SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class: 12Ma

2021

YEAR 12
TERM 2
HSC Task #3

Mathematics Extension 1

General Instructions

- Working Time – 65 minutes
- Write using black pen
- NESA approved calculators may be used
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 1–3, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 38

Section I – 8 marks (Personal Reference sheet)

Leave your Reference Sheet inside the booklet for Question 1

Section II – 30 marks (page 3–13)

- Attempt Questions 1–3

Examiner:

PSP

Marks	
Section I	/8
Section II	/30
Total	/38

NESA Number									

Q1	

YEAR 12 Mathematics Extension 1

TERM 2 HSC TASK #3

Question 1

Section II

Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 1 (10 marks)

(a) Write $\sin\left[2\arccos\left(-\dfrac{2}{3}\right)\right]$ in the form $a\sqrt{b}$ where $a \in \mathbb{Q}$ and $b \in \mathbb{Z}$. **2**

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(b) Given $f(x) = -2x - 1$ and $g(x) = -3x^2$, show that $g \circ f^{-1}(-1) = 0$. **2**

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(c) Find $\dfrac{d}{dx}\left(\arcsin\dfrac{a}{x}\right)$. **2**

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(d) Evaluate $\int_0^{\sqrt{2}} \frac{4}{\sqrt{16-4x^2}}\,dx$. **2**

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(e) Find $\int \frac{4x+8}{x^2+25}\,dx$. **2**

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Use this space to re-write any question. Clearly indicate that you have done so.

NESA Number									

Q2

YEAR 12 Mathematics Extension 1

TERM 2 HSC TASK #3

Question 2

Section II

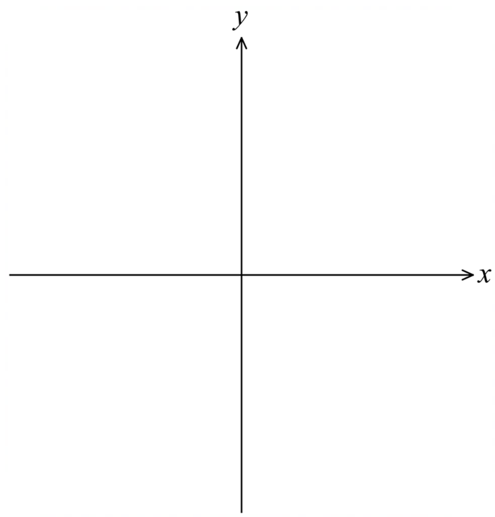
Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 2 (10 marks)

(a) (i) Sketch the graph of $y = \cos^{-1}(-4x)$. **2**

Your diagram should indicate intercepts. Clearly show the domain and range.



(ii) Find the gradient of the tangent to $y = \cos^{-1}(-4x)$ at $x = -\dfrac{1}{8}$. **2**

(b) Use the method of mathematical induction to prove that for all integers $n \geq 2$, the expression $9^n - 8n - 1$ is divisible by 64. **3**

(c) By using the substitution $x = \sin\theta$, show that $\int_0^{\frac{1}{2}} \frac{1}{\left(1-x^2\right)\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{3}}$. **3**

Use this space to re-write any question. Clearly indicate that you have done so.

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NESA Number											Q3	

YEAR 12 Mathematics Extension 1

TERM 2 HSC TASK #3

Question 3

Section II

Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 3 (10 marks)

(a) By using the substitution $y = \sqrt{x}$, find $\displaystyle\int \frac{1}{\sqrt{x}\left(1+x\right)}\,dx$. **3**

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(b) A function is called a self-inverse function if $g\left(g(x)\right) = x$. **2**

Find the value of the constant k so that $g(x) = \dfrac{3x-5}{x+k}$, $x \neq -k$ is a self-inverse function.

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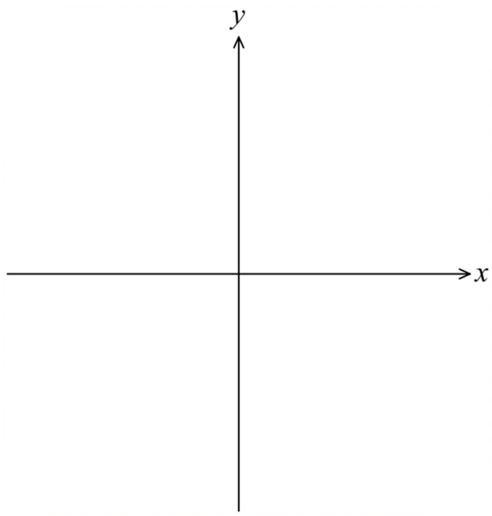
(c) Sketch $y = \cos\left(2\sin^{-1} x\right)$, showing all essential information. **2**



(d) Solve $\cos^{-1} x = \tan^{-1} x$. **3**

Leave answer(s) in exact form.

End of paper

Use this space to re-write any question. Clearly indicate that you have done so.

SYDNEY BOYS HIGH SCHOOL

2021	Year 12 Term 2 HSC Task 3

Mathematics Extension 1

Sample Solutions

NOTE: **Some of you may be disappointed with your mark.
This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.**

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of ink you have used.

Just because you have shown 'working' does not justify that your solution is worth any marks.

Section II

Question 1

(a) Let $\alpha = \cos^{-1}\left(-\frac{2}{3}\right)$, so $\cos\alpha = -\frac{2}{3}$

$$\sin^2\alpha = 1 - \cos^2\alpha$$
$$= 1 - \frac{4}{9}$$
$$= \frac{5}{9}$$

So $\sin\alpha = \frac{\sqrt{5}}{3}$ (2nd quadrant)

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$= 2\,(\frac{\sqrt{5}}{3})(-\frac{2}{3})$$
$$= -\frac{4}{9}\sqrt{5}$$

(b) $f(x) = -2x - 1,\ g(x) = -3x^2$.

So $x = -2f^{-1}(x) - 1$

$$2f^{-1}(x) = -x - 1$$
$$f^{-1}(x) = \frac{-x-1}{2}$$

So $f^{-1}(-1) = \frac{-(-1)-1}{2} = 0$.

Then $g \circ f^{-1}(-1) = g(0) = -3(0)^2 = 0$ as required.

You need to find the inverse of *f*, evaluate it with -1, then compose that result with *g* to find if the result is zero.

(c) Find $\frac{d}{dx}\left(\arcsin\left(\frac{a}{x}\right)\right) = \frac{dy}{du} \times \frac{du}{dx}$ Let $u = x^2 + 25$ $y = \arcsin(u)$

$$= \frac{-a}{x^2\sqrt{1-\frac{a^2}{x^2}}} \qquad \frac{du}{dx} = -\frac{a}{x^2} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}$$

$= \frac{-a}{\sqrt{x^4-a^2x^2}}$ (full marks for this answer)

$$= \frac{-a}{|x|\sqrt{x^2-a^2}}$$

(d) $\int_0^{\sqrt{2}} \frac{4}{\sqrt{16-4x^2}}\,dx = \int_0^{\sqrt{2}} \frac{4}{2\sqrt{2^2-x^2}}\,dx$

$$= 2\int_0^{\sqrt{2}} \frac{1}{\sqrt{2^2-x^2}}\,dx$$

$$= 2\left[\arcsin\left(\frac{x}{2}\right)\right]_0^{\sqrt{2}}$$

$$= 2\left(\arcsin\left(\frac{1}{\sqrt{2}}\right) - \arcsin(0)\right)$$

$$= \frac{\pi}{2}.$$

Please give your answer in radians.

(e) $\int \frac{4x+8}{x^2+25}\,dx = 2\int \frac{2x}{x^2+25}\,dx + 8\int \frac{1}{x^2+5^2}\,dx$

$$= 2\ln(x^2+25) + \frac{8}{5}\arctan\left(\frac{x}{5}\right) + C$$

Remember to look for an opportunity to break up a sum into parts.

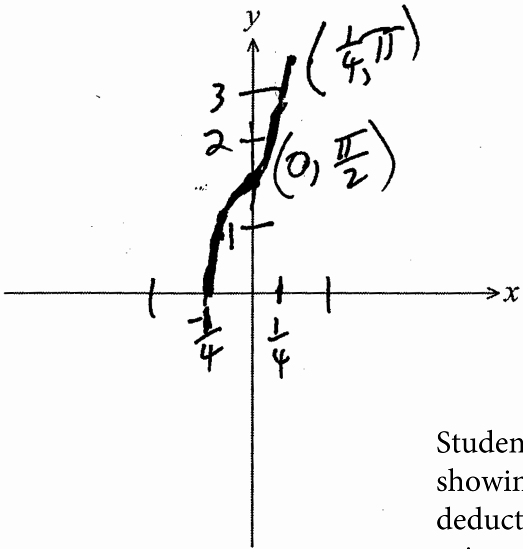
Answer each question in the spaces provided. A blank page is provided at the end of each question to allow rewriting of a part.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 2 (10 marks)

(a) (i) Sketch the graph of $y = \cos^{-1}(-4x)$. **2**

Your diagram should indicate intercepts. Clearly show the domain and range.



$D: -\frac{1}{4} \leq x \leq \frac{1}{4}$

$R: 0 \leq y \leq \pi$

Students lost 1/2 mark for not showing y intercept. One mark deducted for curve not in correct orientation. Also one mark deducted if domain or range was incorrect.

(ii) Find the gradient of the tangent to $y = \cos^{-1}(-4x)$ at $x = -\frac{1}{8}$. **2**

$y' = \frac{-1}{\sqrt{1-(-4x)^2}} \times -4$

$y' = \frac{4}{\sqrt{1-16x^2}}$

$y'\left(-\frac{1}{8}\right) = \frac{4}{\sqrt{1-16\left(\frac{1}{64}\right)}}$

$y'\left(-\frac{1}{8}\right) = \frac{4}{\sqrt{\frac{3}{4}}} = \frac{8}{\sqrt{3}}$

$\Rightarrow$ gradient $= \frac{8}{\sqrt{3}}$

A common error was having the derivative being negative. One mark for correct derivative, one mark for correct substitution.

(b) Use the method of mathematical induction to prove that for all integers $n \geq 2$, the expression $9^n - 8n - 1$ is divisible by 64. **3**

Prove $9^n - 8n - 1 = 64p$ where $p \in Z^+$

Let $n=2 \Rightarrow 9^2 - 16 - 1$
$= 64$. $\therefore$ true for $n=2$

Assume true for $n=k$
ie $9^k - 8k - 1 = 64q$, $q \in Z^+$
$\Rightarrow 9^k - 1 = 64q + 8k$.

Let $n=k+1$ Must show $9^{k+1} - 8(k+1) - 1 = 64r$, $r \in Z^+$

LHS $= 9.9^k - 8k - 9$
$= 9(9^k - 1) - 8k$
$= 9(64q + 8k) - 8k$ (from assumption)
$= 9(64q) + 9 \times 8k - 8k$
$= 64(9q) + 64k$
$= 64(9q + k)$ which is a multiple of 64.
$= 64r$, $r \in Z^+$
$\therefore$ true for $n = k+1$.

$\therefore$ true $\forall n \geq 2$, $n \in Z^+$

This question was done well. Most students got full marks.

(c) By using the substitution $x = \sin\theta$, show that $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{3}}$. **3**

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)\sqrt{1-x^2}}\,dx$$

Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$.

Also $\theta = \sin^{-1} x$

$\Rightarrow$ when $x = 0$, $\theta = 0$

when $x = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

Then $I = \displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{(1-\sin^2\theta)\sqrt{1-\sin^2\theta}} \times \cos\theta\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos\theta\, d\theta}{\cos^2\theta\sqrt{\cos^2\theta}}$

$= \displaystyle\int_0^{\frac{\pi}{6}} \frac{d\theta}{\cos^2\theta}$ ✓ $1\frac{1}{2}$

$= \displaystyle\int_0^{\frac{\pi}{6}} \sec^2\theta\, d\theta$

$= \Big[\tan\theta\Big]_0^{\frac{\pi}{6}}$ ✓ 1

$= \frac{1}{\sqrt{3}} - 0$

$= \frac{1}{\sqrt{3}}$ $\frac{1}{2}$

Again, most students got full marks for this. Those who didn't generally didn't recognise that 1/(cos squared) = sec squared.

YEAR 12 Mathematics Extension 1

TERM 2 HSC TASK #3

Question 3

Solutions

Question 3 (10 marks)

(a)	By using the substitution $y=\sqrt{x}$, find $\int \frac{1}{\sqrt{x}\left(1+x\right)}\,dx$.	**3**

$$y=\sqrt{x}\Rightarrow x=y^2$$
$$\therefore dx=2y\,dy$$

$$\int \frac{1}{\sqrt{x}\left(1+x\right)}\,dx=\int \frac{1}{y\left(1+y^2\right)}2y\,dy$$

$$=\int \frac{2}{1+y^2}\,dy \qquad\qquad (*)$$

$$=2\arctan y+c$$

$$=2\arctan\sqrt{x}+c$$

Comment

Too many students did not leave their final answer in terms of x.

Students who omitted the "+ c" were penalised.

There are some ME 1 students who are not recognising the integral (*) which is a concern.

(b)	A function is called a self-inverse function if $g(g(x))=x$. Find the value of the constant k so that $g(x)=\dfrac{3x-5}{x+k}$, $x\neq -k$ is a self-inverse function.	**2**

Method 1: $g(g(x))=x \Rightarrow g(x)=g^{-1}(x)$

$$y=\frac{3x-5}{x+k}\Rightarrow x=\frac{3y-5}{y+k}$$

$$\therefore xy+xk=3y-5$$

$$y(x-3)=-xk-5$$

$$\therefore y=\frac{-xk-5}{x-3}=\frac{kx-5}{x+k}$$

$$\therefore k=-3$$

Method 2: $g(g(x))=x$

$$g(g(x))=g\left(\frac{3x-5}{x+k}\right)$$

$$=\frac{3\left(\dfrac{3x-5}{x+k}\right)-5}{\left(\dfrac{3x-5}{x+k}\right)+k}\times\frac{x+k}{x+k}$$

$$=\frac{3(3x-5)-5(x+k)}{3x-5+k(x+k)}$$

$$=\frac{4x-(15+5k)}{(3+k)x+(k^2-5)}$$

$$=x$$

$$=\frac{4x-0}{0+4}$$

$$\therefore 15+5k=0 \text{ and } k+3=0 \text{ and } k^2-5=4$$

$$\therefore k=-3$$

Comment: What was meant to be a straightforward problem was for many students not so.

(c)	Sketch $y=\cos\left(2\sin^{-1}x\right)$, showing all essential information.	2

Note: $2\sin^{-1}x$ is odd and $\cos x$ is even
$\therefore \cos\left(2\sin^{-1}x\right)$ is even.

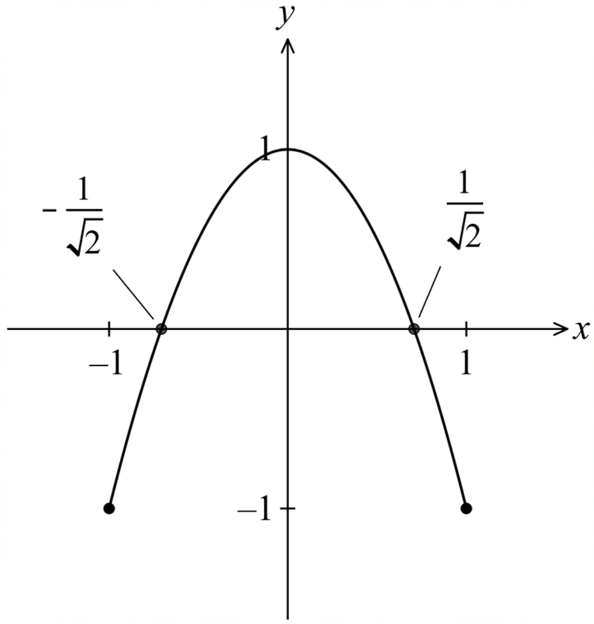
$x\in[-1,1]$
$\therefore 2\sin^{-1}x\in[-\pi,\pi]$
$\therefore \cos\left(2\sin^{-1}x\right)\in[-1,1]$

Let $u=\sin^{-1}x \Rightarrow x=\sin u$

$$y=\cos(2u)$$

$$=1-2\sin^2 u$$

$$=1-2x^2$$



Comment:

The first essential bit of information is the equation of the parabola. This was necessary for full marks.

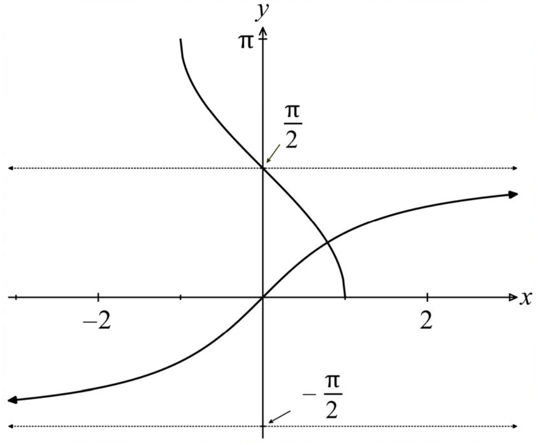
Domain, range and intercepts and symmetry were also essential.

Some students felt that there should be vertical tangents at $x=\pm1$.

$$\frac{dy}{dx}=\frac{-2\sin\left(2\sin^{-1}x\right)}{\sqrt{1-x^2}}$$

At $x=\pm1$, $\frac{dy}{dx}=\frac{0}{0}$ which means that though $\frac{dy}{dx}$ is undefined it is not because of a vertical asymptote.
And so there are no vertical tangents as would be expected for parabolae.

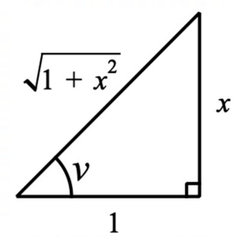
(d)	Solve $\cos^{-1} x = \tan^{-1} x$. Leave answer(s) in exact form.	**3**



From the diagram there is only one positive solution

$\cos^{-1} x = \tan^{-1} x$

Let $v = \tan^{-1} x \Rightarrow x = \tan v$



$\therefore x = \cos\left(\tan^{-1} x\right)$

$\therefore x = \dfrac{1}{\sqrt{1+x^2}}$

$\therefore x^2 = \dfrac{1}{1+x^2}$

$\therefore u = \dfrac{1}{1+u}$ $\quad\left[\text{Let } u = x^2\right]$

$\therefore u + u^2 = 1 \Rightarrow u^2 + u - 1 = 0$

$\therefore u = \dfrac{-1 \pm \sqrt{5}}{2} \Rightarrow x^2 = \dfrac{-1+\sqrt{5}}{2}$ $\quad[\because x > 0]$

$\therefore x = \sqrt{\dfrac{-1+\sqrt{5}}{2}}$

Comment: Students had to justify why there was only one positive solution.

Several students considered the gradients of $\cos^{-1} x$ and $\tan^{-1} x$, but this assumes that $\cos^{-1} x \equiv \tan^{-1} x$, which is clearly not true.
If two functions intersect there is no reason to suspect that their gradients are equal at the point of intersection.

End of solutions