**SYDNEY BOYS HIGH SCHOOL**

| 2020 | YEAR 12<br>TERM 2 HSC TASK 2 |
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# Mathematics Extension 1

**General Instructions**

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black pen.
- NESA approved calculators may be used.
- A reference sheet is provided with this paper.
- Marks may **NOT** be awarded for messy or badly arranged work.
- In Questions 6–8 show ALL relevant mathematical reasoning and/or calculations. The amount of working needed is commensurate with the marks available.

**Total Marks: 50**

**Section I: – 5 marks**

- Attempt Questions 1–5 (pages 2–3)
- Allow about 10 minutes for this section.

**Section II: – 45 marks**

- Attempt Questions 6–8 (pages 4–8)
- Allow about 1 hour and 20 minutes for this section.

**Examiner:**
***A Ward***

**Name: ____________________ Class: 12 Ma _**

| NESA Number | | | | | | | | | |
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## Section I

**5 marks**
**Attempt Questions 1–5**
**Allow about 10 minutes for this section.**

Use the multiple-choice answer sheet for Questions 1–5.

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**1** If $(x-2)$ is a factor of $x^3 - kx^2 + 12x - 8$, what must be the value of $k$ ?

A. –3

B. 3

C. 6

D. –6

**2** Colm has 4 identical blue exercise books and 3 identical red exercise books. In how many different ways could he place the exercise books on a shelf?

A. 144

B. 5040

C. 72

D. 35

**3** A curve is defined by the parametric equations $x = \sin 3t$ and $y = \cos 3t$ .

Which of the following, in terms of $t$, equates to $\frac{dy}{dx}$ ?

A. $\cot 3t$

B. $\tan 3t$

C. $-\tan 3t$

D. $-3\sin 3t$ .

**4** The remainder when a polynomial $P(x)$ is divided by $x^2 - 1$ is $3x - 4$ . What is the remainder when $P(x)$ is divided by $x + 1$?

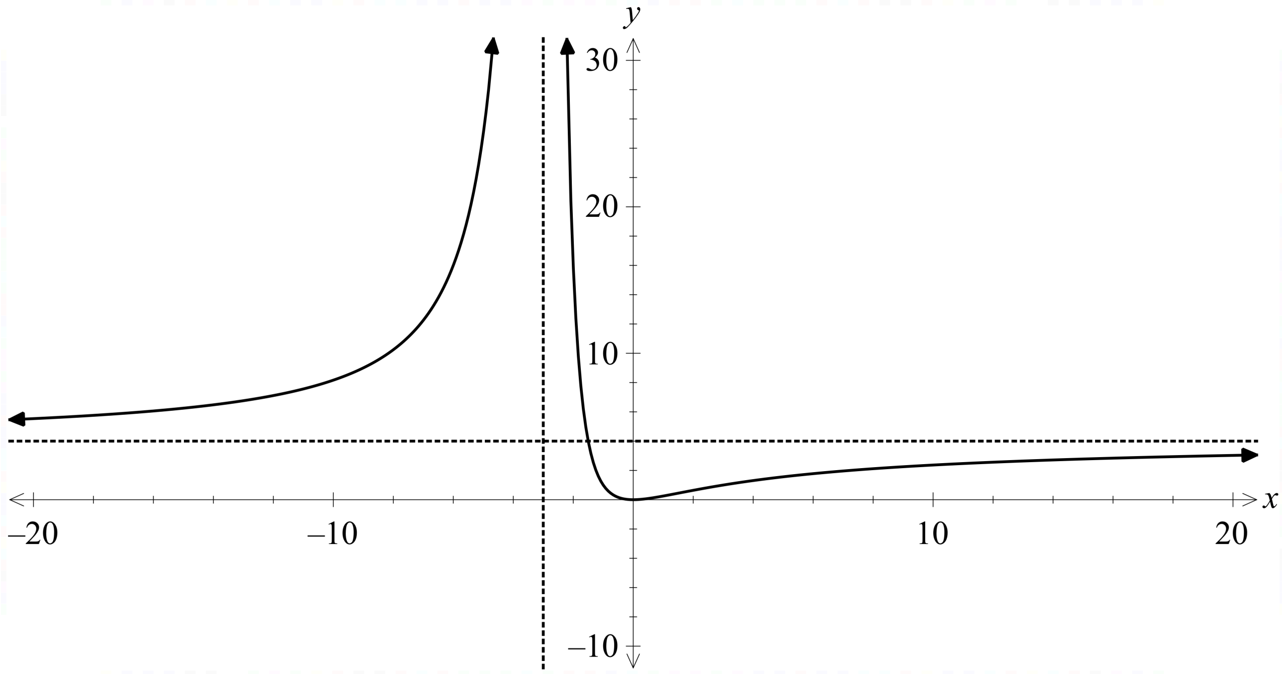
A. $-1$

B. $3x^2 - x - 4$

C. $3x - 4$

D. $-7$

**5** The graph shown below has the equation of the form $y = \dfrac{r - sx^2}{(x+t)^2}$.




A. $r = -2,\ s = 4,\ t = -3$.

B. $r = 0,\ s = -4,\ t = 3$.

C. $r = 0,\ s = 4,\ t = 3$.

D. $r = 2,\ s = -4,\ t = -3$.

**End of Section I**

## Section II

**45 marks**
**Attempt Questions 6–8**
**Allow about 1 hour and 20 minutes for this section.**

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 6–8, your responses should include ALL relevant mathematical reasoning and/or calculations.

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**Question 6** (14 marks) Use a SEPARATE writing booklet

(a) Bill has 10 music tracks on his iPod. Two of them are by Mozart, five are by Eminem and the other three are by various other artists, each different.

(i) Bill wants to play all tracks only once. How many arrangements are there of the ten tracks? **1**

(ii) Bill now wants to play the ten tracks with each artist's tracks being grouped together, each group of tracks in any order.
How many arrangements of the ten tracks are now possible? **1**

(iii) He now decides he only has time for another five tracks. The five will start with one by Mozart and end with one by Eminem, with the three others by any artist in between. How many arrangements of the five are possible? **1**

(b) $P(x) = 2x^3 - 5x^2 - 14x + 8$ has zeros $\alpha$, $\beta$, and $\gamma$. Determine the value of:

(i) $\dfrac{2}{\alpha} + \dfrac{2}{\beta} + \dfrac{2}{\gamma}$ **2**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(c) Prove by Mathematical Induction that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all $n \in \mathbb{Z}^+$. **3**

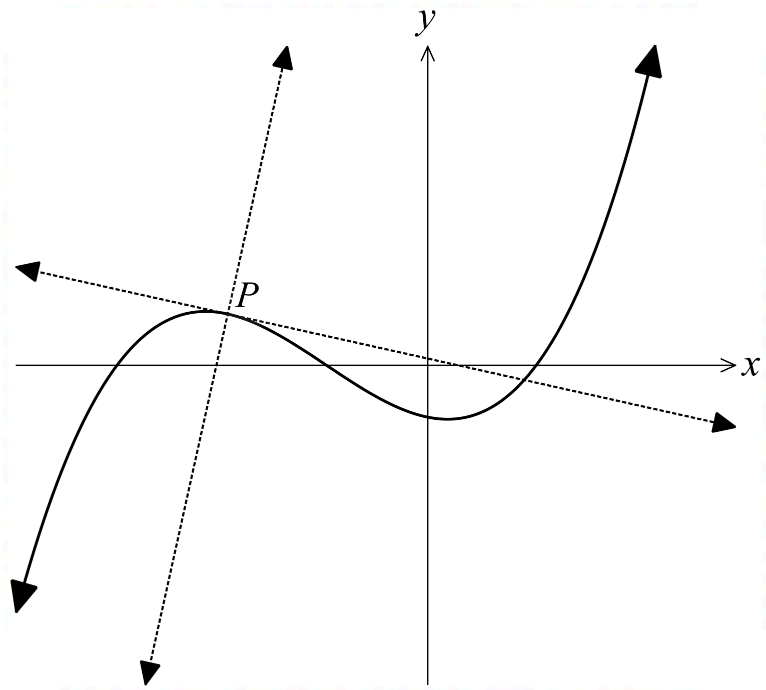
**Question 6 continues on page 5**

Question 6 (continued)

(d) The diagram shows a curve defined by the parametric equations

$$x = 3t - 2 \text{ and } y = t^3 - 2t$$

The tangent and normal to the curve, at $P\left(3t-2, t^3-2t\right)$, are shown below.




(i) Show that the equation of the tangent at $P$ is **2**

$$y + 2t^3 = t^2x - \frac{2}{3}x + 2t^2 - \frac{4}{3}$$

(ii) Find the equation of the normal through $P$. **2**

**End of Question 6**

**Question 7** (15 marks) Use a SEPARATE writing booklet

(a) How many numbers must be chosen from the integers 1 to 100 inclusive to ensure that at least three numbers are consecutive. **2**
Give an explanation for your answer.

(b) Prove by Mathematical Induction that: **3**

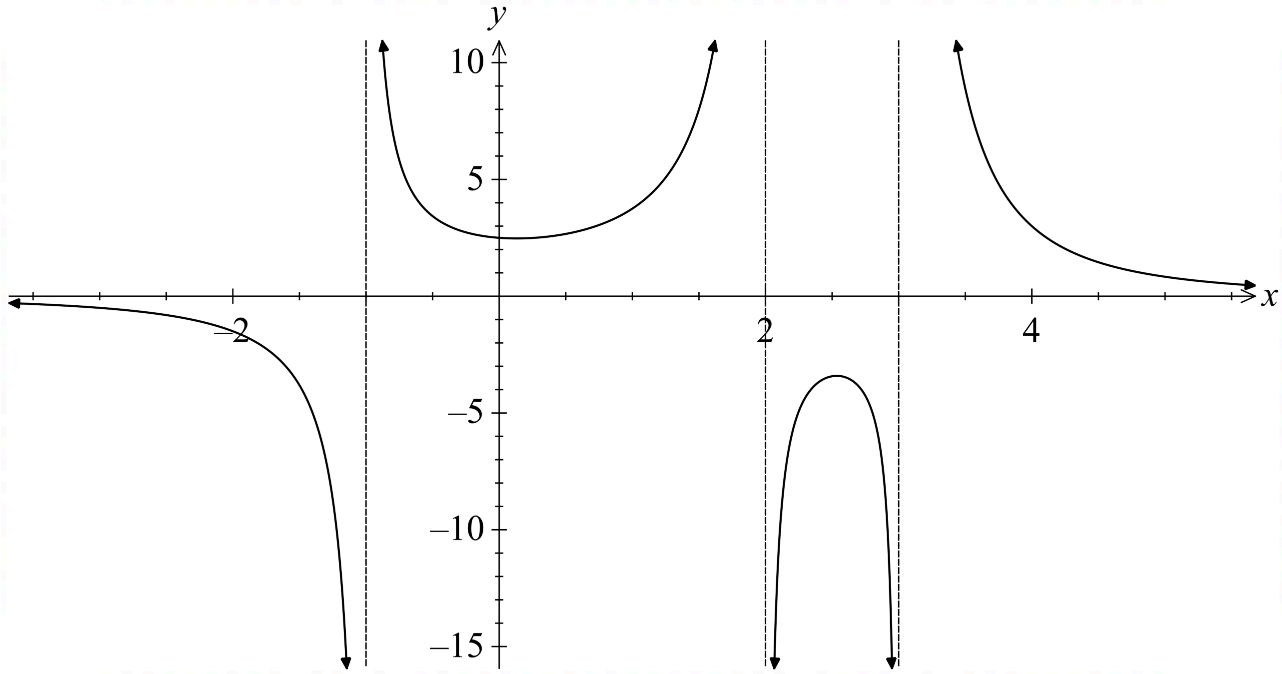
$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+...+\frac{n-1}{2^{n-1}}+\frac{n}{2^n}=2-\frac{(n+2)}{2^n} \text{ for all } n\in\mathbb{Z}^+$$

(c) Factorise $2x^3+3x^2-12x-20$ into real factors, given that it has a double zero. **2**

**Question 7 continues on page 7**

Question 7 (continued)

(d) The diagram below shows the graph of $y = f(x)$.




The diagram above shows asymptotes at $x = -1$, $x = 2$, $x = 3$, and $y = 0$.
There is a minimum turning point just to the right of the $y$-axis.

On the insert page provided, sketch the following:

(i) $y^2 = f(x)$. **2**

(ii) $y = \dfrac{1}{f(x)}$. **2**

(iii) $y = f'(x)$. **2**

(e) The equation of a curve is given by $x^2 + 3xy + 2y^2 = 9$. **2**

Show that $\dfrac{dy}{dx} = -\dfrac{2x+3y}{3x+4y}$.

**End of Question 7**

# Question 8 starts on page 8

**Question 8** (16 Marks) Use a SEPARATE writing booklet

(a) (i) In how many ways can four people be grouped into two pairs to play a set of doubles badminton? **1**

(ii) Eight members of a badminton club meet to play two simultaneous sets of doubles on two indistinguishable courts. **2**
In how many different ways can the members of this club be selected for these doubles games of badminton?

(b) The polynomial equation $x^3 - x^2 - x + 2 = 0$ has roots $\alpha, \beta$ and $\gamma$.

(i) Find the equation of the polynomial with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$. **2**

(ii) Find the equation of the polynomial with roots, **2**

$$\alpha+1,\ \beta+1, \text{ and } \gamma+1.$$

Give your answer in expanded, simplified form.

(c) Three couples are supposed to sit down to dinner at a round table such that no person is to sit next to their partner. In how many ways are there for the six people to sit down at the table under these conditions? **4**

(d) (i) A polynomial $H(x)$ of degree $n$ has $n$ distinct real roots. **2**
Explain why the $r$th derivative of $H(x)$ has $(n - r)$ distinct real roots.

(ii) Consider the polynomial $P(x) = t_n x^n + t_{n-1}x^{n-1} + t_{n-2}x^{n-2} + \ldots + t_1 x + t_0$ **3**

Show that if $(n-1)t_{n-1}^2 < 2nt_n t_{n-2}$ then $P(x)$ cannot have $n$ distinct real roots.

**End of Paper**

SYDNEY
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2020 | YEAR 12 TERM 1
HSC ASSESSMENT TASK 2

# Mathematics Extension 1

## SUGGESTED SOLUTIONS

### MC QUICK ANSWERS

**1** C

**2** D

**3** C

**4** D

**5** B

# Question 6

(a) (i) $10! = 3\,628\,800$
$(= {}^{10}P_{10})$ [1]

(ii) $2! \times 5! \times 5! \times 1 \times 1 \times 1$
$(= {}^{2}P_{2} \times {}^{5}P_{5} \times ({}^{1}P_{1})^{3} \times {}^{5}P_{5})$

$= 28\,800$ [1]

(iii) $2 \times 8 \times 7 \times 6 \times 5$
$({}^{2}P_{1} \times {}^{8}P_{3} \times {}^{5}P_{1})$

$= 3360$ [1]

**Comments:**

Part (i), done well. However, in parts (ii) and (iii), many students have didn't read properly.

(b) $P(x) = 2x^3 - 5x^2 - 14x + 8$

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \left(= \frac{-(-5)}{2}\right)$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} \quad \left(= -\frac{14}{2}\right)$$

$$\alpha\beta\gamma = -\frac{d}{a} \quad \left(= -\frac{8}{2}\right)$$

(i) $$\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} = 2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)$$

$$= 2\left(\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}\right)$$

$$= 2(-7 \div (-4))$$

$$= \frac{7}{2}$$ [2]

(ii) $\alpha^2 + \beta^2 + \gamma^2$

$$\Rightarrow (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \alpha\gamma)$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$$

$$= \left(\frac{-(-5)}{2}\right)^2 - 2(-7)$$

$$= \frac{81}{4} \quad \left(20\ \frac{1}{4}\right)$$ [2]

| **Comments:** |
| --- |
| Done reasonably well. |

(c) 1// Prove true for $n=1$:

$$LHS = 1^3 + (1+1)^3 + (1+2)^3$$
$$= 1 + 8 + 27$$
$$= 36$$
$$= 9 \times 4$$
$\therefore$ divisible by 9.
$\therefore$ true for $n=1$.

2// Assume true for $n=k$, where $k \in \mathbb{Z}^+$
ie; $k^3 + (k+1)^3 + (k+2)^3 = 9M$ (1)
where $M \in \mathbb{Z}$

Prove true for $n=k+1$;
ie; $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9N$ (2)
where $N \in \mathbb{Z}$

From (2)
$$= (k+1)^3 + (k+2)^3 + (k+3)^3 + k^3 - k^3$$
$$= 9M + (k+3)^3 - k^3$$
$$= 9M + k^3 + 9k^2 + 27k + 27 - k^3$$
$$= 9M + 9k^2 + 27k + 27$$
$$= 9(M + k^2 + 3k + 3)$$
$$= 9N$$
(where $N' = M + k^2 + 3k + 3$ [which is an integer as M and k are integers]).

3// $\therefore$ True $\forall n \geq 1$ by Mathematical Induction.

[3]

**Comments:**

Done very well.

(d) (i) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{dy}{dx} \times \frac{1}{\frac{dx}{dt}}$

$x = 3t - 2$ $\qquad$ $y = t^3 - 2t$

$\therefore \frac{dx}{dt} = 3$ $\qquad$ $\therefore \frac{dy}{dt} = 3t^2 - 2$

- gradient of the tangent:

$\therefore \frac{dy}{dx} = \frac{3t^2 - 2}{3}$

$= t^2 - \frac{2}{3}$

- equation of the tangent at P.

$y - y_1 = m(x - x_1)$

$y - (t^3 - 2t) = \left(t^2 - \frac{2}{3}\right)(x - (3t - 2))$

$y - t^3 + 2t = t^2x - 3t^3 + 2t^2 - \frac{2}{3}x + 2t - \frac{4}{3}$

$\therefore y + 2t^3 = t^2x - \frac{2}{3}x + 2t^2 - \frac{4}{3}$ [2]

(ii) • Equation of the normal at P.

$$y - (t^3 - 2t) = \left(\frac{-3}{3t^2 - 2}\right)\left(x - (3t - 2)\right)$$

$$(3t^2 - 2)(y - t^3 + 2t) = -3x + 9t - 6$$

$$3t^2y - 3t^5 + 6t^3 - 2y - 2t^3 - 4t = -3x + 9t - 6$$

$$\therefore 3x + (3t^2 - 2)y - 3t^5 + 8t^3 - 13t + 6 = 0 \quad [2]$$

**Comments:**

Done reasonably well. Some students didn't apply the chain rule properly when evaluating the gradients.

Solutions for Yr 12 HSC Task 2: Question 7

NOTE: When looking at the marking for this question, it is important to note that a tick does not mean a mark was awarded (similarly a cross does not mean a mark was not awarded). It simply means the expression on paper is correct – no more.

a) One could ask a different question of oneself: How many numbers is it possible to choose and not have 3 consecutive numbers (by selection rather than randomly)?
It is possible to choose from every group of 3 numbers (1-3,4-6,…,97-99) the first two numbers and not the last and also choose 100. There are $\frac{99}{3}=33$ groups of 3 numbers from which, 2 numbers were chosen in each. Therefore, we have selected 66 numbers from the groups; and also the number 100. This is a total of 67 numbers, and we do not yet have three consecutive numbers. It is clear though and whatever number is chosen next will lead to at least 4 consecutive numbers (if the number 99 were chosen), which is at least 3 consecutive numbers. The solution is: 68 numbers would need to be chosen to ensure that at least three numbers are consecutive.

Report: Too many students felt that choosing every second number was enough. As you can see, it is possible to choose many more numbers and still not have 3 consecutive numbers. A few students got close to this but failed to see they could add one more number in a specific way and still not have 3 consecutive numbers. Only a small percentage (less than 10% is my educated guess) got full marks for this question, although it is a very simple pigeonhole principle question. One can find much harder ones in a textbook.

b) Step 1: Prove it is true for $n = 1$

$$\frac{1}{2^1}=2-\frac{(1+2)}{2^1}$$

$$\frac{1}{2}=2-\frac{3}{2}$$

$$\frac{1}{2}=\frac{1}{2}$$

Therefore, it is true for $n = 1$.

Step 2: Assume it is true for $n = k$

$$\frac{1}{2}+\frac{2}{2^2}+\ldots+\frac{k-1}{2^{k-1}}+\frac{k}{2^k}=2-\frac{(k+2)}{2^k}$$

Step 3: Prove it is true for $n = k + 1$

$$\frac{1}{2}+\frac{2}{2^2}+\ldots+\frac{k-1}{2^{k-1}}+\frac{k}{2^k}+\frac{k+1}{2^{k+1}}=2-\frac{(k+1+2)}{2^{k+1}}$$

$$LHS=2-\frac{(k+2)}{2^k}+\frac{k+1}{2^{k+1}}$$ (using the assumption in Step 2)

$$LHS=2-\frac{2(k+2)}{2^{k+1}}+\frac{k+1}{2^{k+1}}$$

$$LHS=2+\frac{k+1-2(k+2)}{2^{k+1}}$$

$LHS = 2 + \frac{-k-3}{2^{k+1}}$

$LHS = RHS$

Step 4: Final Statement

$\frac{1}{2} + \frac{2}{2^2} + ... + \frac{n-1}{2^{n-1}} + \frac{n}{2^n} = 2 - \frac{(n+2)}{2^n}$, for all $n \in \mathbb{Z}^+$, is true for $n = 1$. It is also true for $n = k + 1$ (assuming it is true for $n = k$), therefore $\frac{1}{2} + \frac{2}{2^2} + ... + \frac{n-1}{2^{n-1}} + \frac{n}{2^n} = 2 - \frac{(n+2)}{2^n}$ is true for all $n \in \mathbb{Z}^+$.

Report: This was generally well done. If students made a mistake but went on with their working regardless (i.e. there was no covering up or 'fudging' going on) I was generous with marking. If students made a mistake in their earlier working and obviously ignored it at the end such that they miraculously had the correct answer despite it being impossible to do so considering their working – I could not apply e.c.f. and marks were not as forthcoming.

It is also worth note that most of the marks here are for the mathematics in the question, but if I were also marking this question based on layout and presentation of the proof, almost nobody would get full marks.

c) Roots describe the solutions to the equation $P(x) = 0$. (One would find the zeroes of $P(x)$). Zeroes are the value for $x$ which makes a polynomial zero (if one is given an expression then one is looking for its zeroes).

Differentiating the expression gives $6x^2 + 6x - 12 = 6(x^2 + x - 2)$
This factorises to $6(x + 2)(x - 1)$
Let the original expression be equal to $P(x)$
$P(1) = 2(1)^3 + 3(1)^2 - 12(1) - 20 = -27$, this is not a zero of $P(x)$
$P(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 20$
$= 0$
Therefore $-2$ is a zero of the original expression and, furthermore, must be a double zero (at least – though by inspection, it is clear it cannot have a multiplicity of 3).
In order to simplify things, turn the expression into a monic (this will not affect what the zeroes

are): $x^3 + \frac{3}{2}x^2 - 6x - 10$

There are three zeroes of the original expression (since it has degree three), two of which are $-2$.
The product of the zeroes is 10.
Let $\alpha$ be the unknown zero.

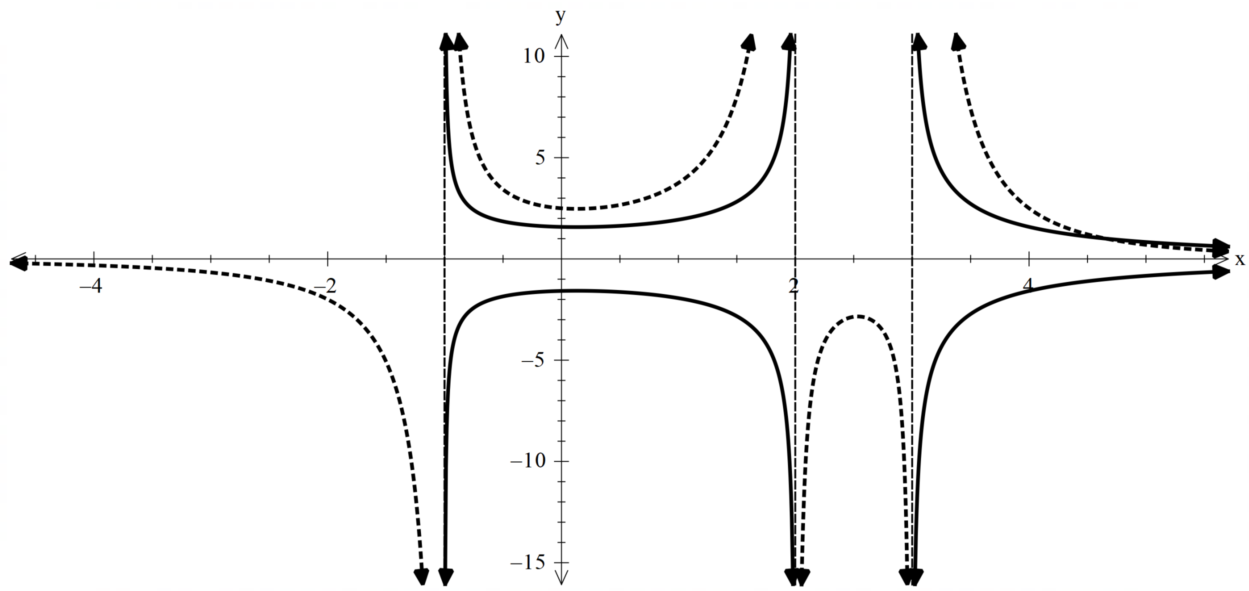
Then: $10 = (-2)^2\alpha \rightarrow \alpha = \frac{5}{2}$

The zeroes are $\frac{5}{2}$ and $-2$, where $-2$ has a multiplicity of 2.

The factorised form of the expression is: $(2x-5)(x+2)^2$

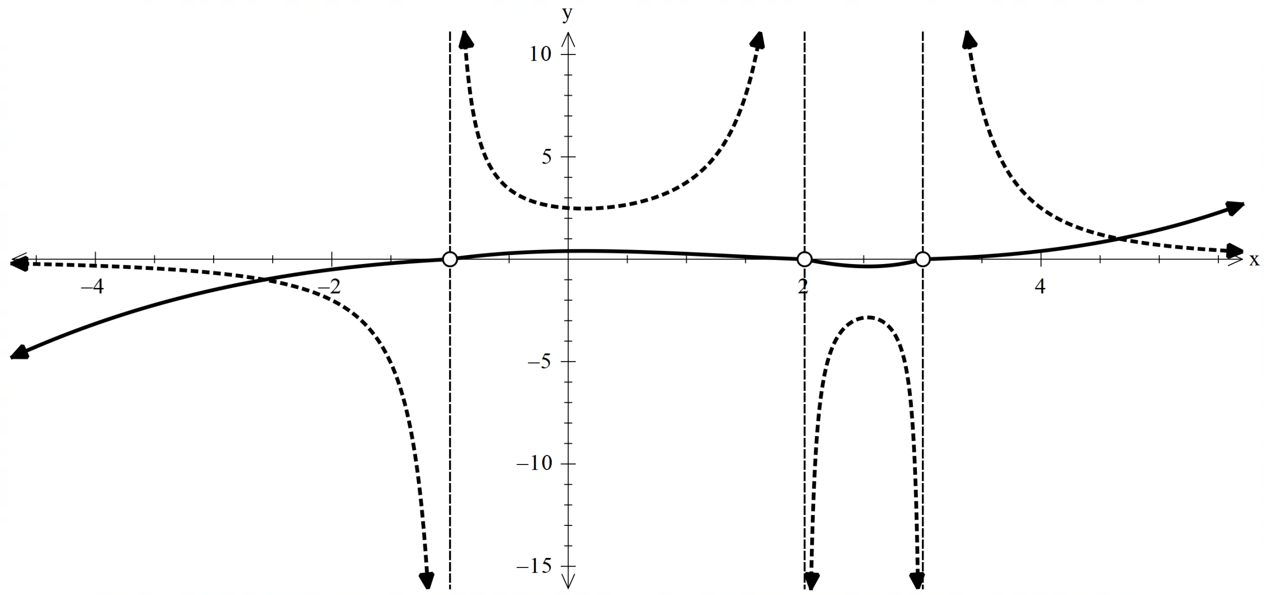
Report: This question generally received full marks. However, only a small percentage of students used the hint that there exists a double zero. Many used the remainder theorem and educated guesses. Once a zero was found, these then used long division (to good effect I might add) in order to obtain other factors. Many students went straight to using the sum and product of roots. This often led to difficult algebra and a few were not able to go on with it.

d) i)



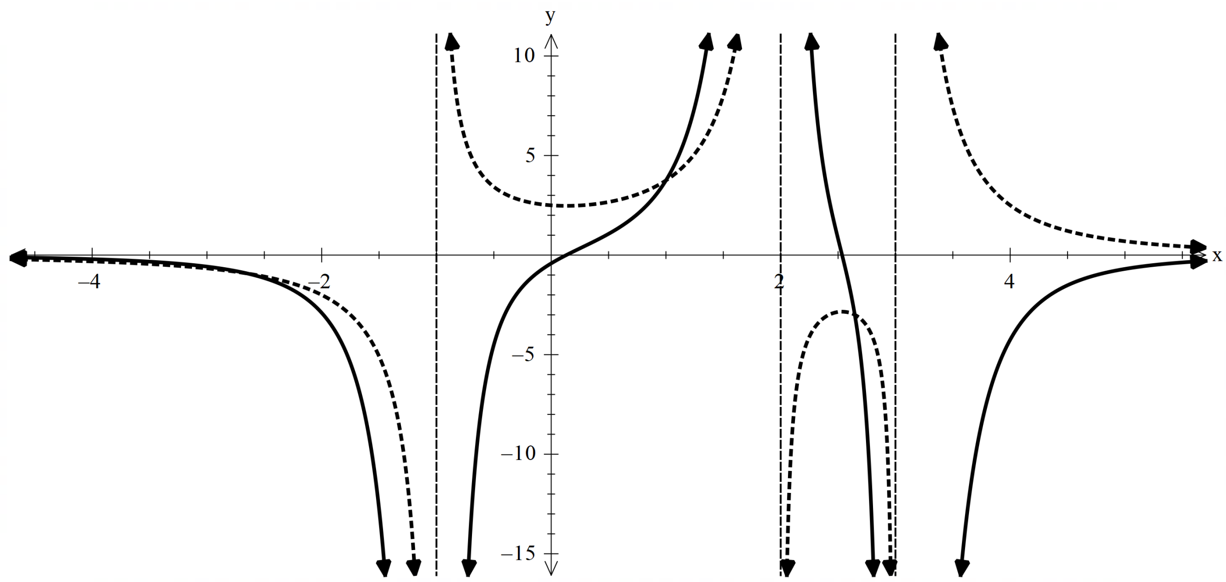

Report: Most students did well on this. Occasionally students were not careful enough to show that when $f(x) = 1$ then the square root of $f(x)$ also equals 1.

ii)




Report: This was not done very well, and many mistakes were made. The main mistake was that in the outer regions (far left and right) students failed to realise (hence show) that the reciprocal of $f(x)$ must intercept $f(x)$ at $y = 1$ (far right) and again at $y = –1$ (far left), since the reciprocal of 1 and –1 are 1 and –1 respectively. Another mistake was the maximum and minimum values of the middle left and right regions were made too high and low respectively.

iii)




Report: In the question it made it clear the there is a stationary point just to the right of the origin and students had to show this. The two middle regions were often shown as linear in students' responses. I'm assuming they assumed these to curves are parabolas (the derivative of a parabola is linear). Any reasonable examination of these should have dispelled this idea (I'll explain later). Many students just drew over the curve in the far-left region. Whilst the derivative here will have a similar shape, there is a very good reason why it cannot possibly be exactly the same. This would mean the derivative is equal to the original function. It must then be of the form $a \times e^{x-b}$, where $a$ and $b$ are real numbers and $a < 0$. This function and the parabola $ax^2 + bx + c$ ($a$, $b$, and $c$ are real numbers, $a \neq 0$) cannot have asymptotes, hence these are not the functions in those regions. Students were marked wrong for making these assumptions.

e) $\frac{d}{dx}\left(x^2+3xy+2y^2=9\right)$

The first term inside the brackets is a simple application of the derivative.

$$\frac{d}{dx}\left(x^2\right)=2x$$

The second term will be an application of Leibnitz Rule (product) and one should use the chain rule on $y$ – consider it as a function in $x$)

$$\frac{d}{dx}(3xy)=3\left(\left(\frac{d}{dx}x\right)\times y+x\times\left(\frac{d}{dy}y\right)\times\frac{dy}{dx}\right)$$

$$=3\left(y+x\times\frac{dy}{dx}\right)$$

The third term will be an application of the chain rule only.

$$\frac{d}{dx}\left(2y^2\right)=2\times\left(\frac{d}{dy}y^2\right)\times\frac{dy}{dx}$$

$$=4y\times\frac{dy}{dx}$$

The last term (the constant on the right-hand side) is once a simple application of the derivative.

$$\frac{d}{dx}(9)=0$$

Then $\frac{d}{dx}\left(x^2+3xy+2y^2=9\right)$ is: $2x+3\left(y+x\times\frac{dy}{dx}\right)+4y\times\frac{dy}{dx}=0$

One must expand this result and collect like terms (considering the derivative as a term).

$$2x+3y+\frac{dy}{dx}(3x+4y)=0$$

Solving for the derivative:

$$\frac{dy}{dx}(3x+4y)=-2x-3y$$

$$\frac{dy}{dx}=-\frac{2x+3y}{3x+4y}$$

Report: Some students tried to separately take the derivative with respect to $x$ and then $y$ (considering all else constant) and then used the chain rule (as one would if it were a parametric equation) to put them back together. It works for parametric equations because $x$ and $y$ are both functions of another (single) variable. One cannot apply it here. It was necessary to use implicit differentiation and show this in an obvious way, then use algebra to show the result, to get full marks. Another problem with this question was that many students were not comfortable using the $\frac{d}{dx}$ operator and used $\frac{dy}{dx}=$ then an expression that uses derivatives and another equals sign. This is clumsy use of notation and needs to be fixed up before HSC. One student even added it to their expression and hence got something that was not quite right. Had they used the operator correctly, that individual would have gotten full marks.

# Question 8 SOLUTIONS

(a) (i) In how many ways can four people be grouped into two pairs to play a set of doubles badminton? **1**

Let the players be ABCD

Choose 2 players to form a doubles team in $\binom{4}{2}$ ways.

But when AB is selected as a team CD is selected by default.

$\therefore$ there are $\binom{4}{2} \div 2 = 3$ ways to group the four people.

**Comment:**

Many students have forgotten that without specific information relating to the teams, then the groups are to be regarded as indistinguishable.

(ii) Eight members of a badminton club meet to play two simultaneous sets of doubles on two indistinguishable courts.
In how many different ways can the members of this club be selected for these doubles games of badminton? **2**

Let the players be ABCDEFGH

First choose 4 people to play on one court in ${}^8C_4$ ways.
But when ABCD are picked for one court then EFGH are selected by default for the other court. As the courts are indistinguishable there are $\frac{{}^8C_4}{2}$ ways to do this.

Now that the players have been assigned to the two courts then from part (a), there are 3 ways to group the players on each court.

$\therefore$ there are $\frac{{}^8C_4}{2} \times 3^2 = 315$ ways to organise the games.

**Comment:**
This was not done well by most students.

At least a $\frac{1}{2}$ was given where students tried to make a link to the result in part (a).

8. (a) (ii) **Alternative**

Pick two players in ${}^8C_2$ ways and their competitors in ${}^6C_2$ ways.

So for the first court there are $\dfrac{{}^8C_2 \times {}^6C_2}{2!}$ ways of organising the doubles match as their position relative to the net is irrelevant.

Similarly, for the second court, there are $\dfrac{{}^4C_2 \times {}^2C_2}{2!}$ to organise the remaining players.

Given that the courts are indistinguishable,
there are $\dfrac{{}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{2! \times 2! \times 2!} = 315$ different ways to organise the sets.

(b) The polynomial equation $x^3 - x^2 - x + 2 = 0$ has roots $\alpha, \beta$ and $\gamma$.

(i) Find the equation of the polynomial with roots $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. **2**

Let $P(x) = x^3 - x^2 - x + 2$.

$\therefore P\left(\dfrac{1}{x}\right) = 0$ has roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$, and $\dfrac{1}{\gamma}$.

$$P\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - \left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right) + 2$$

$$= \frac{1}{x^3} - \frac{1}{x^2} - \frac{1}{x} + 2$$

$$= \frac{1}{x^3}\left(1 - x - x^2 + 2x^3\right)$$

$\therefore$ the polynomial equation is $1 - x - x^2 + 2x^3 = 0$

(ii) Find the equation of the polynomial with roots, **2**

$\alpha + 1$, $\beta + 1$, and $\gamma + 1$.

Give your answer in expanded, simplified form.

$\therefore P(x-1) = 0$ has roots $\alpha + 1$, $\beta + 1$, and $\gamma + 1$.

$$P(x-1) = (x-1)^3 - (x-1)^2 - (x-1) + 2$$

$$= x^3 - 3x^2 + 3x - 1 - \left(x^2 - 2x + 1\right) - x + 1 + 2$$

$$= x^3 - 4x^2 + 4x + 1$$

$\therefore$ the polynomial equation is $x^3 - 4x^2 + 4x + 1 = 0$

**Alternative Methods on next pages**

### Alternative Method 1 – Substitution

(b) (i) As the new roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$, then let $y = \frac{1}{x}$.

$\therefore x = \frac{1}{y}$

$\therefore x^3 - x^2 - x + 2 = 0 \Leftrightarrow \left(\frac{1}{y}\right)^3 - \left(\frac{1}{y}\right)^2 - \left(\frac{1}{y}\right) + 2 = 0$

$\therefore \frac{1}{y^3} - \frac{1}{y^2} - \frac{1}{y} + 2 = 0$

$\therefore \frac{1}{y^3}\left(1 - y - y^2 + 2y^3\right) = 0$

$\therefore 1 - y - y^2 + 2y^3 = 0$

(ii) As the new roots are $\alpha + 1$, $\beta + 1$, and $\gamma + 1$, then let $y = x + 1$.

$\therefore x = y - 1$

$\therefore x^3 - x^2 - x + 2 = 0 \Leftrightarrow (y-1)^3 - (y-1)^2 - (y-1) + 2 = 0$

$\therefore y^3 - 3y^2 + 3y - 1 - \left(y^2 - 2y + 1\right) - y + 1 + 2 = 0$

$\therefore y^3 - 4y^2 + 4y + 1 = 0$

### Comment:

This question was generally well done.

The common mistake was collecting like terms.

### Alternative Method 2

For $x^3 - x^2 - x + 2 = 0$: $\sum\alpha = 1$. $\sum\alpha\beta = -1$, and $\alpha\beta\gamma = -2$.

(b) (i) For $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$: $\sum\frac{1}{\alpha} = \frac{\sum\alpha\beta}{\alpha\beta\gamma} = \frac{1}{2}$, $\sum\frac{1}{\alpha\beta} = \frac{\sum\alpha}{\alpha\beta\gamma} = -\frac{1}{2}$, and $\frac{1}{\alpha\beta\gamma} = -\frac{1}{2}$

Equation needed: $$x^3 - \left(\sum\frac{1}{\alpha}\right)x^2 + \left(\sum\frac{1}{\alpha\beta}\right)x - \frac{1}{\alpha\beta\gamma} = 0$$

$$\therefore x^3 - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{2} = 0$$

$$\therefore 2x^3 - x^2 - x + 1 = 0$$

(ii) For $\alpha + 1$, $\beta + 1$, and $\gamma + 1$:

$$\sum(\alpha+1) = \left(\sum\alpha\right) + 3 = 4$$

$$\sum(\alpha+1)(\beta+1) = 3 + 2\sum\alpha + \sum\alpha\beta = 3 + 2 - 1 = 4$$

$$(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma + \sum\alpha\beta + \left(\sum\alpha\right) + 1 = -2 - 1 + 1 + 1 = -1$$

Equation needed: $$x^3 - \left(\sum(\alpha-1)\right)x^2 + \left(\sum(\alpha-1)(\beta-1)\right)x - (\alpha-1)(\beta-1)(\gamma-1) = 0$$

$$\therefore x^3 - 4x^2 + 4x + 1 = 0$$

### Comments:

Surprisingly to the marker, many who used this last method did it correctly.
That said, many students need to know the expansion of $(x-a)(x-b)(x-c)$ without having to do it the long way.

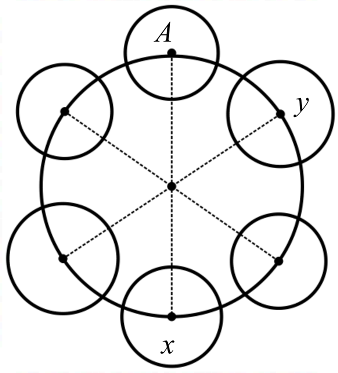
(c) Three couples are supposed to sit down to dinner at a round table such that no person is to sit next to their partner. In how many ways are there for the six people to sit down at the table under these conditions? **4**

Let the partnerships be $(A, x)$, $(B, y)$ and $(C, z)$

**Method 1:** Direct Method

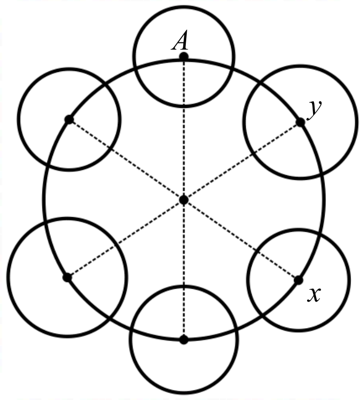
Seat A first:

**Case 1:** $x$ sits directly opposite $A$




There are 4 people who could sit to $A$'s left e.g. $y$
This means that are only 2 spots where $B$ could sit.
This leaves only 2 chairs for the final couple $(C, z)$ and this can be done in 2 ways.
Total for this subcase: $4 \times 2 \times 2 = 16$ ways.

**Case 2:** $x$ does not sit directly opposite $A$.
There are two spots for $x$.



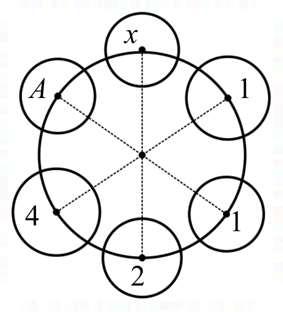

There are 4 people who could sit between $A$ and $x$ e.g $y$
Now two situations arise: either $B$ sits opposite $y$ or doesn't.
If $B$ doesn't sit opposite $y$ then $C$ and $z$ will be forced to sit adjacent to one another.
So $B$ must sit opposite $y$ and then there are two options for $(C, z)$
Total for this subcase: $2 \times 4 \times 2 = 16$ ways.

Total ways = 16 + 16 = 32 ways.

# Question 8 SOLUTIONS (continued)

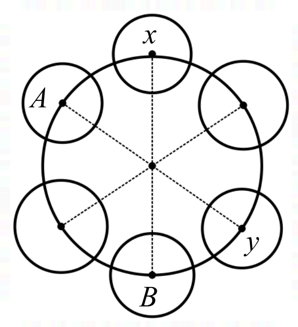
**Method 2:** Indirect Method

How many ways can only 1 couple sit together?



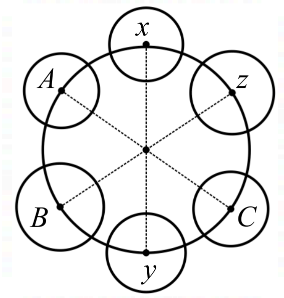

Pick a couple to sit together in ${}^{3}C_{1}$ ways. The couple can swap seats in 2! ways.
In the diagram above, there are 4 people to sit next to $A$.
This means that there are only 2 people who can sit next to this person.
The remaining seats are now fixed.
$\therefore$ total number of ways = ${}^{3}C_{1} \times 2! \times 4 \times 2 = 48$ ways.

How many ways can only 2 couples sit together?




Pick two couples to sit together in ${}^{3}C_{2}$ ways.
The couples can swap seats with their parts in $2! \times 2!$ ways.
The remaining couple ($C$, $z$) are forced to sit in the remaining seats.
$\therefore$ total number of ways = ${}^{3}C_{2} \times 2! \times 2! \times 2 = 24$ ways.

How many ways can the 3 couples sit together?




Sit one couple down. The remaining couples can be sat down in 2! ways.
The couples can swap seats with their parts in $2! \times 2! \times 2!$ ways.
$\therefore$ total number of ways = $2! \times 2! \times 2! \times 2 = 16$ ways.

There are 5! ways for the couples to sit down without any restriction.
$\therefore$ total number of ways that the couples don't sit together = $5! - \left(48 + 24 + 16\right) = 32$

**Method 3:** Inclusion-Exclusion Principle.

Let $X$ be the event that $(A, x)$ are seated together, $Y$ be the event that $(B, y)$ are seated together, and $Z$ the event that $(C, z)$ are kept together.

What is $\mathrm{n}\left(X \cup Y \cup Z\right)$?

**Ans:** The number of ways could at least one couple be seated together.

$$\mathrm{n}\left(X \cup Y \cup Z\right) = \mathrm{n}\left(X\right) + \mathrm{n}\left(Y\right) + \mathrm{n}\left(Z\right)$$
$$- \mathrm{n}\left(X \cap Y\right) - \mathrm{n}\left(Y \cap Z\right) - \mathrm{n}\left(Z \cap X\right)$$
$$+ \mathrm{n}\left(X \cap Y \cap Z\right)$$

$\mathrm{n}\left(X\right)$: Fix the couple $(A, x)$. They can swap seats in 2! ways.

The remaining 4 people can be arranged in 4! ways.

$\therefore\ \mathrm{n}\left(X\right) = 2 \times 24 = 48$ ways

$\therefore\ \mathrm{n}\left(X\right) + \mathrm{n}\left(Y\right) + \mathrm{n}\left(Z\right) = 3 \times 48 = 144$ ways

$\mathrm{n}\left(X \cap Y\right)$: Fix the couple $(A, x)$. They can swap seats in 2! ways.

The other couple has a choice of 3 positions and this couple can swap seats in 2! ways.
The remaining two people can be seated in 2! ways.

$\therefore\ \mathrm{n}\left(X \cap Y\right) = 2 \times 3 \times 2 \times 2 = 24$ ways

$\therefore\ \mathrm{n}\left(X \cap Y\right) + \mathrm{n}\left(Y \cap Z\right) + \mathrm{n}\left(Z \cap X\right) = 3 \times 24 = 72$ ways.

$\mathrm{n}\left(X \cap Y \cap Z\right)$: Fix the couple $(A, x)$. They can swap seats in 2! ways.

The remaining couples can be arranged in 2! ways.
These remaining couples can swap seats in 2! ways each.

$\therefore\ \mathrm{n}\left(X \cap Y \cap Z\right) = 2^4 = 16$

$\therefore\ \mathrm{n}\left(X \cup Y \cup Z\right) = 144 - 72 + 16 = 88$

No restrictions: 5! = 120

How many ways can the couples be arranged with them not seated together = 5! – 88 = 32

**Comment:**

Students need to realise that it is their responsibility to communicate their logic.
Random diagrams and illegible text does not help communicate their logic.

This question was worth 4 marks, so most students who tried to do it in "one line" generally did not score any marks.

Students who got a result more than 5! did not score any marks.

Many students do not know what the complement really means. Students who thought that the complement of 'no couples together' was 'having them all together' could only get a maximum of 1 mark.

| | | | |
|---|---|---|---|
| (d) | (i) | A polynomial $H(x)$ of degree $n$ has $n$ distinct real roots.<br>Explain why the $r$th derivative of $H(x)$ has $(n - r)$ distinct real roots. | **2** |

If a polynomial has $n$ distinct real roots, its graph must cross the $x$-axis exactly $n$ times.

This means it must have $(n - 1)$ relative maximum or minimum turning points between each of these zeros and each of these $(n - 1)$ turning points will form real, distinct zeros on the graph of the first derivative.

Hence, the first derivative of this polynomial has $(n - 1)$ real roots.

Now, by using the same reasoning as above, between the $(n - 1)$ real roots of the first derivative there are $(n - 2)$ turning points for the second derivative and each of these $(n - 2)$ turning points will form real, distinct zeros on the graph of the second derivative.

This pattern will continue resulting in exactly $(n - r)$ distinct real roots on the graph of the $r^{\text{th}}$ derivative.

### Comment:

There are many students who don't know the correct notation for the $k$th derivative.

$P^k(x) = \underbrace{P \circ P \circ P \circ \ldots \circ P}_{k}(x)$ whereas $P^{(k)}(x) = \dfrac{d^k}{dx^k}\big(P(x)\big)$

Students need to learn the name of the theorems that they are quoting AND quote them properly.

Many made allusions to the corollary to the Fundamental Theorem of Algebra i.e. a degree $n$ polynomial has exactly $n$ roots. However, this could mean that the roots are not real, however this question needed that $P^{(k)}(x)$ has $k$ distinct, real roots.
Students who did this only scored a maximum of 1 mark.

| | | | |
|---|---|---|---|
| (d) | (ii) | Consider the polynomial $P(x)=t_nx^n+t_{n-1}x^{n-1}+t_{n-2}x^{n-2}+...+t_1x+t_0$<br><br>Show that if $(n-1)t_{n-1}^2<2nt_nt_{n-2}$ then $P\left(x\right)$ cannot have $n$ distinct real roots. | 3 |

$$P(x)=t_nx^n+t_{n-1}x^{n-1}+t_{n-2}x^{n-2}+...+t_1x+t_0$$

$$P'(x)=nt_nx^{n-1}+(n-1)t_{n-1}x^{n-2}+(n-2)t_{n-2}x^{n-3}+...+t_1$$

$$P^{(2)}(x)=n(n-1)t_nx^{n-2}+(n-1)(n-2)t_{n-1}x^{n-3}+(n-2)(n-3)t_{n-2}x^{n-4}+...$$

$$=\frac{n!}{(n-2)!}t_nx^{n-2}+\frac{(n-1)!}{(n-3)!}t_{n-1}x^{n-3}+\frac{(n-2)!}{(n-4)!}t_{n-2}x^{n-4}+...$$

$$P^{(3)}(x)=n(n-1)(n-2)t_nx^{n-3}+(n-1)(n-2)(n-3)t_{n-1}x^{n-4}+(n-2)(n-3)(n-4)t_{n-2}x^{n-3}+...$$

$$=\frac{n!}{(n-3)!}t_nx^{n-3}+\frac{(n-1)!}{(n-4)!}t_{n-1}x^{n-4}+\frac{(n-2)!}{(n-5)!}t_{n-2}x^{n-5}+...$$

$$\therefore P^{(r)}(x)=\frac{n!}{(n-r)!}t_nx^{n-r}+\frac{(n-1)!}{(n-r-1)!}t_{n-1}x^{n-r-1}+\frac{(n-2)!}{(n-r-2)!}t_{n-2}x^{n-r-2}+...$$

$$\therefore P^{(n-2)}(x)=\frac{n!}{2!}t_nx^2+\frac{(n-1)!}{1!}t_{n-1}x+\frac{(n-2)!}{0!}t_{n-2}$$

$$=\frac{n!}{2!}t_nx^2+(n-1)!t_{n-1}x+(n-2)!t_{n-2}$$

$$P^{(n-2)}(x)=0\Leftrightarrow\frac{n!}{2!}t_nx^2+(n-1)!t_{n-1}x+(n-2)!t_{n-2}=0$$

$$\therefore\frac{n(n-1)}{2!}t_nx^2+(n-1)t_{n-1}x+t_{n-2}=0\qquad\left[\div(n-2)!\right]$$

$$\therefore n(n-1)t_nx^2+2(n-1)t_{n-1}x+2t_{n-2}=0$$

Consider the discriminant: $\Delta=\left[2(n-1)t_{n-1}\right]^2-4\times2t_{n-2}\times n(n-1)t_n$

$$=4(n-1)\left[(n-1)t_{n-1}^2-2nt_{n-2}t_n\right]$$

(d) (ii) (continued)

Given that $(n-1)t_{n-1}^2 - 2nt_n t_{n-2} < 0$ then $\Delta < 0$

$\therefore$ $P^{(n-2)}(x) = 0$ must have two non-real roots.

$\therefore$ $P(x) = 0$ does not have $n$ real roots.

$\therefore$ $P(x) = 0$ does not have $n$ distinct real roots.

**Alternative**

$P'(x) = nt_n x^{n-1} + (n-1)t_{n-1}x^{n-2} + (n-2)t_{n-2}x^{n-3} + \ldots + t_1$

This has $(n-1)$ roots $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{n-1}$.

Consider $\sum \alpha^2$ :

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$

$$= \left(-\frac{(n-1)t_{n-1}}{nt_n}\right)^2 - 2\left(\frac{(n-2)t_{n-2}}{nt_n}\right)$$

$$= \left(\frac{1}{nt_n}\right)^2 \times \left((n-1)t_{n-1}^2 - 2n(n-2)t_n t_{n-2}\right)$$

If $(n-1)t_{n-1}^2 - 2nt_n t_{n-2} < 0$ then $(n-1)t_{n-1}^2 < 2nt_n t_{n-2}$

* Now $2nt_n t_{n-2} < 2n(n-2)t_n t_{n-2}$ and so $(n-1)t_{n-1}^2 < 2nt_n t_{n-2} < 2n(n-2)t_n t_{n-2}$

$\therefore \sum \alpha^2 = \left(\frac{1}{nt_n}\right)^2 \times \left((n-1)t_{n-1}^2 - 2n(n-2)t_n t_{n-2}\right) < 0$

This means that $P'(x)$ does not have $(n-1)$ real roots.

For $P(x)$ to have $n$ distinct real roots then $P'(x)$ must have $(n-1)$ distinct, real roots.

So if $(n-1)t_{n-1}^2 - 2nt_n t_{n-2} < 0$ then $P(x)$ cannot have $n$ distinct roots as $\sum \alpha^2 < 0$.

* The statement is true for $n > 2$. The question can be shown to be true for $n = 1$ or 2. Trivial for $n = 1$ and for $n = 2$, $(n-1)t_{n-1}^2 - 2nt_n t_{n-2} < 0$ becomes the discriminant.

**Comment:**

Many students had the 'direction' of their logic wrong – they weren't penalised this time.
If *X* is the statement that '$(n-1)t_{n-1}^2 - 2nt_n t_{n-2} < 0$' and *Y* is the statement that '$P(x)$ cannot have $n$ distinct roots', then the question means that students are to assume *X* and then prove *Y*.
Most students who tried this question assumed *Y* and then proved *X*.

**End of Solutions**

**1** If $(x-!)$ is a factor of $x^3 - kx^2 + 12x - 8$, what must be the value of $k$ ?

A. –3

B. 3

C. 6

D. –6

$2^3 - k \times 2^2 + 12 \times 2 - 8 = 0 \Rightarrow 4k = 24$

**2** Colm has 4 identical blue exercise books and 3 identical red exercise books.
In how many different ways could he place the exercise books on a shelf?

A. 144

B. 5040

C. 72

D. 35

This is about arrangements with repetition: $\dfrac{7!}{3! \times 4!} = 35$

**3** A curve is defined by the parametric equations *x* = !"#$*t* and $y = \cos 3t$ .
Which of the following, in terms of $t$, equates to $\dfrac{dy}{dx}$ ?

A. $\cot 3t$

B. $\tan 3t$

C. $-\tan 3t$

D. $-3\sin 3t$ .

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3\sin 3t}{3\cos 3t}$$

**4** The remainder when a polynomial $P(x)$ is divided by *x*^! −" is !*x*−" .
What is the remainder when $P(x)$ is divided by *x*+! ?

A. −!

B. "*x*^! − *x* −#

C. !*x*−"

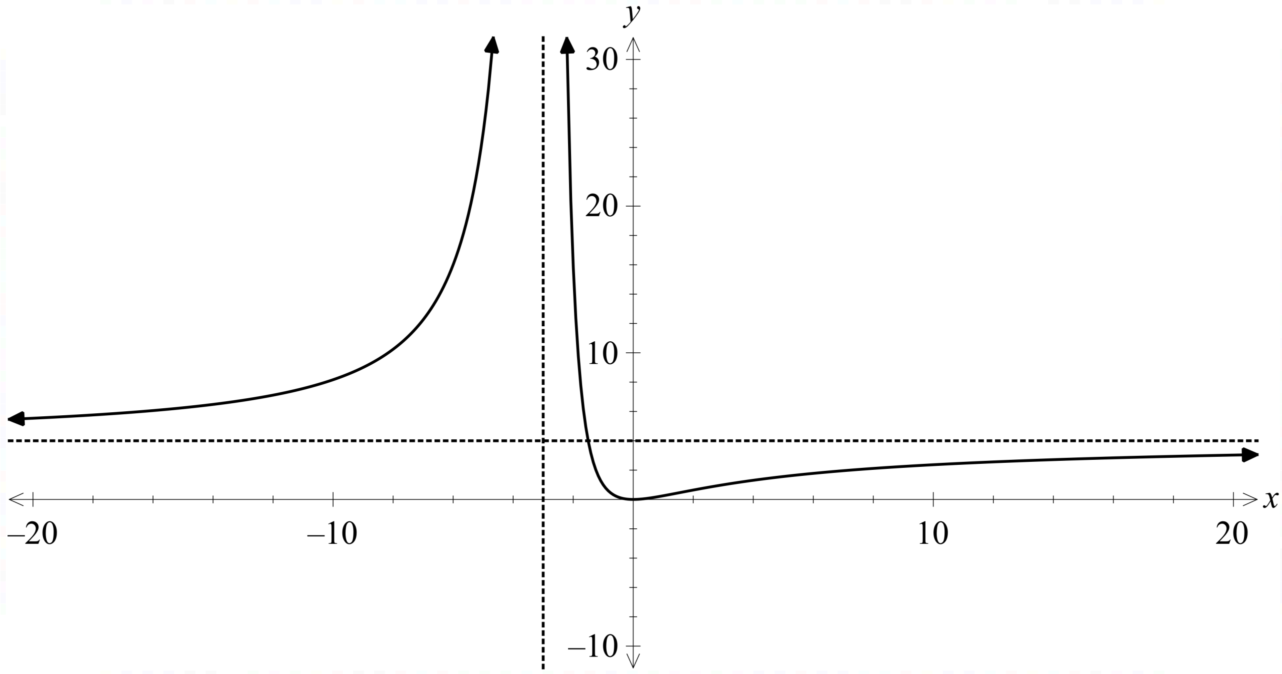
D. −!

**Note:** A and D are the only possible options

$P(x) = (x^2 - 1)\,Q(x) + 3x - 4$

$\therefore\ P(-1) = -7$

**5** The graph shown below has the equation of the form $y = \dfrac{r - sx^2}{(x+t)^2}$.




A. r = –!" s = #" t = –$.

B. $r = 0,\ s = -4,\ t = 3$.

C. $r = 0,\ s = 4,\ t = 3$.

D. r = !" s = –#" t = –$.

From the diagram, the asymptote is at $x = -3 \Rightarrow t = 3$

B or C
There is a stationary point at $x = 0 \Rightarrow r = 0$ (alas B or C again)

There is a horizontal asymptote at $y = 4 \Rightarrow s = -4$

**End of Section I Solutions**