SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class:

2021

YEAR 12
TERM 1
HSC ASSESSMENT
TASK #2

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- **Working time – 90 minutes**
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–13, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 64

Section I – 10 marks (pages 2–5)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 54 marks (pages 6–11)
- Attempt Questions 11–13
- Allow about 1 hour and 15 minutes for this section

Examiner:

AMG

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

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1 Given $f(x) = -x^2 + 2x + 3$, which of the following is true for the graph of $y = \dfrac{1}{f(x)}$?

A. Asymptotes at $x = 1$ and $x = -3$.

B. Intercepts at $x = 1$ and $x = -3$.

C. Asymptotes at $x = -1$ and $x = 3$.

D. Intercepts at $x = -1$ and $x = 3$.

2 What is the angle between the vectors $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 18 \end{pmatrix}$?

A. $\arccos\left(\dfrac{39}{65}\right)$

B. $\arccos\left(-\dfrac{39}{65}\right)$

C. $\arccos\left(\dfrac{33}{65}\right)$

D. $\arccos\left(-\dfrac{33}{65}\right)$

3 The polynomial $x^3 + 3x^2 + 2x - 1 = 0$ has roots α, β, and γ.

Which of the following polynomials has roots $\dfrac{2}{\alpha}$, $\dfrac{2}{\beta}$, and $\dfrac{2}{\gamma}$?

A. $x^3 + 4x^2 - 12x + 8 = 0$

B. $x^3 - 4x^2 - 12x - 8 = 0$

C. $8x^3 - 12x^2 - 4x + 1 = 0$

D. $x^3 + 12x^2 + 4x - 1 = 0$

4 A force of magnitude 4 newtons acts in the north-easterly direction and another force of 3 newtons acts in the easterly direction.
What is the magnitude, in newtons, of the resultant of these two forces?

A. $25+12\sqrt{2}$

B. $\sqrt{25+12\sqrt{2}}$

C. $\sqrt{25-12\sqrt{2}}$

D. $25-12\sqrt{2}$

5 The equation $8x^3-54x+k=0$ has a double root. What are the possible values of k?

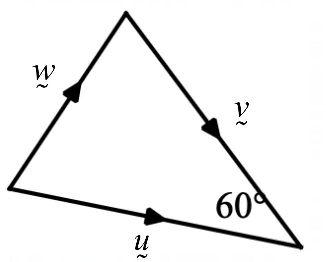
A. ± 4

B. ± 9

C. ± 40.5

D. ± 54

6 Vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$, and $\underset{\sim}{w}$ are shown below.



Which of the following statements is true?

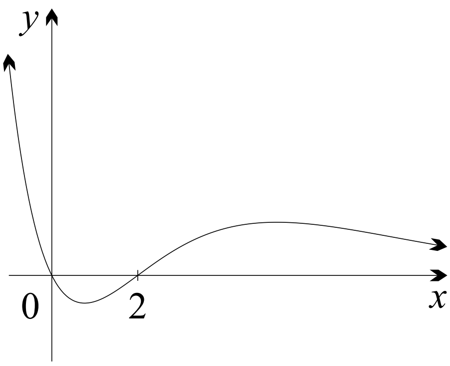
A. $|\underset{\sim}{w}|^2=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2$

B. $|\underset{\sim}{w}|^2=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2-|\underset{\sim}{u}\cdot\underset{\sim}{v}|$

C. $|\underset{\sim}{w}|^2=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2+|\underset{\sim}{u}\underset{\sim}{v}|$

D. $|\underset{\sim}{w}|^2=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2-|\underset{\sim}{u}||\underset{\sim}{v}|$

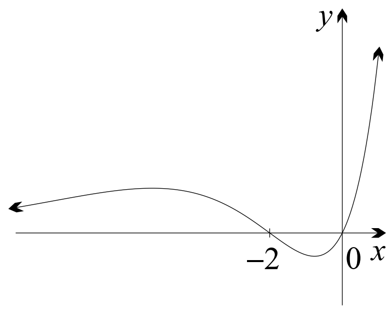
7



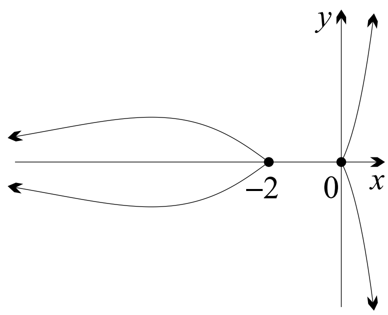
Above is the graph of $y = f(x)$.

Which of the following best shows the graph of $|y| = f(-x)$?

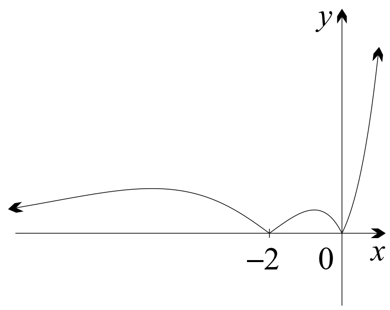
A.



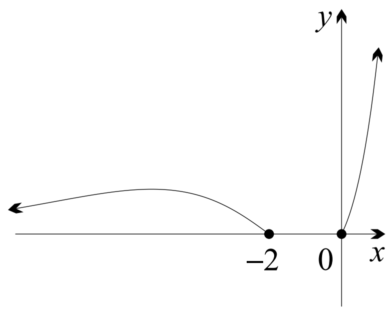
B.



C.



D.



8 The polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ is an odd polynomial with at least one multiple root. Gainesforde was asked to give some facts about the curve and replied:

I $y = P(x)$ must pass through the origin.

II If $P'(k) = 0$ then $P(-k) = 0$.

Which of Gainesforde's statements must always be correct?

A. I only

B. II only

C. Both I and II

D. Neither I nor II

9 Consider the relation $x^3 + y^3 = 3xy$.

Which of the following is the correct expression for $\dfrac{dy}{dx}$?

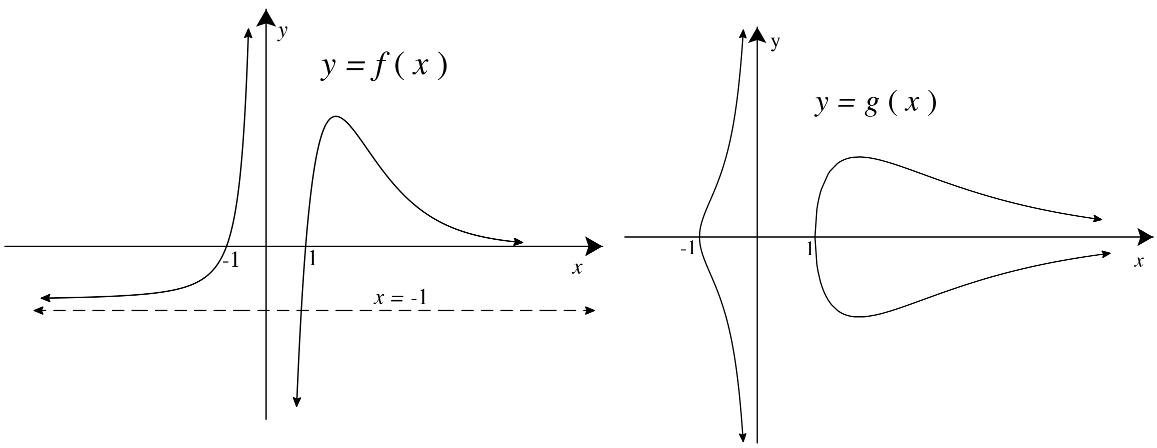
A. $\dfrac{y - x^2}{y^2 - x}$

B. $\dfrac{y^2 - x}{y - x^2}$

C. $\dfrac{x^2 + y^2}{x}$

D. $\dfrac{x}{x - y^2}$

10 The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



Which of the following best describes the relationship between $f(x)$ and $g(x)$?

A. $\left| f(x) \right| = g(x)$

B. $g(x) = \ln\left[f(x) \right]$

C. $f(x) = \pm \ln\left| g(x) \right|$

D. $g(x) = \pm\sqrt{f(x)}$

Section II

54 marks
Attempt Questions 11–13
Allow about 1 hour and 15 minutes for this section

In Questions 11-13, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (19 marks) Start a NEW Answer Booklet

(a) The cubic equation $2x^3 - 5x^2 + 3x - 2 = 0$ has roots α, β, and γ.
Find the value of

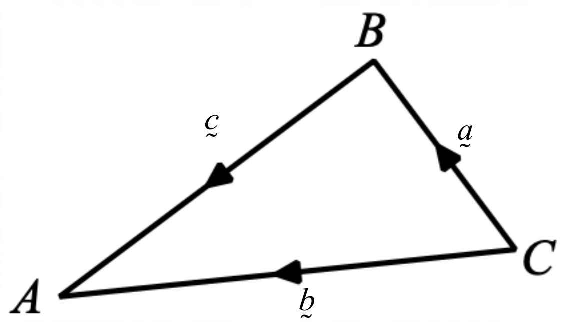
(i) $\alpha + \beta + \gamma$ **1**

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **2**

(iii) $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **2**

(b) When a polynomial $P(x)$ is divided by $(x+1)(x-3)$, the remainder is $2x+7$. **2**
What is the remainder when $P(x)$ is divided by $x-3$?

(c) In the diagram below, $\overrightarrow{CB} = \underset{\sim}{a}$, $\overrightarrow{CA} = \underset{\sim}{b}$, and $\overrightarrow{BA} = \underset{\sim}{c}$.



(i) Show that $|\underset{\sim}{b}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a} + 2(\underset{\sim}{a} \cdot \underset{\sim}{c}) + \underset{\sim}{c} \cdot \underset{\sim}{c}$ **2**

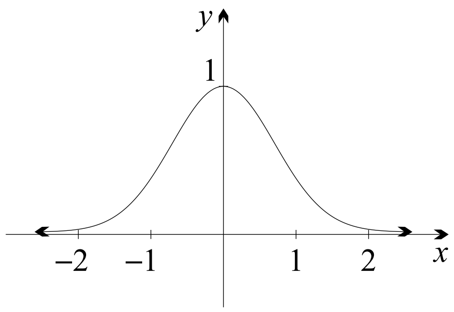
(ii) Hence, if ΔABC has a right angle at B, show that $|\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{c}|^2$ **1**

Question 11 continues on page 7

Question 11 (continued)

For the following graphs draw your answers on the insert sheet provided.

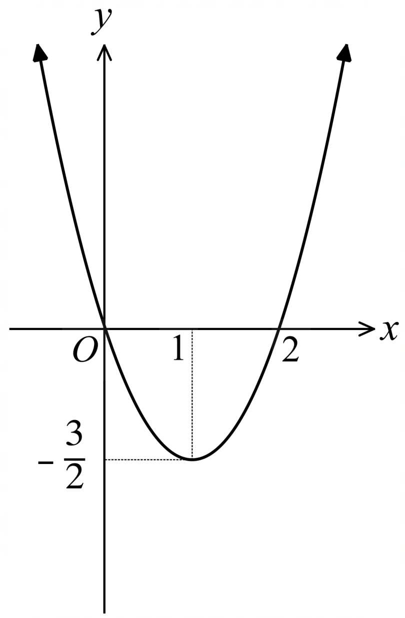
(f) The graph of $y = e^{-x^2}$ is drawn below. **2**



On the sheet provided, sketch the graph of $y = \left(1 - x^2\right)e^{-x^2}$ without the aid of calculus.

Show all asymptotes and intercepts.

(g) The graph of $y = f(x)$ is drawn below.



On the sheet provided sketch the following:

(i) $y^2 = f\left(x\right)$. **2**

(ii) $y = f(x)^2$. **2**

(iii) $y = \dfrac{1}{f(2x)}$. **3**

End of Question 11

Question 12 (17 marks) Start a NEW Answer Booklet

(a) (i) Prove by induction that **3**

$$2+3\times2+4\times2^2+...+(n+1)2^{n-1}=n2^n$$

for all integers $n\geq1$.

(ii) Hence show that **2**

$$(n+2)2^n+(n+3)2^{n+1}+...+(2n+1)2^{2n-1}=n2^n\left(2^{n+1}-1\right).$$

(b) Show that $(x-1)^3$ is a factor of $P(x)=x^{2n}-nx^{n+1}+nx^{n-1}-1$, for integers $n\geq2$. **2**

(c) Solve $\log_2(x+3)+\log_2(x^2+5x-4)=3$ **3**

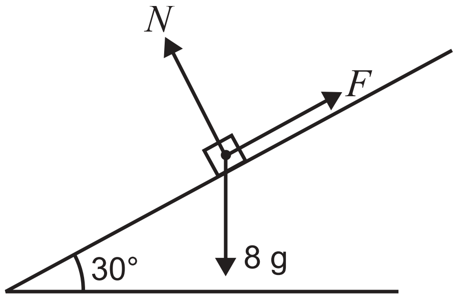
Question 12 continues on page 9

(d) Given vectors $\overrightarrow{OA} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, $\overrightarrow{OC} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, and $\overrightarrow{OA} = 2\overrightarrow{CB}$.

(i) Find the vector projection of $\overrightarrow{OC}$ in the direction of $\overrightarrow{OA}$. **2**

(ii) Hence find the area of trapezium *OABC*. **3**

(e) A block of mass 8 kg is at rest on a plane inclined at an angle of 30° to the horizontal. **2**



In the diagram, N newtons is the normal reaction of the plane on the block, and F newtons is the frictional force on the block up the plane.

If the block is in equilibrium, find the exact value of $\dfrac{F}{N}$.

End of Question 12

Question 13 (18 marks) Start a NEW Answer Booklet

(a) Prove, using induction, that $5^n - (-1)^n$ is divisible by 6, where n is a positive integer **3**

(b) Suppose m is a constant greater than 1.

Let $g(x) = \dfrac{1}{1 + \cot^m x}$ where $0 < x \le \dfrac{\pi}{2}$. You may assume that $\lim_{x \to 0} g(x) = 0$.

(i) Show that $g(x) + g\left(\dfrac{\pi}{2} - x\right) = 1$ for $0 < x \le \dfrac{\pi}{2}$. **2**

(ii) Sketch $y = g(x)$ for $0 < x \le \dfrac{\pi}{2}$. **2**

[There is no need to find $g'(x)$ but assume that $y = g(x)$ has a horizontal tangent at $x = \dfrac{\pi}{2}$. Your graph must exhibit the property of part (i).]

(iii) If $f(x)$ is defined as **1**

$$f(x) = \begin{cases} 0, & x = 0 \\ g(x), & 0 < x \le \dfrac{\pi}{2} \end{cases}$$

evaluate $\displaystyle\int_0^{\frac{\pi}{2}} f(x)\,dx$.

(c) A curve is defined by the parametric equations $x = t + \dfrac{1}{t}$ and $y = t^2 + \dfrac{1}{t^2}$ for $t \ne 0$.

(i) Show that the Cartesian equation of the curve is $y = x^2 - 2$. **2**

(ii) By considering the values of k for which $x = k$ has real solutions, where k is a constant, find any restrictions on x. **2**

(iii) Sketch the curve, showing any domain restrictions implied by part (ii). **1**

Question 13 continues on page 11

Question 13 (continued)

(d) (i) Find the three solutions of $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$. **3**

You may assume that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(ii) Hence, show that $\cot\dfrac{\pi}{18}\cot\dfrac{5\pi}{18}\cot\dfrac{7\pi}{18} = \sqrt{3}$. **2**

End of paper

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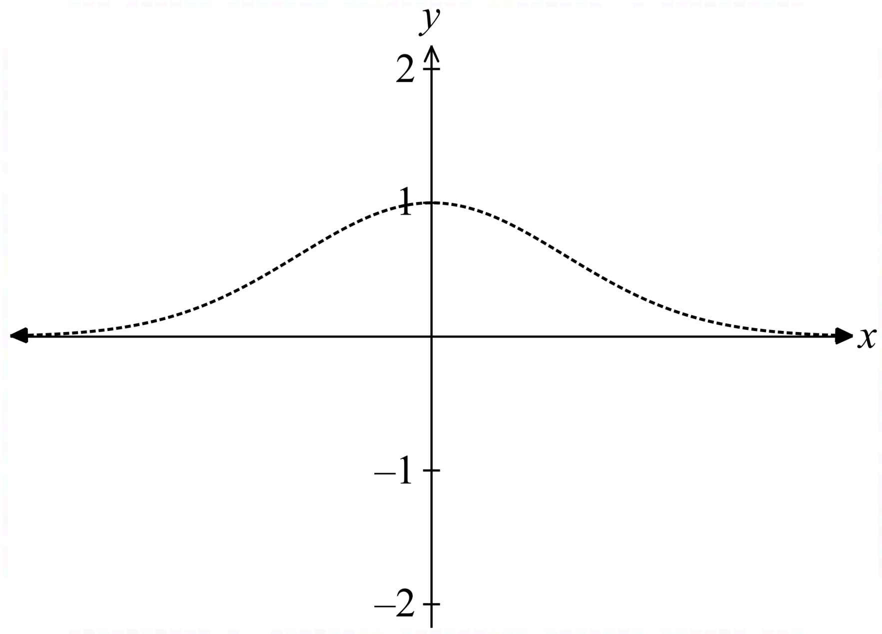
Question 11 Place this sheet inside the booklet for Q11

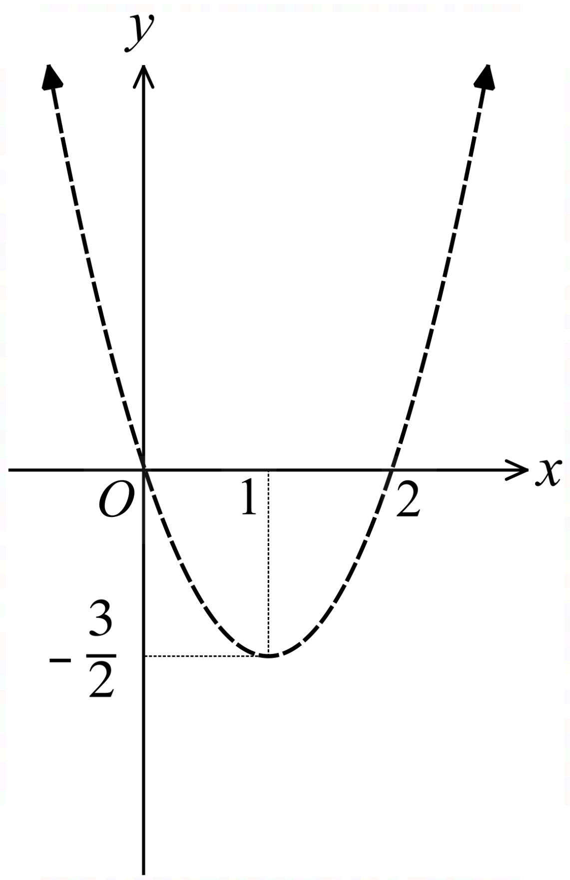
(f) The graph of $y = e^{-x^2}$ is drawn below. **2**



Sketch the graph of $y = \left(1 - x^2\right)e^{-x^2}$ without the aid of calculus. Show all asymptotes and intercepts.

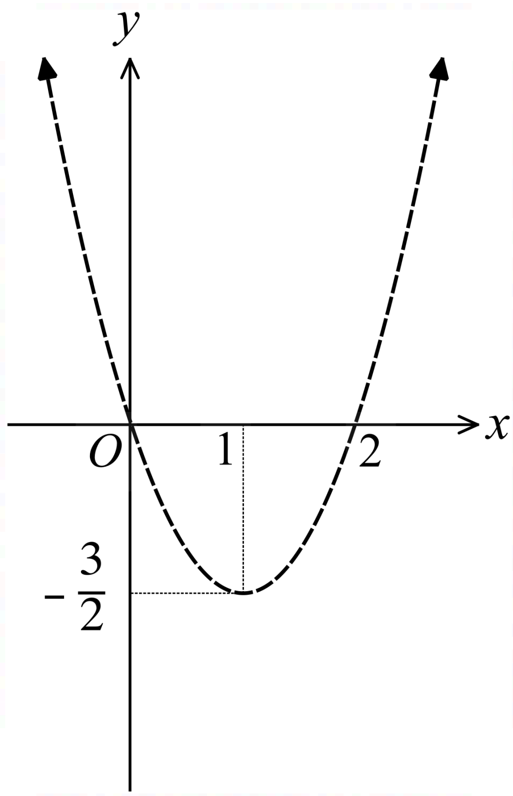
(g) Sketch the following:

(i) $y^2 = f\left(x\right)$ **2**

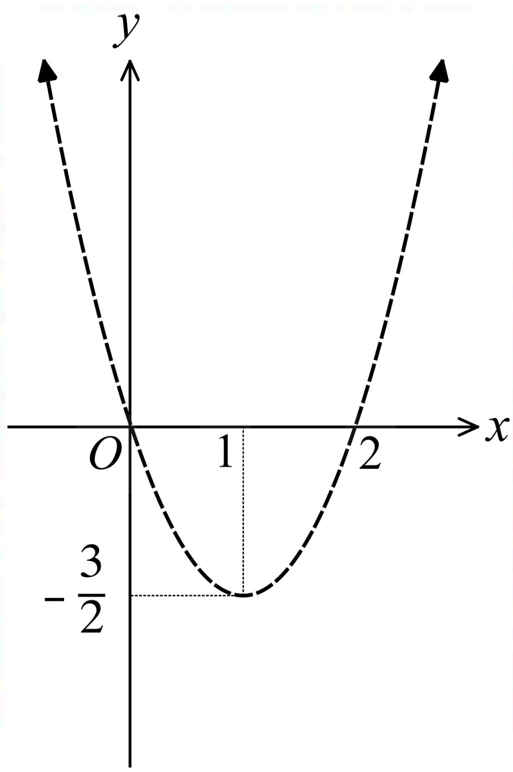


Question 11 (continued)

(g) (ii) $y = f(x)^2$ **2**



(iii) $y = \dfrac{1}{f(2x)}$ **3**



Place this sheet inside the booklet for Q11

SYDNEY
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2021	YEAR 12 TERM 1 ASSESSMENT TASK 2

Mathematics Extension 1

Sample Solutions

NOTE: **Some of you may be disappointed with your mark.**
This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of ink you have used.

Just because you have shown 'working' does not justify that your solution is worth any marks.

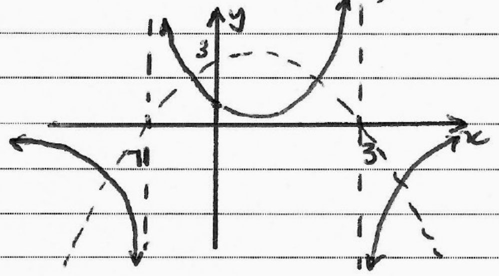
Quick MC Answers

1. C
2. C
3. B
4. B
5. D
6. D
7. B
8. A
9. A
10. D

2021 Y12 Task 2 ME1 Multiple Choice Solutions

Mean (out of 10): 7.41

1.
$$f(x) = -x^2+2x+3$$
$$= -(x^2-2x-3)$$
$$= -(x-3)(x+1)$$



Asymptotes at $x=-1$ and $x=3$

C

A	5
B	0
C	156
D	12

2.
$$\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$
$$-3\times1+2\times18 = \sqrt{13}\cdot\sqrt{325}\ \cos\theta$$
$$33 = \sqrt{13}\cdot\sqrt{13\cdot25}\ \cos\theta$$
$$\therefore 33 = 13.5\ \cos\theta$$
$$\therefore \cos\theta = \frac{33}{65}$$
$$\therefore \theta = \arccos\frac{33}{65}$$

C

A	5
B	2
C	156
D	11

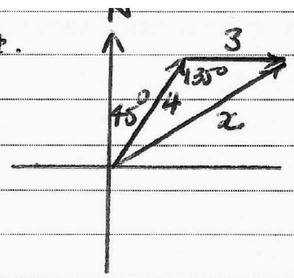
3. Let $u = \frac{2}{x}$
$$\therefore x = \frac{2}{u}$$
$$\therefore \left(\frac{2}{u}\right)^3 + 3\left(\frac{2}{u}\right)^2 + 2\left(\frac{2}{u}\right) - 1 = 0$$
$$\therefore \frac{8}{u^3} + \frac{12}{u^2} + \frac{4}{u} - 1 = 0$$
$$\therefore 8 + 12u + 4u^2 - u^3 = 0$$
$$\therefore u^3 - 4u^2 - 12u - 8 = 0$$

$\therefore$ Required polynomial is
$$x^3 - 4x^2 - 12x - 8 = 0$$

B

A	21
B	138
C	7
D	7

4.



$$x^2 = 3^2+4^2-2\times3\times4\times\cos135^\circ$$
$$= 9+16-24\times\frac{-1}{\sqrt{2}}$$
$$= 25+12\sqrt{2}$$
$$\therefore x = \sqrt{25+12\sqrt{2}}\ \ N.$$

B

A	19
B	120
C	29
D	5

5. $P(x) = 8x^3 - 54x + k$

$P'(x) = 24x^2 - 54$

Let α be the double root

$\therefore 24\alpha^2 - 54 = 0$

$\therefore \alpha^2 = \frac{54}{24} = \frac{9}{4}$

$\therefore \alpha = \pm\frac{3}{2}$.

$P(\alpha) = 0$

$\therefore 8\left(\pm\frac{3}{2}\right)^3 - 54\left(\pm\frac{3}{2}\right) + k = 0$

$8 \times \left(\pm\frac{27}{8}\right) - (\pm 81) + k = 0$

$\therefore k = \pm 81 - (\pm 27)$

$= 81 - 27$ OR $-81 + 27$

$= \pm 54$

D

A	7
B	14
C	11
D	141

6. Using the cosine rule,

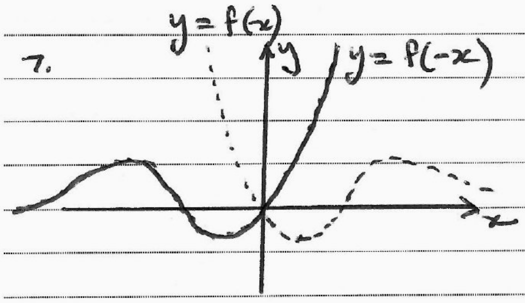
$|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2 \times |\underset{\sim}{u}||\underset{\sim}{v}| \times \cos 60^\circ$

$\therefore |\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - |\underset{\sim}{u}||\underset{\sim}{v}|$

D

A	2
B	56
C	12
D	101

7.



$|y| = f(-x) \Rightarrow f(-x) \geq 0$

$y = \pm f(-x)$ (where $f(-x) \geq 0$)

B

A	4
B	125
C	36
D	9

8. As $P(x)$ is an odd polynomial, it is continuous and will pass through the origin (as constant term is 0)

$\therefore$ I is true.

Odd functions have point symmetry through the origin.
$P(-x) = -P(x)$.

Consider the polynomial

$$P(x) = (x-1)^2(x+1)^2 x(x-2)(x+2)$$

$$\Big(= \big((x-1)(x+1)\big)^2 x(x^2-4)$$

$$= x(x^2-1)^2(x^2-4)$$

$$= x(x^4-2x^2+1)(x^2-4)$$

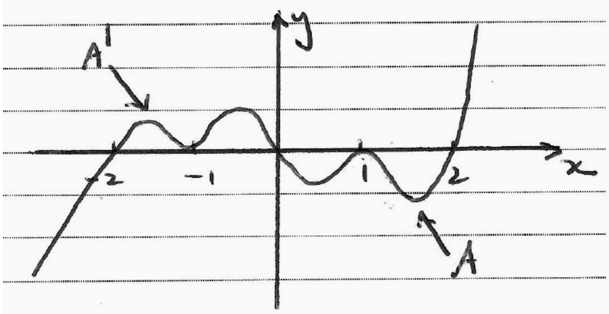
$$= x(x^6-2x^4+x^2-4x^4+8x^2-4)$$

$$= x(x^6-6x^4+9x^2-4)$$

$$= x^7-6x^5+9x^3-4x$$

an odd polynomial$\Big)$

This polynomial has a multiple root.



At A $P'(k) = 0$

At A' $P(-k) \neq 0$

II is not always correct

$\therefore$ I only A

A	61
B	23
C	26
D	64

9. $x^3 + y^3 = 3xy$

$\therefore 3x^2 + 3y^2 y' = 3(y.1 + x.y')$

$\therefore x^2 + y^2 y' = y + xy'$

$\therefore y^2 y' - xy' = y - x^2$

$\therefore y'(y^2 - x) = y - x^2$

$\therefore y' = \dfrac{y - x^2}{y^2 - x}$ A

A	150
B	13
C	8
D	3

10. $g(x)$ exists when $f(x) \geq 0$

There is no scale on the y axis but $y = g(x)$ suggests that when $f(x) > 1$, $g(x) < f(x)$ and when $f(x) < 1$ $g(x) > f(x)$.
This suggests that $g(x)$ involves $\sqrt{f(x)}$.
The symmetry in the x-axis suggests $g(x) = \pm\sqrt{f(x)}$

D

A	11
B	6
C	15
D	142

2021 SBHS YR12 Ext 1 Task 2 Q11 Solutions and Comments

General feedback:

- Students who didn't use the insert sheet risked losing marks due to being unable to compare the transformations of $f(x)$ with the graph of $f(x)$ itself.
- Overall, the cohort demonstrated a poor understanding of Vectors and Advanced Graphing Techniques, so these are topics to consider revisiting.

<u>2021 SBHS YR12 Ext 1 Task 2 Q11 Solutions and Comments</u>

Solutions in pencil or with white out/liquid paper **<u>CAN'T</u>** appeal for marks.

A. The cubic equation $2x^3 - 5x^2 + 3x - 2 = 0$ has roots α, β and γ.
Find the value of:

I. $\alpha + \beta + \gamma$ 1

Solution	Comment(s)
$\alpha + \beta + \gamma = -\frac{b}{a}$ $= \frac{5}{2}$	No half marks. Students generally performed well on this question.

II. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

Solution	Comment(s)
$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$ $= \frac{\left(\frac{c}{a}\right)}{\left(-\frac{d}{a}\right)}$ $= \frac{c}{-d}$ $= \frac{3}{2}$	Students generally performed well on this question.

III. $(\alpha - 1)(\beta - 1)(\gamma - 1)$ 2

Solution	Comment(s)
$(\alpha - 1)(\beta - 1)(\gamma - 1)$ $= (\alpha - 1)(\beta\gamma - \beta - \gamma + 1)$ $= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$ $= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma) - 1$ $= -\frac{d}{a} - \frac{c}{a} - \frac{b}{a} - 1$ $= \frac{2}{2} - \frac{3}{2} + \frac{5}{2} - 1$ $= 1$	Students generally performed well on this question.

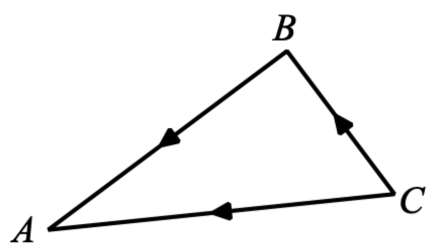
2021 SBHS YR12 Ext 1 Task 2 Q11 Solutions and Comments

Solutions in pencil or with white out/liquid paper **CAN'T** appeal for marks.

B. When a polynomial $P(x)$ is divided by $(x+1)(x-3)$, the remainder is $2x+7$. What is the remainder when $P(x)$ is divided by $x-3$? 2

Solution	Comment(s)
By the polynomial division algorithm: $P(x) = Q(x)D(x) + R(x)$ $\quad = Q(x)(x+1)(x-3) + (2x+7)$ By the remainder theorem: $P(3) = Q(3)(3+1)(3-3) + (2\times 3+7)$ $\quad = 0 + 13$ $\quad = 13$	Common error(s): • Substituting $x = -1$ instead of $x = 3$.

C. In the diagram below, $\overrightarrow{CB} = \boldsymbol{a}$, $\overrightarrow{CA} = \boldsymbol{b}$ and $\overrightarrow{BA} = \boldsymbol{c}$.



I. Show that $|\boldsymbol{b}|^2 = \boldsymbol{a}\cdot\boldsymbol{a} + 2(\boldsymbol{a}\cdot\boldsymbol{c}) + \boldsymbol{c}\cdot\boldsymbol{c}$. 2

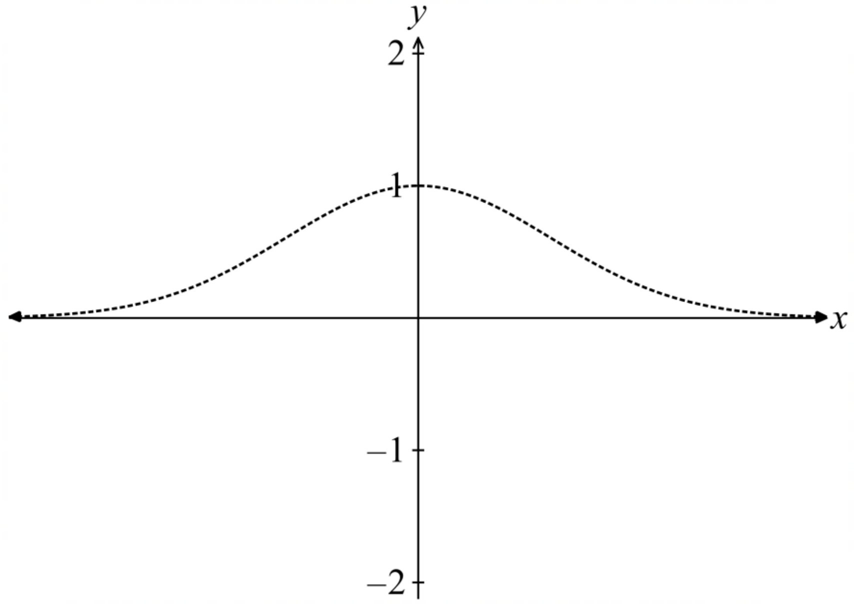
Solution	Comment(s)
From vector addition: $\boldsymbol{a}+\boldsymbol{c} = \boldsymbol{b}$ Hence: $|\boldsymbol{b}|^2 = \boldsymbol{b}\cdot\boldsymbol{b}$ $\quad = (\boldsymbol{a}+\boldsymbol{c})\cdot(\boldsymbol{a}+\boldsymbol{c})$ $\quad = \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{a} + \boldsymbol{c}\cdot\boldsymbol{c}$ $\quad = \boldsymbol{a}\cdot\boldsymbol{a} + 2(\boldsymbol{a}\cdot\boldsymbol{c}) + \boldsymbol{c}\cdot\boldsymbol{c}$	Some students opted for the less efficient method of using the cosine rule. Many students risked being penalised from showing an insufficient amount of working out.

II. If ΔABC has a right angle at B, show that $|\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{c}|^2$. 1

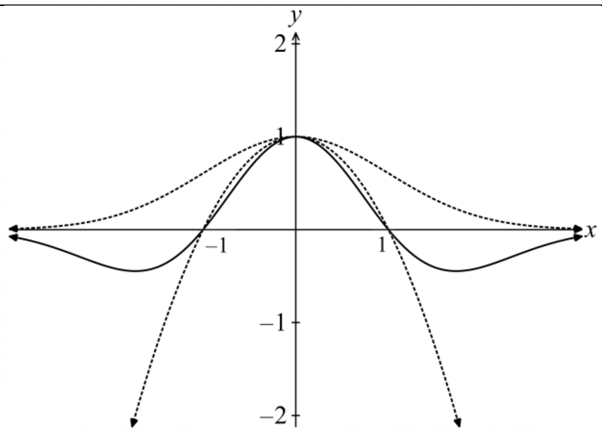
Solution	Comment(s)
If $\angle ABC = 90°$, then $\boldsymbol{a}\cdot\boldsymbol{c} = \boldsymbol{0}$. Hence: $|\boldsymbol{b}|^2 = \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{c}\cdot\boldsymbol{c}$ $\quad = |\boldsymbol{a}|^2 + |\boldsymbol{c}|^2$	No half marks. Students who answered using Pythag theorem could not score any marks.

Solutions in pencil or with white out/liquid paper **CAN'T** appeal for marks.

D. The graph of $y = e^{-x^2}$ is drawn below. 2



On the sheet provided, sketch the graph of $y = (1 - x^2)e^{-x^2}$ without the aid of calculus.
Show all asymptotes and intercepts.

<table>
<tr><td>Solution</td><td>Comment(s)</td></tr>
<tr><td></td><td>Graph of $y = 1 - x^2$ shown for reference, no marks deducted if not included in student answer.

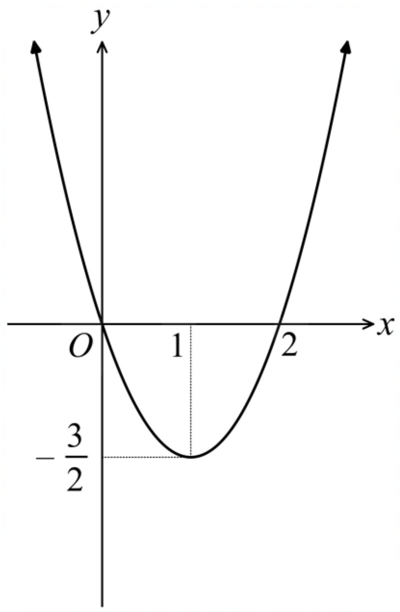
To avoid being penalised, students must clearly show:
• Reflective symmetry about $x = 0$.
• Maximum turning point $(0, 1)$.
• x intercepts $(1, 0)$ and $(-1, 0)$.
• $y \to 0^-$ as $x \to \pm\infty$.
• $(1 - x^2)e^{-x^2} \leq e^{-x^2}$ for $-1 \leq x \leq 1$.

Common error(s):
• Not checking the limiting behaviour of y as $x \to \pm\infty$.</td></tr>
</table>

2021 SBHS YR12 Ext 1 Task 2 Q11 Solutions and Comments

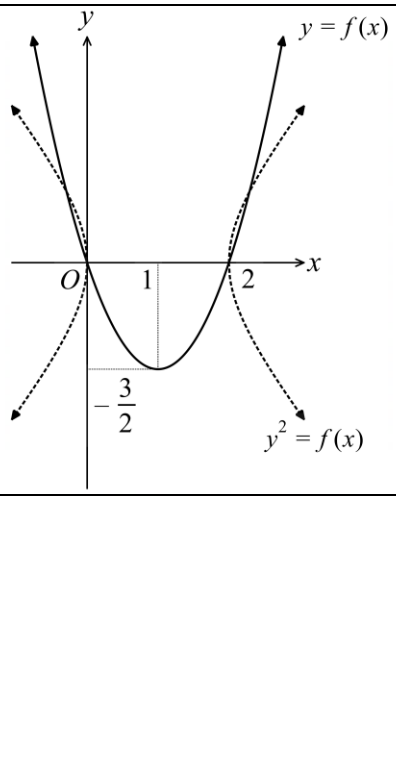
Solutions in pencil or with white out/liquid paper **CAN'T** appeal for marks.

E. The graph of $y = f(x)$ is drawn below.



On the sheet provided, sketch the following:

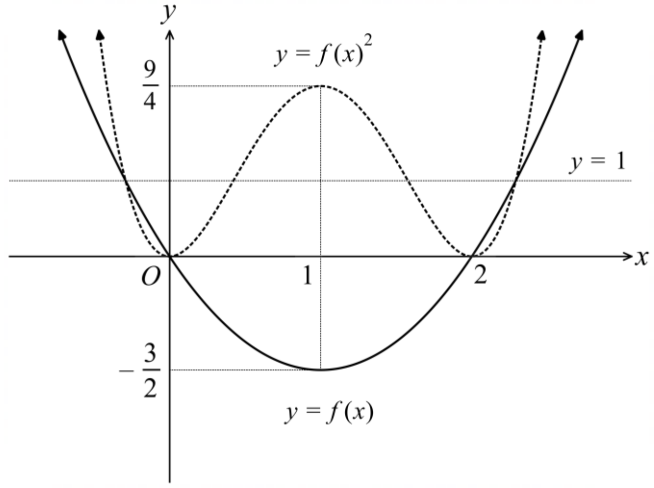
I. $y^2 = f(x)$ 2

Solution	Comment(s)
	To avoid being penalised, students must clearly show: • Reflective symmetry about $x = 1$. • Reflective symmetry about $y = 0$. • General shape of curve. Disappointingly, a range of nonsensical sketches were submitted from a fair number of students.

2021 SBHS YR12 Ext 1 Task 2 Q11 Solutions and Comments

Solutions in pencil or with white out/liquid paper **CAN'T** appeal for marks.

II. $y = f(x)^2$ 2

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>


</td><td>
Line $y = 1$ shown for reference, no marks deducted if not included in student answer.

To avoid being penalised, students must clearly show:

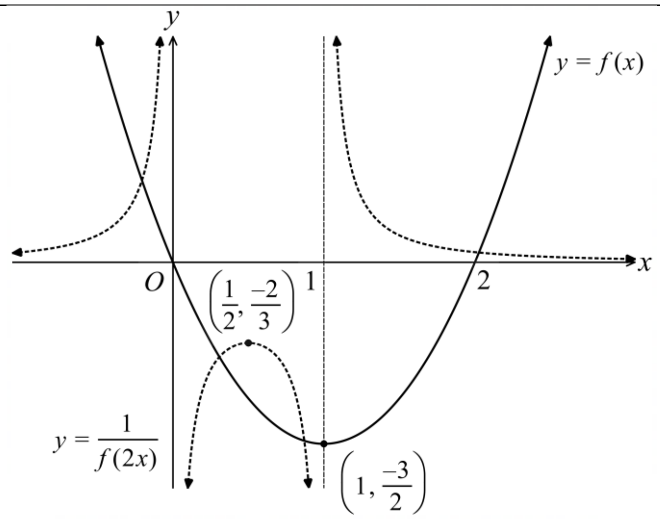
Maximum turning point $\left(1, \frac{9}{4}\right)$.
Double roots at $x = 0$ and $x = 2$.
Reflective symmetry about $x = 1$.
$f(x)^2 < f(x)$ for $0 < f(x) < 1$.
$f(x)^2 > f(x)$ for $f(x) > 1$.
General shape of curve.

Common error(s):

Not considering the change in behaviour of $f(x)^2$ for $0 < f(x) < 1$ and $f(x) > 1$.

Many students risked being penalised for not clearly showing the double roots at $x = 0$ and $x = 2$.
</td></tr>
</table>

III. $y = \dfrac{1}{f(2x)}$ 3

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr><td>


</td><td>
To avoid being penalised, students must clearly show:

Maximum turning point $\left(\frac{1}{2}, \frac{-2}{3}\right)$.
Vertical asymptotes $x = 0$ and $x = 1$.
Horizontal asymptote $y = 0$.
Reflective symmetry about $x = \frac{1}{2}$.
General shape of curve.

Common error(s):

Not dilating $y = \frac{1}{f(x)}$ to $y = \frac{1}{f(2x)}$.
Not stating or incorrectly stating the coordinates of the maximum turning point.

Unfortunately, some students were penalised for not specifying the x coordinate of the maximum turning point.
</td></tr>
</table>

1(a) Prove by induction

$2 + 3\times2 + 4\times2^2 + \dots + (n+1)2^{n-1} = n2^n, \ n \geq 1$

Let $n=1$ $\quad$ LHS $= 2$
RHS $= 1\times2^1 = 2$ $\quad \therefore$ true for $n=1$

Assume true for $n=k$

$2 + 3\times2 + 4\times2^2 + \dots + (k+1)2^{k-1} = k2^k$

Let $n=k+1$.

Must show

$2 + 3\times2 + 4\times2^2 + \dots + (k+1)2^{k-1} + (k+2)2^k = (k+1)2^{k+1}$

$$\begin{aligned} \text{LHS} &= 2 + 3\times2 + 4\times2^2 + \dots + (k+1)2^{k-1} + (k+2)2^k \\ &= k2^k + (k+2)2^k \quad \text{from assumption} \\ &= (2k+2)2^k \\ &= 2(k+1)2^k \\ &= (k+1)2^{k+1} = \text{RHS} \end{aligned}$$

$\therefore$ true for $n=k+1$

$\therefore$ true for all $n\geq1$, by mathematical induction.

This was mainly done well. However some students reduced the step 'Let k=1' to $(k+1)2^{k-1}=k2^k$ without the summation sign. They did the same thing for the n=k+1 step. This was penalised.

3

(a)

(ii) Hence show $(n+2)2^n + (n+3)2^{n+1} + \dots + (2n+1)2^{2n-1} = n2^n(2^{n+1}-1)$

From (i) For $n=1$ to $(2n+1)$

$$\Rightarrow 2 + 3\times 2 + \dots + (n+1)2^{n-1} + (n+2)2^n + (n+3)2^{n+1} + \dots + (2n+1)2^{2n-1} = (2n).2^{2n}$$

$\therefore$ $(n+2)2^n + (n+3)2^{n+1} + \dots + (2n+1)2^{2n-1} =$ (Sum from $n=1$ to $n=2n$) minus (Sum from $n=1$ to $(n+1)$)

$$= 2n(2^{2n}) - n2^n = n2^{2n+1} - n2^n$$

$$= n2^n(2^{n+1}-1)$$

2

If students did not use the result from part (i) then no marks. Also those who assumed the statement in (ii) in order to prove it received no marks.

(b) Show $(x-1)^3$ is a factor of $P(x) = x^{2n} - nx^{n+1} + nx^{n-1} - 1$, $n \geq 2$

$P(1) = 1 - n + n - 1 = 0 \Rightarrow (x-1)$ is a factor of $P(x)$

$$P'(x) = 2nx^{2n-1} - n(n+1)x^n + n(n-1)x^{n-2}$$

$$P''(x) = 2n(2n-1)x^{2n-2} - n^2(n+1)x^{n-1} + n(n-1)(n-2)x^{n-3}$$

Then $P''(1) = 2n(2n-1) - n^2(n+1) + n(n-1)(n-2)$

$$= 4n^2 - 2n - n^3 - n^2 + n(n^2 - 3n + 2)$$

$$= 4n^2 - 2n - n^3 - n^2 + n^3 - 3n^2 + 2n$$

$$= 0$$

$\therefore$ $x=1$ is a triple root

$\Rightarrow (x-1)^3$ is a factor of $P(x)$

2

Many tried to show this was true for n=2 etc. This received no marks. Many also tried to prove this by induction but without success unless they found the first and second derivative in each step.

(c) Solve $\log_2(x+3) + \log_2(x^2+5x-4) = 3$

$\Rightarrow \log_2(x+3)(x^2+5x-4) = 3$

$\Rightarrow (x+3)(x^2+5x-4) = 8$

$\Rightarrow x^3+5x^2-4x+3x^2+15x-20=0$

$\Rightarrow x^3+8x^2+11x-20=0$

Let $x=1 \Rightarrow P(1) = 1+8+11-20=0$

$\Rightarrow (x-1)$ is a factor.

$\Rightarrow (x-1)(ax^2+bx+c) = x^3+8x^2+11x-20=0$

$\Rightarrow a=1, c=20$

Compare x^2 terms: $bx^2-ax^2=8x^2$

$b-a=8 \Rightarrow b-1=8 \Rightarrow \underline{b=9}$

$\therefore (x-1)(x^2+9x+20)=0$

$(x-1)(x+4)(x+5)=0$

$\Rightarrow x=1$ or -4 or -5

But only $\underline{x=1}$ is a soln (other values give log(negative))

Finding solutions was generally done well except for those who didn't know their log rules. Only a small number of students realised that not all of the solutions of the polynomial were solutions to the original log equation.

3

(d) $\vec{OA} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ $\quad \vec{OC} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ $\quad \vec{OA} = 2\vec{CB}$

$$\Rightarrow \vec{CB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\vec{OB} = \vec{OC} + \vec{CB}$$

$$\Rightarrow \vec{OB} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}.$$

(i) Let $\vec{OA} = \underset{\sim}{a}$, $\vec{OB} = \underset{\sim}{b}$, $\vec{OC} = \underset{\sim}{c}$

$$\text{proj}_{\vec{OA}} \vec{OC} = (\underset{\sim}{c} \cdot \hat{\underset{\sim}{a}})\hat{\underset{\sim}{a}}$$

Now $\hat{\underset{\sim}{a}} = \dfrac{6\underset{\sim}{i} + 8\underset{\sim}{j}}{10} = \frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$

$$\therefore \text{proj}_{\vec{OA}} \vec{OC} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \left(\frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}\right)$$

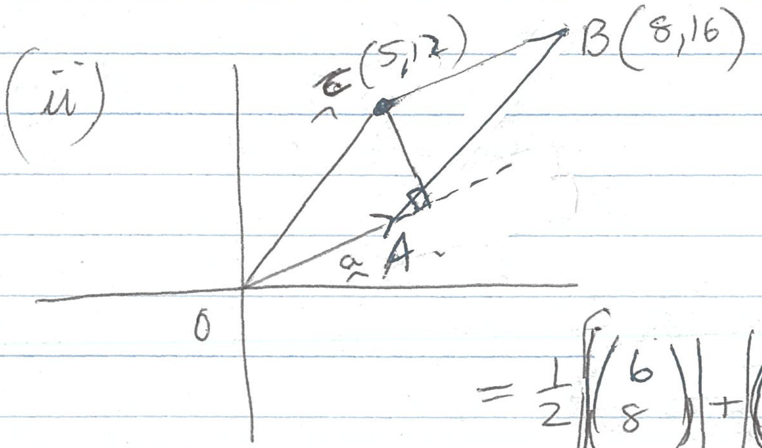
$$= \frac{63}{5}\left(\frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}\right)$$

2

$$= \frac{189}{25}\underset{\sim}{i} + \frac{252}{25}\underset{\sim}{j}.$$

This question was done well.

(ii)



TRAP. OABC

$$\boxed{A = \frac{1}{2}(a+b)h}.$$

$$= \frac{1}{2}(OA + CB)\left|\underset{\sim}{c} - \text{proj}_{OA}\vec{OC}\right|$$

$$= \frac{1}{2}\left[\left|\begin{pmatrix} 6 \\ 8 \end{pmatrix}\right| + \left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|\right] \times \left|\begin{pmatrix} 5 \\ 12 \end{pmatrix} - \begin{pmatrix} \frac{189}{25} \\ \frac{252}{25} \end{pmatrix}\right|$$

$$= \frac{1}{2}(10 + 5) \times \left|\begin{matrix} \frac{-64}{25} \\ \frac{-48}{25} \end{matrix}\right|$$

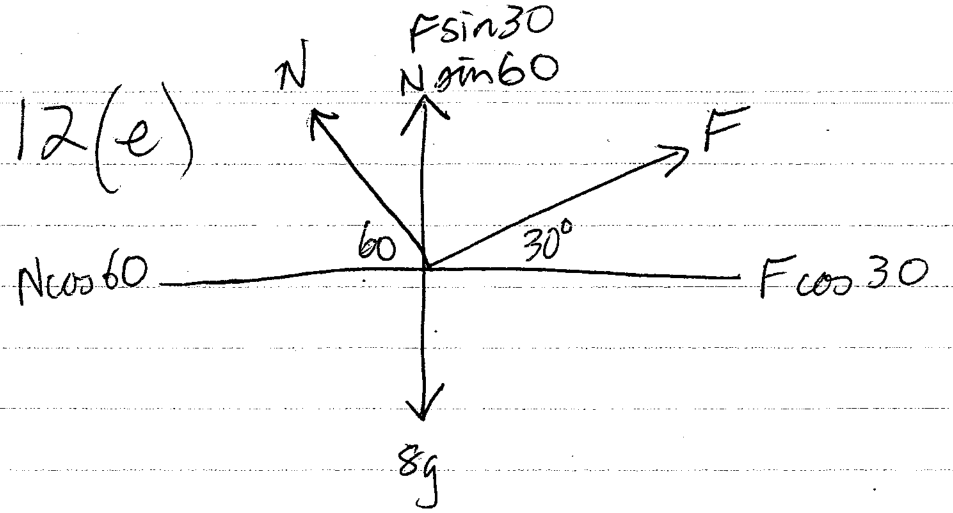
3

$$= \frac{15}{2} \times \sqrt{\frac{6400}{625}} = \frac{15}{2} \times \frac{80}{25}$$

$$= 24 \text{ sq. units}$$

This question was done poorly. Only a handful of students did it correctly. Most could not find the height of the trapezium or used the wrong vectors.

12(e)



Resolve forces in horizontal and vertical directions.

In equilibrium $\Rightarrow F\cos 30 = N\cos 60$ (horizontal forces)

$$\Rightarrow \frac{N}{N} = \frac{\cos 60}{\cos 30} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Only about half of the students got this correct- there were many different incorrect diagrams and results.

Question 13 **SOLUTIONS**

(a)	Prove, using induction, that $5^n-(-1)^n$ is divisible by 6, where n is a positive integer	**3**

Test $n = 1$: $5^1-(-1)^1=6$

$\therefore$ true for $n = 1$

Assume true for $n = k$ i.e. $5^k-(-1)^k=6M,\ M\in\mathbb{Z}$

Need to prove true for $n = k + 1$ i.e. $5^{k+1}-(-1)^{k+1}=6N,\ N\in\mathbb{Z}$

Method 1:

$$5^{k+1}-(-1)^{k+1}=5.5^k-(-1)^{k+1}$$

$$=5\left[6M+(-1)^k\right]-(-1)^{k+1} \qquad \left[\text{by assumption}\right]$$

$$=6\times5M+5\times(-1)^k-(-1)^{k+1}$$

$$=6\times5M+5\times(-1)^k-(-1)(-1)^k$$

$$=6\times5M+5\times(-1)^k+(-1)^k$$

$$=6\left[5M+(-1)^k\right]$$

$$=6N,\text{ where } N=5M+(-1)^k\in\mathbb{Z}$$

Method 2:

$$5^{k+1}-(-1)^{k+1}=5.5^k-(-1)^{k+1}$$

$$=5\left[5^k-(-1)^k\right]+5(-1)^k-(-1)^{k+1} \qquad \left[\text{by assumption}\right]$$

$$=5\times6M+5\times(-1)^k-(-1)^{k+1}$$

continues as Method 1

So the statement is true for $n = k + 1$ if it is true for n = k.
So by the principle of mathematical induction the statement is true for all positive integers.

Comment

Many students are not stating what they intend to prove – so how does anyone know if it has been proved?
Some students are not sticking to the script with mathematical induction
e.g instead of 'Assume true for $n = k$' they just wrote '$n = k$'. These students did not score maximum marks.

(b) Suppose m is a constant greater than 1.

Let $g(x) = \dfrac{1}{1+\cot^m x}$ where $0 < x \le \dfrac{\pi}{2}$. You may assume that $\lim\limits_{x\to 0} g(x) = 0$.

(i) Show that $g(x) + g\left(\dfrac{\pi}{2} - x\right) = 1$ for $0 < x \le \dfrac{\pi}{2}$. **2**

$$\begin{aligned}
\text{LHS} &= g(x) + g\left(\frac{\pi}{2} - x\right) \\
&= \frac{1}{1+\cot^m x} + \frac{1}{1+\cot^m\left(\frac{\pi}{2}-x\right)} \\
&= \frac{1}{1+\cot^m x} + \frac{1}{1+\tan^m x} && \left[\cot x = \tan\left(\frac{\pi}{2} - x\right)\right] \\
&= \frac{1+\cot^m x + 1 + \tan^m x}{\left(1+\cot^m x\right)\left(1+\tan^m x\right)} \\
&= \frac{1+\cot^m x + 1 + \tan^m x}{1+\cot^m x + 1 + \tan^m x} && \left[2+\cot^m x + \tan^m x \ne 0 \text{ for } x \in \left(0, \frac{\pi}{2}\right]\right] \\
&= 1 \\
&= \text{RHS}
\end{aligned}$$

Comment

This is a 'Show that' question. This question relies on the complementary nature of $\cot x$ and $\tan x$
cot$x$ = complement of tan$x$.

This reason was crucial to the solution and its absence was reflected in the mark.

Similarly for sin$x$ and cos$x$.

(b)	(ii)	Sketch $y = g(x)$ for $0 < x \le \dfrac{\pi}{2}$. [There is no need to find $g'(x)$ but assume that $y = g(x)$ has a horizontal tangent at $x = \dfrac{\pi}{2}$. Your graph must exhibit the property of part (i).]	**2**

$g(x) + g\left(\dfrac{\pi}{2} - x\right) = 1$ means that $\dfrac{g\left(\frac{\pi}{4} - x\right) + g\left(\frac{\pi}{4} + x\right)}{2} = g\left(\dfrac{\pi}{4}\right)$ $\left[= \dfrac{1}{2}\right]$

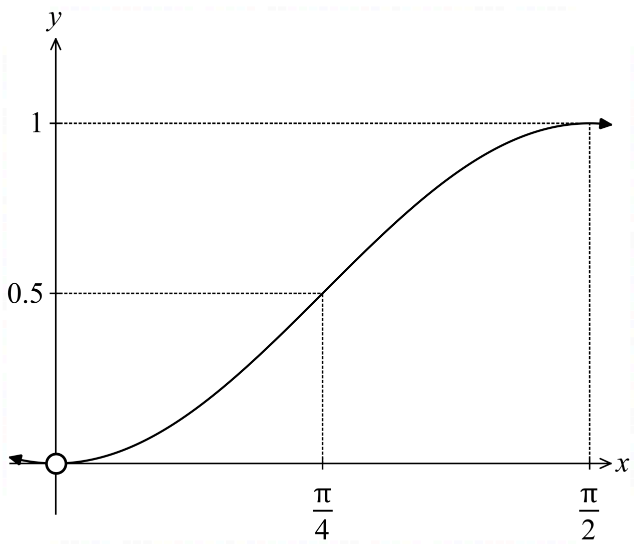
$\therefore$ point symmetry about $\left(\dfrac{\pi}{4},\ g\left(\dfrac{\pi}{4}\right)\right)$.

Also, differentiating both sides of $g(x) + g\left(\dfrac{\pi}{2} - x\right) = 1$ wrt x gets:

$$g'(x) - g'\left(\frac{\pi}{2} - x\right) = 0 \Rightarrow g'(x) = g'\left(\frac{\pi}{2} - x\right)$$

This means that $g'(x) \to 0$ as $x \to 0^+$.

Conclusion: The graph has point symmetry about $\left(\dfrac{\pi}{4},\ g\left(\dfrac{\pi}{4}\right)\right)$.



Comment

First, $\cot\dfrac{\pi}{2} = \dfrac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \dfrac{0}{1} = 0$.

Second, the graphs needed to have the following 1. Point symmetry about $x = \dfrac{\pi}{4}$;

2. $\left(\dfrac{\pi}{2}, 1\right)$ 'coloured' in; 3. $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$ indicated.

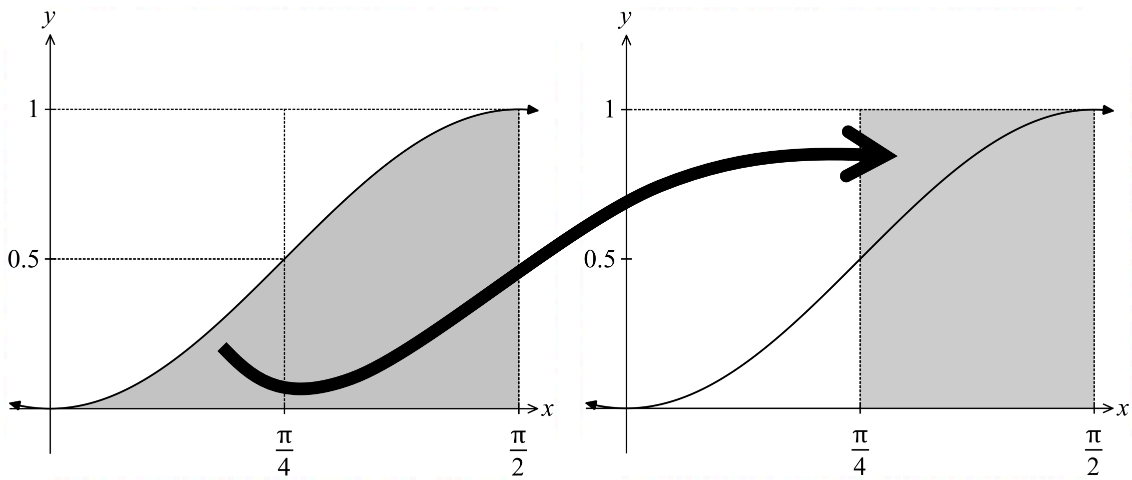
NESA recommends a $\dfrac{1}{3}$ of a page for a diagram, though the PSP rule of thumb is that it must be bigger than PSP's thumb.

(b) (iii) If $f(x)$ is defined as **1**

$$f(x)=\begin{cases} 0, & x=0 \\ g(x), & 0<x\le\dfrac{\pi}{2} \end{cases}$$

evaluate $\displaystyle\int_0^{\frac{\pi}{2}} f(x)\,dx$.

Using the point symmetry about $x=\dfrac{\pi}{4}$:



$$\therefore \int_0^{\frac{\pi}{2}} f(x)\,dx = 1\times\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{4}$$

Comment

For one mark, there were many that tried to evaluate a primitive rather than try and use their sketch.

(c) A curve is defined by the parametric equations $x = t + \frac{1}{t}$ and $y = t^2 + \frac{1}{t^2}$ for $t \neq 0$.

(i) Show that the Cartesian equation of the curve is $y = x^2 - 2$. **2**

$$x^2 = \left(t + \frac{1}{t}\right)^2$$

$$= t^2 + 2 + \frac{1}{t^2} \qquad (*)$$

$$= t^2 + \frac{1}{t^2} + 2$$

$$= y + 2$$

$$\therefore y = x^2 - 2$$

Comment

This is a 'Show that' question. Students had to show an algebraic result that they might have done elsewhere without any working.
There are other variations of this, but the line indicated by (*), or its equivalency, needed to be seen.

Many students attempted parametric differentiation, though I am still baffled as to why.

(ii) By considering the values of k for which $x = k$ has real solutions, where k is a constant, find any restrictions on x. **2**

$$t + \frac{1}{t} = k \Rightarrow t^2 - kt + 1 = 0$$

$$\therefore \Delta = k^2 - 4$$

$x = k$ has real solutions if $\Delta \geq 0$

$$\Delta \geq 0 \Rightarrow k^2 - 4 \geq 0$$

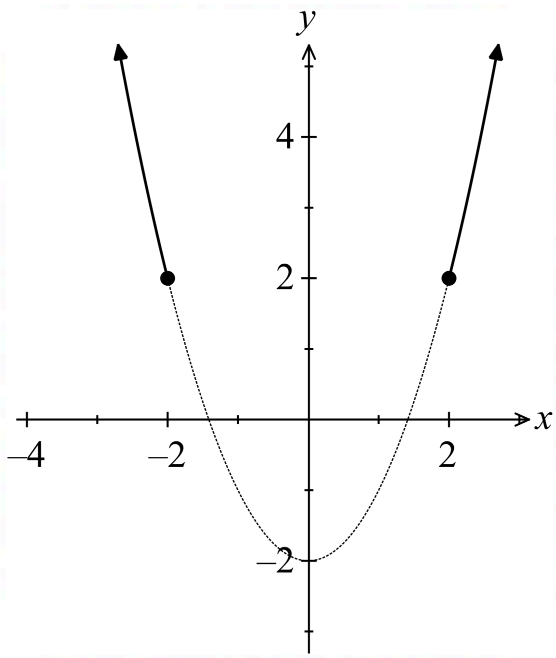
$$\therefore k \leq -2, k \geq 2$$

$$\therefore x \leq -2, x \geq 2$$

Comment

Some really bad algebraic mistakes atypical of an Extension student were present.
Some students chose to go the rational functions approach, but they had more work to do in order to justify the 2 marks available.

(c)	(iii)	Sketch the curve, showing any domain restrictions implied by part (ii).	**1**



Comment

Students had to sketch a graph that was clearly related to the parabola $y = x^2 - 2$.
Several parabolae that were drawn had positive y-intercepts, along with other 'random' graphs they did not score well.

(d)	(i)	Find the three solutions of $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$. You may assume that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.	**3**

$$x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0 \Rightarrow \sqrt{3}\left(1 - 3x^2\right) = 3x - x^3$$

$$\therefore \frac{3x - x^3}{1 - 3x^2} = \sqrt{3}$$

Let $x = \tan\theta$

$$\therefore \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3} \Rightarrow \tan 3\theta = \sqrt{3}$$

$$\tan 3\theta = \sqrt{3} \Rightarrow 3\theta = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \ldots$$

$-(*)$

$$\therefore \theta = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \ldots$$

$$\therefore x = \tan\frac{\pi}{9}, \tan\frac{4\pi}{9}, \tan\frac{7\pi}{9}, \tan\frac{10\pi}{9}, \ldots$$

Now a cubic equation has exactly three solutions

Note: $\tan\dfrac{10\pi}{9} = \tan\left(\pi + \dfrac{\pi}{9}\right) = \tan\dfrac{\pi}{9}$

$\therefore x = \tan\dfrac{\pi}{9}, \tan\dfrac{4\pi}{9}, \tan\dfrac{7\pi}{9}$ are the three solutions.

Comment

Notation abuse: $\tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$ NOT $\tan^{-1}\sqrt{3} = \dfrac{\pi}{3}, \dfrac{4\pi}{3}, \dfrac{7\pi}{3}, \ldots$

Some students found three values of θ, but didn't find *x*.

(d)	(ii)	Hence, show that $\cot\frac{\pi}{18}\cot\frac{5\pi}{18}\cot\frac{7\pi}{18}=\sqrt{3}$.	**2**

From part (i), the three roots are $x=\tan\frac{\pi}{9},\ \tan\frac{4\pi}{9},\ \tan\frac{7\pi}{9}$

$\therefore\ \tan\frac{\pi}{9}\tan\frac{4\pi}{9}\tan\frac{7\pi}{9}=-\sqrt{3}$ $\quad$ [product of roots]

$\therefore\ \tan\frac{\pi}{9}\tan\frac{4\pi}{9}\left(-\tan\frac{2\pi}{9}\right)=-\sqrt{3}$ $\quad$ $\left[\tan(\pi-x)=-\tan x\right]$

$\therefore\ \tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}=\sqrt{3}$

$\therefore\ \cot\left(\frac{\pi}{2}-\frac{\pi}{9}\right)\cot\left(\frac{\pi}{2}-\frac{2\pi}{9}\right)\tan\left(\frac{\pi}{2}-\frac{4\pi}{9}\right)=\sqrt{3}$ $\quad$ $\left[\tan x=\cot\left(\frac{\pi}{2}-x\right)\right]$

$\therefore\ \cot\frac{7\pi}{18}\cot\frac{5\pi}{18}\cot\frac{\pi}{18}=\sqrt{3}$

Comment

Too many students tried to fudge this.

Students could have quickly checked (with their calculator) that $x=\cot\frac{\pi}{18}$ is NOT a solution of the original equation.

End of solutions