 + 2e - 2}{e+8}$$

$$= -\frac{18}{e+8}$$

(d) (i) $\frac{1+x}{1-x} = \frac{-(1-x) + 2}{1-x}$

$$= -\frac{(1-x)}{1-x} + \frac{2}{1-x}$$

$$= -1 + \frac{2}{1-x}$$

(ii) $\int \frac{1+x}{1-x}\,dx = \int\left(-1 + \frac{2}{1-x}\right)dx$

$= -x + -2\ln|1-x| + c$

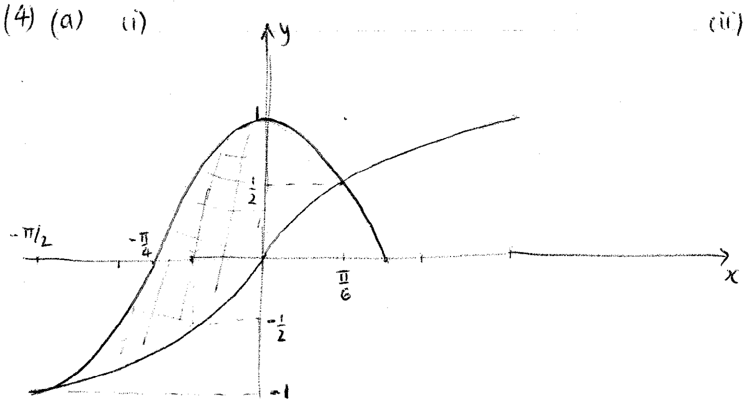
$= -x - 2\ln|1-x| + c$

3(e) $\sqrt{3}\tan\theta = 1$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\therefore \theta = n\pi + \frac{\pi}{6}$

(4) (a) (i)



(ii) $\sin(-\frac{\pi}{2}) = -1$

$\cos(2\times -\frac{\pi}{2}) = \cos(-\pi) = -1$

$\sin\frac{\pi}{6} = \frac{1}{2}$

$\cos(2\times\frac{\pi}{6}) = \cos\frac{\pi}{3} = \frac{1}{2}$

(iii) Area $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (\cos 2x - \sin x)\,dx = \left.\frac{1}{2}\sin 2x + \cos x\right]_{-\frac{\pi}{2}}^{\pi/6}$

$= \left(\frac{1}{2}\sin\frac{\pi}{3} + \cos\frac{\pi}{6}\right) - (0)$

$= \frac{1}{2}\times\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \frac{3}{2}\times\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$

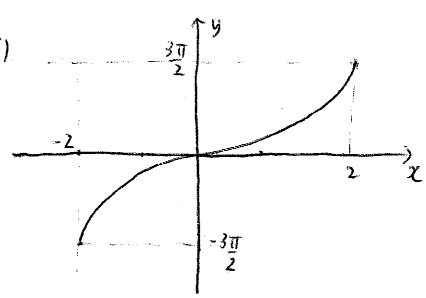
(b) $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

(i) $f(2) = 3\sin^{-1}(1) = 3\times\frac{\pi}{2} = \frac{3\pi}{2}$

(ii) $-1 \le \frac{x}{2} \le 1 \Rightarrow -2 \le x \le 2$

$-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2} \Rightarrow -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(iii)



4(c) $\tan\left[2\tan^{-1}\left(-\frac{1}{2}\right)\right]$ $\qquad$ $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$

let $\alpha = \tan^{-1}\left(-\frac{1}{2}\right)$

$\therefore \tan\alpha = -\frac{1}{2}$

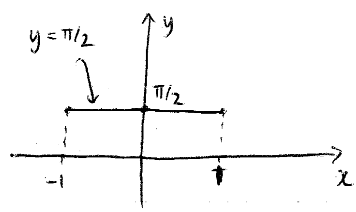
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\times\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)^2} = \frac{-1}{1-\frac{1}{4}} = \frac{-1}{3/4} = -\frac{4}{3}$$

(d) $g(x) = \cos^{-1}x + \sin^{-1}x$

(i) $g'(x) = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0$

(ii) $\therefore g(x) = \text{constant}$ for $-1 \le x \le 1$

$g(0) = \cos^{-1}(0) + \sin^{-1}(0) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$



(5) (a) $f(x) = x\sin^{-1}(x^2)$

(i) $-1 \le x^2 \le 1 \Rightarrow -1 \le x \le 1$

$\Rightarrow -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii) $f'(x) = \sin^{-1}(x^2) + x\left(\frac{2x}{\sqrt{1-x^4}}\right)$

$= \sin^{-1}(x^2) + \frac{2x^2}{\sqrt{1-x^4}}$

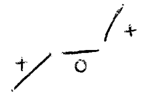
$=$ even function

(iii)

x	0^-	0	0^+
$f'(x)$	$+$	0	$+$

$\leftarrow$ ie $f'\left(-\frac{1}{2}\right) = f'\left(\frac{1}{2}\right)$

$\therefore$ HPOI



(iv) $f'(1) = \sin^{-1}1 + \frac{2}{\sqrt{0}}$

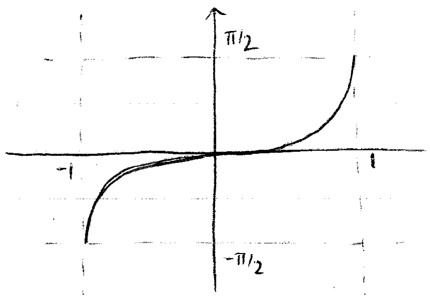
$\therefore$ vertical tangent

[OR: $\sin^{-1}(x^2)$ has a double root at $x=0$ $\because$ even

$\therefore x\sin^{-1}(x^2)$ has a triple root

$\therefore$ HPOI]

5 (a) (v) $f(x)$ is an odd function



(b) (i) $x^2+2x-3 = (x^2+2x+1) - 4$
$= (x+1)^2 - 4$

$\therefore \int \frac{x}{\sqrt{x^2+2x-3}}\,dx = \int \frac{x\,dx}{\sqrt{(x+1)^2-4}}$

(ii) $2\sec u = x+1$
$\therefore 2\sec u \tan u\, du = dx$

$\tan^2 u + 1 = \sec^2 u$

$[4\sec^2 u - 4 = 4(\sec^2 u - 1)$
$= 4\tan^2 u]$

$x+1$, $\sqrt{(x+1)^2-4}$, u, 2

$\therefore \tan u = \frac{\sqrt{x^2+2x-3}}{2}$

$\int \frac{x\,dx}{\sqrt{(x+1)^2-4}}$

$= \int \frac{(2\sec u - 1)\, 2\sec u \tan u\, du}{\sqrt{4\sec^2 u - 4}}$

$= \int \frac{\cancel{2}(2\sec u - 1)\sec u \cancel{\tan u}\, du}{\cancel{2\tan u}}$

$= \int (2\sec^2 u - \sec u)\, du$

$= 2\tan u - \int \sec u\, du$

$= \sqrt{x^2+2x-3} - \underbrace{\int \sec u\, du}$

not an Ext 1 integral

[could be done reversing the substitution BUT worth more than 2 marks!]

$= \sqrt{x^2+2x-3} - \ln|\sec u + \tan u| + C$

$= \sqrt{x^2+2x-3} - \ln\left|\frac{x+1}{2} + \frac{\sqrt{x^2+2x-3}}{2}\right| + C$

$= \sqrt{x^2+2x-3} - \ln\left|x+1 + \sqrt{x^2+2x-3}\right| + K$

(5) (c) $y = \dfrac{1}{\sqrt{1+x^2}}$

$$\therefore V = \pi \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} y^2\,dx$$

$$= \pi \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+x^2}\,dx$$

$$= \pi \ \tan^{-1}(x) \Big]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \pi \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\tfrac{1}{\sqrt{3}}\right)\right]$$

$$= \pi \left[\frac{\pi}{3} - \frac{\pi}{6} \right]$$

$$= \pi \left[\frac{\pi}{6} \right]$$

$$= \frac{\pi^2}{6} \ \text{c.u.}$$

(6) a) $\displaystyle\int_0^{\sqrt{2}} x \sqrt[3]{x^2+1}\,dx = \frac{1}{2}\int_0^{\sqrt{2}} \sqrt[3]{x^2+1}\,(2x\,dx)$

$u = x^2+1 \quad \therefore du = 2x\,dx$

$x = 0 \Rightarrow u = 1$

$x = \sqrt{2} \Rightarrow u = 3$

$$= \frac{1}{2}\int_1^3 u^{\frac{1}{3}}\,du$$

$$= \frac{1}{2} \times \frac{3}{4} u^{4/3} \Big]_1^3$$

$$= \frac{3}{8}\left[3^{4/3} - 1\right]$$

$$= \frac{3}{8}\left[\sqrt[3]{81} - 1\right]$$

$$= \frac{3}{8}\left[3\sqrt[3]{3} - 1\right]$$

$27 \times 3 = 81$

6 (b) $\cos 3x = \sin x$

(i) $\cos(\frac{\pi}{2} - A)$

$= \cos\frac{\pi}{2}\cos A + \sin\frac{\pi}{2}\sin A$

$= 0 \times \sin A + 1 \times \sin A$

$= \sin A$

(ii) $\cos 3x = \sin x$

$\Rightarrow \cos 3x = \cos(\frac{\pi}{2} - x)$

$\therefore 3x = 2n\pi \pm (\frac{\pi}{2} - x)$

$3x = 2n\pi + \frac{\pi}{2} - x$	$3x = 2n\pi - \frac{\pi}{2} + x$
$4x = (4n+1)\pi$	$2x = \frac{(4n-1)\pi}{2}$
$x = (\frac{4n+1}{4})\pi$	$x = \frac{(4n-1)\pi}{4}$

$\therefore x = (\frac{4n \pm 1}{4})\pi$

(c) (i) $\frac{d(\tan^3 x)}{dx} = 3\tan^2 x \cdot \sec^2 x$

$= 3(\sec^2 x - 1)\sec^2 x$

$= 3\sec^4 x - 3\sec^2 x$

(ii) $\therefore \tan^3 x = \int (3\sec^4 x - 3\sec^2 x)\, dx$

$\therefore \tan^3 x \Big]_0^{\pi/4} = 3\int_0^{\pi/4} \sec^4 x\, dx - 3\int_0^{\pi/4} \sec^2 x\, dx$

$\therefore 3\int_0^{\pi/4} \sec^4 x\, dx = \left[\tan^3 \frac{\pi}{4} - \tan^3(0)\right] + 3\tan x \Big]_0^{\pi/4}$

$= (1 - 0) + 3(1 - 0)$

$= 4$

$\therefore \int_0^{\pi/4} \sec^4 x\, dx = \frac{4}{3}$

$\cos 2A = 1 - 2\sin^2 A$

(6) (d) $\displaystyle\int_0^{\pi/6} \sin^2 x \, dx$

$= \dfrac{1}{2}\displaystyle\int_0^{\pi/6} 2\sin^2 x \, dx$

$= \dfrac{1}{2}\displaystyle\int_0^{\pi/6} (1 - \cos 2x) \, dx$

$= \dfrac{1}{2}\left[x - \dfrac{1}{2}\sin 2x\right]_0^{\pi/6}$

$= \dfrac{1}{2}\left[\pi/6 - \dfrac{1}{2}\sin \pi/3\right]$

$= \pi/12 - \dfrac{1}{4} \times \dfrac{\sqrt{3}}{2}$

$= \pi/12 - \dfrac{\sqrt{3}}{8}$

$= \dfrac{2\pi - 3\sqrt{3}}{24}$