SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class:
12MX

2022

YEAR 12
TERM 2
HSC Task #3

Mathematics Extension 1

General Instructions

- Working Time – 65 minutes
- Write using black pen
- NESA approved calculators may be used
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–18, show ALL relevant mathematical reasoning and/or calculations

Total Marks: **Section I – 10 marks (Personal Reference sheet)**

Leave your Personal Reference Sheet this booklet.

Section II – 10 marks (pages 2–4)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section III – 30 marks (pages 5–14)
- Attempt Questions 11–18
- Allow about 45 minutes for this section

Examiner:

PSP

Marks	
Section I	/10
Section II	/10
Section III	/30
Total	/50

Section II

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the exact value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\cos^{-1}\left(\frac{1}{2}\right)$

A. $\frac{\pi}{12}$ B. $\frac{\pi}{24}$ C. $\frac{\pi^2}{24}$ D. $\frac{\pi^2}{12}$

2 What is the domain of $y=\frac{1}{2}\arccos\left(1-3x\right)$?

A. $0\le x\le\frac{\pi}{2}$

B. $0\le x\le\frac{2}{3}$

C. $0\le x\le 2\pi$

D. $-\frac{2}{3}\le x\le 0$

3 What is the derivative of $\tan^{-1}\left(\frac{1}{x}\right)$?

A. $\frac{1}{1+x^2}$

B. $-\frac{1}{1+x^2}$

C. $\frac{x^2}{1+x^2}$

D. $-\frac{x^2}{1+x^2}$

4 Which of the following functions is the inverse of $f\left(x\right)=3-\frac{1}{2x+6}$

A. $f^{-1}\left(x\right)=-3-\frac{1}{2x-6}$

B. $f^{-1}\left(x\right)=-3+\frac{1}{2x-6}$

C. $f^{-1}\left(x\right)=3-\frac{1}{2x-6}$

D. $f^{-1}\left(x\right)=3+\frac{1}{2x-6}$

5 What is the range of the function $f(x) = \arcsin x + \arctan x$?

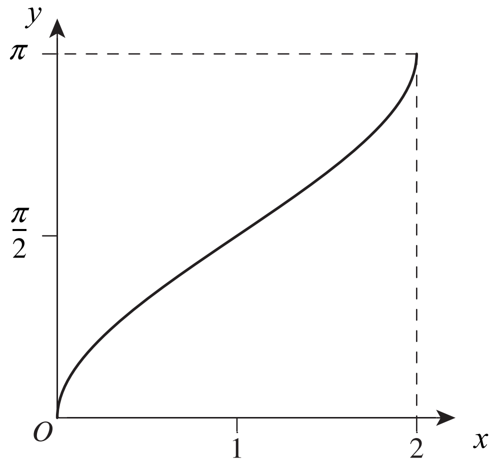
A. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

B. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

C. $(-\pi, \pi)$

D. $\left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

6 Which of the following functions best describes this graph?



A. $y = \arccos(1-x)$

B. $y = \arccos x$

C. $y = \arccos(x-1)$

D. $1 - \arccos x$

7 Which of the following is true about the identity $\sin^{-1}(\sin x) = x$?

A. False for all real x.

B. True for all real x.

C. False for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

D. True for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

8 Which integral is obtained when the substitution $u = 1 - 2x$ is applied to $\int x\sqrt{1-2x}\,dx$?

A. $-\frac{1}{4}\int \sqrt{u}(u-1)\,du$

B. $-\frac{1}{2}\int \sqrt{u}(u-1)\,du$

C. $\frac{1}{4}\int \sqrt{u}(u-1)\,du$

D. $\frac{1}{2}\int \sqrt{u}(u-1)\,du$

9 What is the number of solutions to $\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}(x)$?

A. 0 B. 1 C. 2 D. 3

10 The function $f(x) = x^2 - 4x + 7$ has an inverse function $f^{-1}(x)$ in the domain $[2, \infty)$.
What is the value of $f^{-1}(f(m))$, where *m* is a real number such that $m < 0$?

A. $2 - m$ B. $2 + m$ C. $4 - m$ D. $4 + m$

NESA Number									

YEAR 12 Mathematics Extension 1

TERM 2 HSC TASK #3

Part A

Section III

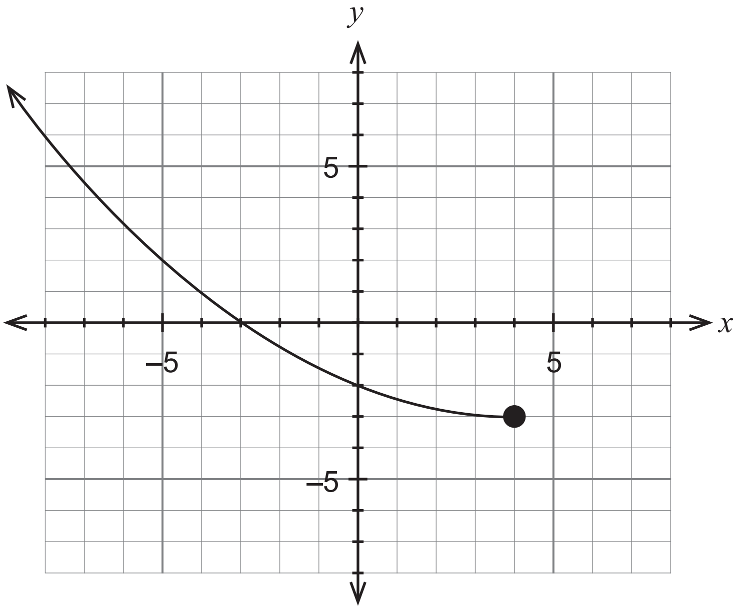
Part A – 15 marks
Attempt Questions 11 – 15

Answer each question in the spaces provided. A blank page is provided at the end of this Part to allow rewriting of a question.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (4 marks)

The graph of $y = g(x)$ is shown below.



(a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. **2**

(b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$, where $x \le 4$, find the rule for $g^{-1}(x)$. **2**

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Question 12 (3 marks)

Let $y = \arccos\left(\dfrac{x}{2}\right)$.

(a) Find $\dfrac{dy}{dx}$. **1**

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(b) Evaluate $\displaystyle\int_0^1 \arccos\left(\frac{x}{2}\right) dx$. **2**

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Question 13 (3 marks)

Evaluate $\displaystyle\int_{\frac{1}{2}}^{1} 4t(2t-1)^5\, dt$ by using the substitution $u = 2t - 1$. **3**

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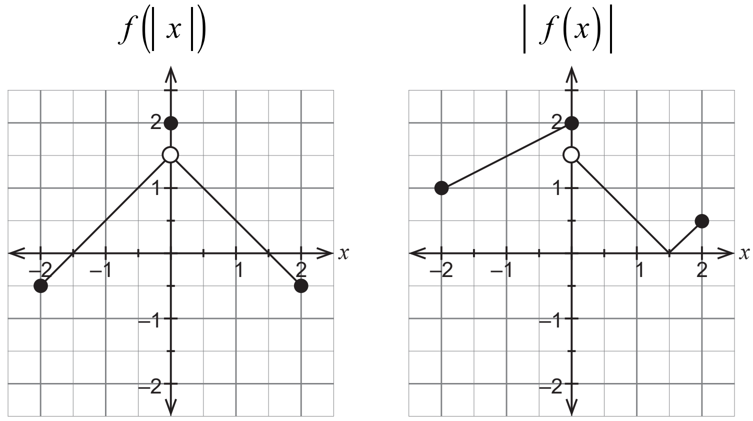
Question 14 (3 marks)

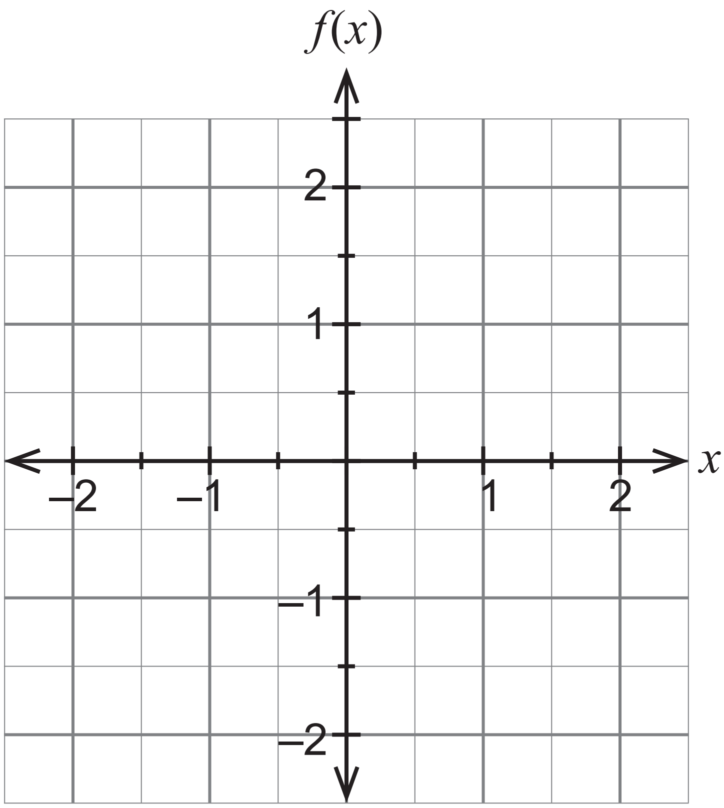
Evaluate $\int_0^{\frac{1}{2}} \frac{4x^2-1}{4x^2+1}\,dx$. **3**

Question 15 (2 marks)

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are drawn below. **2**



Given that $y = f^{-1}(x)$ exists, sketch a possible graph for $y = f(x)$ on the axes below.
Justify your answer by considering $y = f^{-1}(x)$.



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End of Section A

Use this space to re-write any question. Clearly indicate that you have done so.

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YEAR 12 Mathematics Extension 1

TERM 2 HSC TASK #3

Part B

Section III

Part B **15 marks**
Attempt Questions 16 – 18

Answer each question in the spaces provided. Two blank pages are provided at the end of this Part to allow rewriting of a question.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 16 (3 marks)

Evaluate $\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx$ using the substitution $x = t^2 + 1$. **3**

Question 17 (4 marks)

Prove, using mathematical induction, that $\sum_{k=1}^{n} \arctan\left(\frac{1}{2k^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. **4**

You may use $\arctan p + \arctan q = \arctan\left(\frac{p+q}{1-pq}\right)$, where $p, q > 0$ and $pq < 1$. (Do NOT prove)

Question 18 (8 marks)

A function f is defined by $f(x) = \arcsin\left(\dfrac{x^2 - 1}{x^2 + 1}\right)$, where $x \leq 0$.

(a) What is the equation of the horizontal asymptote? **1**

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(b) (i) Show that $f'(x) = \dfrac{2x}{|x|(x^2 + 1)}$. **3**

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(ii) Why does $f^{-1}(x)$ exist? **1**

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Question 18 continues on page 15

Question 18 (continued)

(c) Find a rule for $f^{-1}(x)$. **2**

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(d) State the domain of $f^{-1}(x)$. **1**

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End of paper

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SYDNEY BOYS HIGH SCHOOL

2022 | YEAR 12
TERM 2
ASSESSMENT TASK 3

Mathematics Extension 1

Sample Solutions

NOTE: Some of you may be disappointed with your mark.

This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown 'working' does not justify that your solution is worth any marks.

Put Your NAME and CLASS on the cover page

MC Answers

1	D	**6**	A
2	B	**7**	D
3	B	**8**	C
4	A	**9**	D
5	D	**10**	C

Section II

1 What is the exact value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\cos^{-1}\left(\frac{1}{2}\right)$

A. $\frac{\pi}{12}$ B. $\frac{\pi}{24}$ C. $\frac{\pi^2}{24}$ D. $\frac{\pi^2}{12}$

C or D must have the right answer due to the "π^2".

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{4}\times\frac{\pi}{3}=\frac{\pi^2}{12}$

2 What is the domain of $y=\frac{1}{2}\arccos(1-3x)$?

A. $0\le x\le\frac{\pi}{2}$ B. $0\le x\le\frac{2}{3}$

C. $0\le x\le 2\pi$ D. $-\frac{2}{3}\le x\le 0$

B or D must have the right answer due to the "3".

$-1\le 1-3x\le 1\Rightarrow -2\le -3x\le 0$

$\therefore 0\le x\le\frac{2}{3}$

3 What is the derivative of $\tan^{-1}\left(\frac{1}{x}\right)$?

A. $\frac{1}{1+x^2}$ B. $-\frac{1}{1+x^2}$

C. $\frac{x^2}{1+x^2}$ D. $-\frac{x^2}{1+x^2}$

Students hopefully remember that $\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)=\text{constant}, x\ne 0$

4 Which of the following functions is the inverse of $f(x)=3-\frac{1}{2x+6}$

A. $f^{-1}(x)=-3-\frac{1}{2x-6}$ B. $f^{-1}(x)=-3+\frac{1}{2x-6}$

C. $f^{-1}(x)=3-\frac{1}{2x-6}$ D. $f^{-1}(x)=3+\frac{1}{2x-6}$

Asymptotes "switch", so the answer has to be A or B.

$y=3-\frac{1}{2x+6} \quad \Rightarrow 2x+6=-\frac{1}{y-3}$

$\therefore 2x=-6-\frac{1}{y-3} \Rightarrow x=-3-\frac{1}{2y-6}$

$\therefore f^{-1}(x)=-3-\frac{1}{2x-6}$

5 What is the range of the function $f(x)=\arcsin x+\arctan x$?

A. $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ B. $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

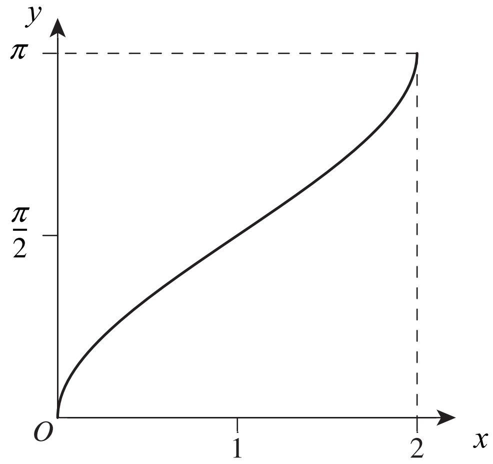
C. $(-\pi,\pi)$ D. $\left[-\frac{3\pi}{4},\frac{3\pi}{4}\right]$

Both functions are odd and 1 : 1.
The common domain is $-1\le x\le 1$.

$\arcsin x$: $\quad -1\le x\le 1, -\frac{\pi}{2}\le x\le\frac{\pi}{2}$

$\arctan x$: $\quad -1\le x\le 1, -\frac{\pi}{4}\le x\le\frac{\pi}{4}$

6 Which of the following functions best describes this graph?



A. $y = \arccos(1-x)$ B. $y = \arccos x$

C. $y = \arccos(x-1)$ D. $1 - \arccos x$

The domain, $0 \le x \le 2$, dictates the answer.

7 Which of the following is true about the identity $\sin^{-1}(\sin x) = x$?

A. False for all real x. B. True for all real x.

C. False for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. D. True for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Definitions!

8 Which integral is obtained when the substitution $u = 1-2x$ is applied to $\int x\sqrt{1-2x}\,dx$?

A. $-\frac{1}{4}\int \sqrt{u}(u-1)\,du$ B. $-\frac{1}{2}\int \sqrt{u}(u-1)\,du$

C. $\frac{1}{4}\int \sqrt{u}(u-1)\,du$ D. $\frac{1}{2}\int \sqrt{u}(u-1)\,du$

$u = 1-2x \Rightarrow du = -2dx$

$\int x\sqrt{1-2x}\,dx = \int \frac{u-1}{-2}\sqrt{u}\,.-\frac{1}{2}du$

9 What is the number of solutions to $\sin^{-1}\left(\frac{3x}{5}\right)+\sin^{-1}\left(\frac{4x}{5}\right)=\sin^{-1}(x)$?

A. 0 B. 1 C. 2 D. 3

0 is an obvious solution.

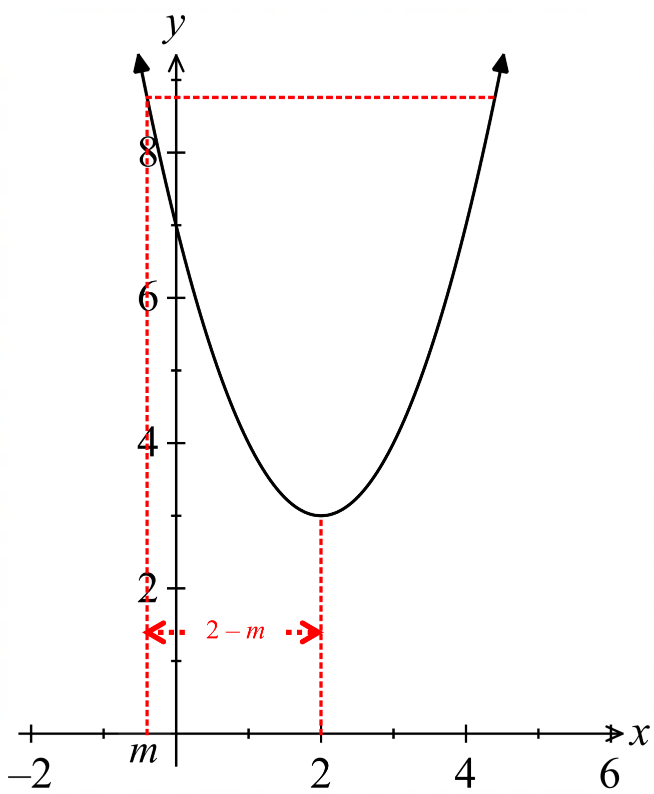
$x = 1$ is a solution, either by calculator, or realising, that using a diagram, $\sin^{-1}\left(\frac{3}{5}\right)$ and $\sin^{-1}\left(\frac{4}{5}\right)$ are complementary angles (or using $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$).

Given the point symmetry (odd) of $\sin^{-1}(x)$ then -1 is also a solution.

10 The function $f(x)=x^2-4x+7$ has an inverse function $f^{-1}(x)$ in the domain $[2,\infty)$.
What is the value of $f^{-1}(f(m))$, where m is a real number such that $m<0$?

A. $2-m$ B. $2+m$ C. $4-m$ D. $4+m$

Draw a diagram and then use symmetry of the parabola about $x = 2$.

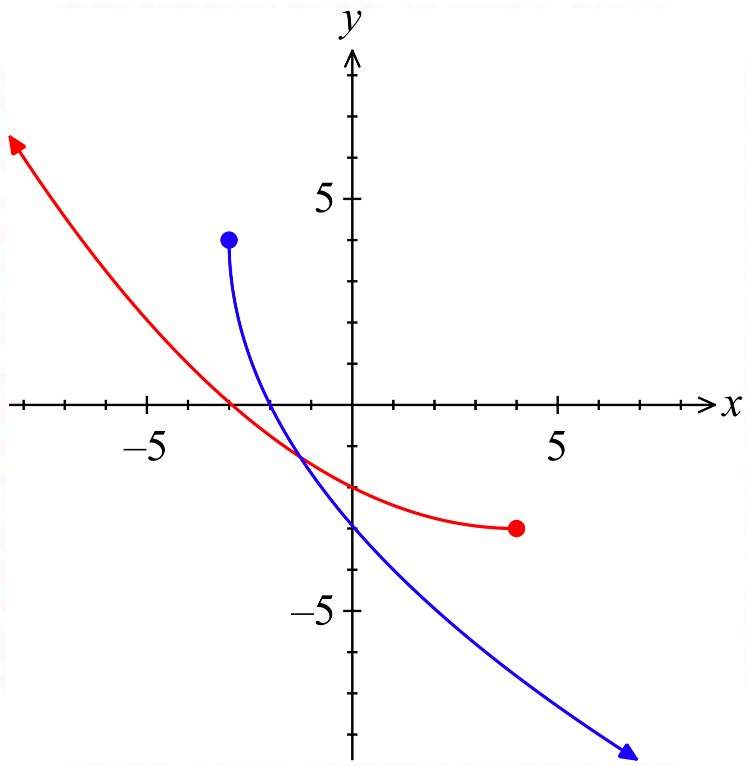


Now $x=2+2-m=4-m$ is in the required domain.
$\therefore f^{-1}(f(m))=f^{-1}(f(4-m))=4-m$

Section III Part A Solutions

Question 11 (4 marks)

The graph of $y = g(x)$ is shown below.



(a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. **2**

(b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$, where $x \le 4$, find the rule for $g^{-1}(x)$. **2**

$$g(x): x \le 4;\ y \ge -3$$
$$g^{-1}(x): x \ge -3;\ y \le 4$$

Let $y = \frac{1}{16}(x-4)^2 - 3 \Rightarrow (x-4)^2 = 16(y+3)$

$$x - 4 = \pm 4\sqrt{y+3}$$

$$\therefore g^{-1}(x) = 4 - 4\sqrt{x+3},\ x \ge -3$$

Comment

Part (a) was generally well done

With Part (b), only the students who included the domain got full marks.

Question 12 (3 marks)

Let $y = \arccos\left(\dfrac{x}{2}\right)$.

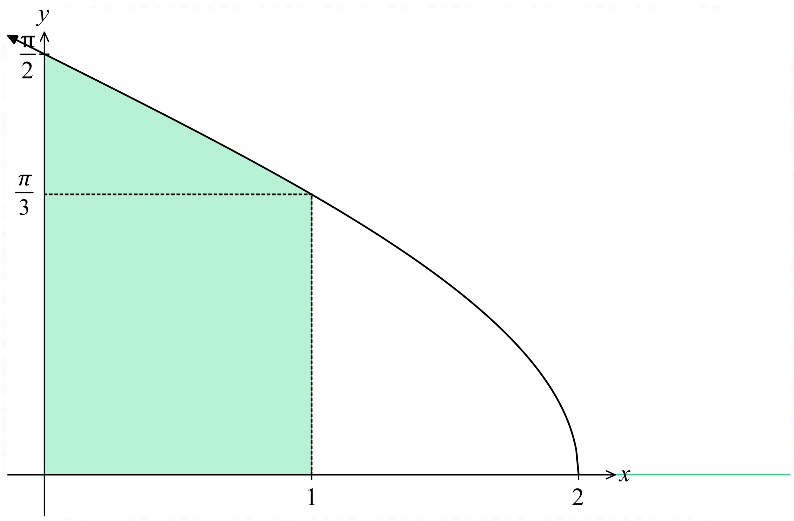
(a) Find $\dfrac{dy}{dx}$. **1**

$$\frac{dy}{dx} = -\frac{1}{2} \times \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$= -\frac{1}{\sqrt{4-x^2}}$$

(b) Evaluate $\displaystyle\int_0^1 \arccos\left(\frac{x}{2}\right) dx$. **2**

The required area is shown in the diagram below



$y = \arccos\left(\dfrac{x}{2}\right) \Rightarrow x = 2\cos y$.

The area is given by:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\cos y \, dy + \frac{\pi}{3} = \left[2\sin y\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \frac{\pi}{3}$$

$$= 2 - \sqrt{3} + \frac{\pi}{3}$$

Alternative solution over the page

Question 12 (b) Alternative solution

This is using the principle of integration by parts.

$$\int_0^1 \arccos\left(\frac{x}{2}\right) dx = \int_0^1 1 \times \arccos\left(\frac{x}{2}\right) dx$$

$$= \left[x \arccos\left(\frac{x}{2}\right) \right]_0^1 - \int_0^1 x \times \frac{-1}{\sqrt{4-x^2}} \, dx$$

$$= \frac{\pi}{3} - \frac{1}{2} \int_0^1 \frac{-2x}{\sqrt{4-x^2}} \, dx$$

$$= \frac{\pi}{3} - \frac{1}{2} \left[2\sqrt{4-x^2} \right]_0^1$$

$$= \frac{\pi}{3} - \left(\sqrt{3} - 2 \right)$$

$$= \frac{\pi}{3} + 2 - \sqrt{3}$$

Comment

The question was done very well if students saw to work with the inverse i.e. looking sideways. Some students used integration by parts.

Question 13 (3 marks)

Evaluate $\int_{\frac{1}{2}}^{1} 4t(2t-1)^5 \, dt$ by using the substitution $u = 2t-1$. **3**

$u = 2t-1 \Rightarrow du = 2dt \Rightarrow dt = \frac{1}{2}du$

$2t = u+1 \Rightarrow 4t = 2(u+1)$

$t: \quad \frac{1}{2} \sim 1$

$u: \quad 0 \sim 1$

$$\int_{\frac{1}{2}}^{1} 4t(2t-1)^5 \, dt = \int_0^1 2(u+1)u^5 \, .\frac{1}{2}du$$

$$= \int_0^1 u^5 + u^6 \, du$$

$$= \left[\frac{1}{6}u^6 + \frac{1}{7}u^7\right]_0^1$$

$$= \frac{13}{42}$$

Comment

Students need to set their work out neatly.

Question 14 (3 marks)

Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{4x^2-1}{4x^2+1}\,dx$. **3**

$$\int_0^{\frac{1}{2}} \frac{4x^2-1}{4x^2+1}\,dx = \int_0^{\frac{1}{2}} \frac{\left(4x^2+1\right)-2}{4x^2+1}\,dx$$

$$= \int_0^{\frac{1}{2}} 1-\frac{2}{4x^2+1}\,dx$$

$$= \int_0^{\frac{1}{2}} 1-\frac{2}{\left(2x\right)^2+1}\,dx$$

$$= \left[x-\tan^{-1}2x\right]_0^{\frac{1}{2}}$$

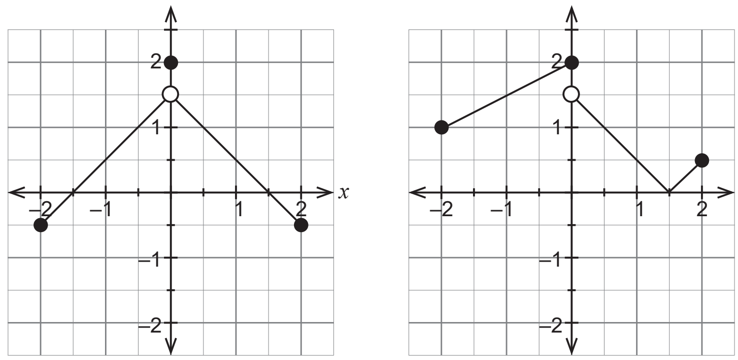
$$= \frac{1}{2}-\frac{\pi}{4}$$

Comment
Students who split the integrand were generally successful.

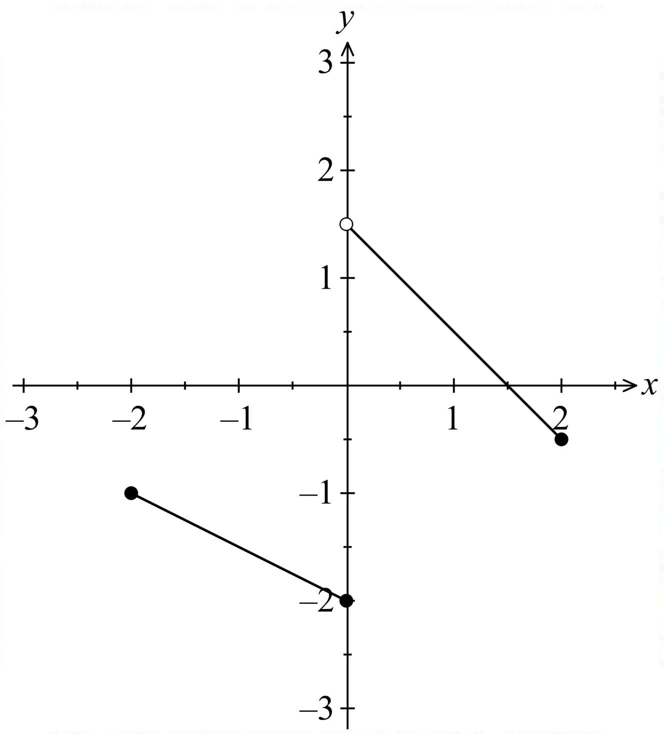
Question 15 (2 marks)

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are drawn below. **2**

$f(|x|)$ $\qquad\qquad$ $|f(x)|$



Given that $y = f^{-1}(x)$ exists, sketch a possible graph for $y = f(x)$ on the axes below.
Justify your answer by considering $y = f^{-1}(x)$.



As $y = f^{-1}(x)$ exists, it must be 1 : 1

Comment
There is more than one possible graph. The $x > 0$ part is taken from the top left graph.
The $x \leq 0$ part is made so that it will pass the horizontal line test.

Question 16 (3 marks) Write your NESA number and write it legibly

Evaluate $\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx$ using the substitution $x = t^2 + 1$. 3

$$\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx$$

$$x = t^2+1$$
$$dx = 2t\,dt$$

When $x=2$, $t^2=1 \Rightarrow t=\pm 1$

$x=10$, $t^2=9$, $\Rightarrow t=\pm 3$.

For convenience take the positive root of t.

$$\int_1^3 \frac{t^2+1}{\sqrt{\cancel{t^2}}}(2\cancel{t}\,dt)$$

$$= 2\int_1^3 (t^2+1)\,dt$$

$$= 2\left[t^3/3 + t\right]_1^3$$

$$= 2\left[(9+3) - 4/3\right]$$

$$= 21\tfrac{1}{3}.$$

Same solution if the negative root of t is taken

ie $\int_{-1}^{-3} \frac{t^2+1}{\sqrt{t^2}}(2t\,dt) = 21\tfrac{1}{3}$.

(Q 16)

This question was well attempted on the whole, with many more candidates scoring full marks.

Common error

$$\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx$$

$$= \frac{1}{2}\int_1^3 \left(\frac{t^2}{t} + \frac{1}{t}\right)dt$$

$$= \frac{1}{2}\left[\frac{t^2}{2} + \ln|t|\right]_1^3$$

$$= 2 + \ln\sqrt{3}$$

Question 17 (4 marks)

Prove, using mathematical induction, that $\sum_{k=1}^{n} \arctan\left(\frac{1}{2k^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. **4**

You may use $\arctan p + \arctan q = \arctan\left(\frac{p+q}{1-pq}\right)$, where $p, q > 0$ and $pq < 1$. (Do NOT prove)

Let $S(n)$ be the proposition that

For $n = 1$, LHS $= \tan^{-1}\left(\frac{1}{2 \times 1^2}\right) = \tan^{-1}\frac{1}{2}$.

$= \tan^{-1}\left(\frac{1}{1+1}\right)$

$=$ RHS $\therefore S(1)$ is true

Assume $S(k)$ is true

$$\sum_{n=1}^{k} \tan^{-1}\left(\frac{1}{2k^2}\right) = \tan^{-1}\left(\frac{k}{k+1}\right)$$

Considering prove time for $n = k+1$

Now $S(k+1)$

$$= \sum_{k=1}^{n=k+1} \tan^{-1}\left(\frac{1}{2k^2}\right)$$

$$= \underbrace{\sum_{k=1}^{k} \tan^{-1}\left(\frac{1}{2k^2}\right)} + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$

From inductive hypothesis

$$= \tan^{-1}\left(\frac{n}{n+1}\right) + \tan^{-1}\left(\frac{1}{2(n+1)^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{n}{n+1} + \frac{1}{2(n+1)^2}}{1 - \frac{n}{n+1} \times \frac{1}{2(n+1)^2}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2n(n+1)+1}{2(n+1)^2}}{1 - \frac{n}{2(n+1)^3}}\right)$$

$$= \tan^{-1}\left[\frac{(n+1)(2n^2+2n+1)}{2(n+1)^3 - n}\right]$$

$$= \tan^{-1}\left[\frac{(n+1)(2n^2+2n+1)}{n+2\ (2n^2+2n+1)}\right].$$

Proof

Method 1: Test $m=-2$: $2(-2+1)^3-(-2)=0$

$2(m+1)^3-m=(m+2)(2m^2+Bm+1)$

Consider the coefficient of m^2: $2\times 3=B+4$

$\therefore 2(m+1)^3-m=(m+2)(2m^2+2m+1)$

Method 2: Long division

Method 3: Expand

$$\text{LHS}=\arctan\left(\frac{(m+1)(2m^2+2m+1)}{2(m+1)^3-m}\right)$$

$$=\arctan\left(\frac{(m+1)(2m^2+2m+1)}{(m+2)(2m^2+2m+1)}\right)$$

$$=\arctan\left(\frac{m+1}{m+2}\right)$$

$$=\text{RHS}$$

(Q 17)

This question was a disaster for half of the students, probably because the algebraic is difficult to trace through

The algebra involved in long division is not done properly: $\Rightarrow$ Revision in polynomial (which is not stressed enough in the new syllabus

Method (1) Test $m=-2$. (a suitable substitution) and equating coefficient of m^2.

Method (2) Long division.

Most candidate stops at;

$$\tan^{-1}\left[\frac{(m+1)(2m^2+2m+1)}{2(m+1)^3-m}\right]$$

and could not go on to $P(k+1)$ form —— lots of fudging.

Question 18 (8 marks)

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, where $x \leq 0$.

(a) What is the equation of the horizontal asymptote? 1

$$\lim_{x\to-\infty} \sin^{-1}\left(\frac{(x^2+1)-2}{x^2+1}\right) = \lim_{x\to-\infty} \sin^{-1}\left(1-\frac{2}{x^2+1}\right)$$

$$= \sin^{-1}(1)$$

$$= \pi/2 .$$

(b) (i) Show that $f'(x) = \frac{2x}{|x|(x^2+1)}$. 3

$$f(x) = \sin^{-1}\left(1-\frac{2}{x^2+1}\right), \quad f'(x) = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

Where $f(x) = \left(1-\frac{2}{x^2+1}\right)$

$$f'(x) = \frac{\frac{4x}{(x^2+1)^2}}{\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2+1)}}}$$

$$= \frac{4x}{(x^2+1)\sqrt{(2x^2)(2)}}$$

$$= \frac{2x}{|x|(x^2+1)} \quad \because \sqrt{x^2} = |x|$$

When $x<0$, $f'(x)<0$

$\therefore$ f is decreasing

$\Longrightarrow$ satisfies the criterion for the existence of the inverse

either • one-to-one,

• f is increasing or decreasing.

(ii) Why does $f^{-1}(x)$ exist? 1

$\therefore$ $f^{-1}(x)$ exists

Question 18 (continued)

(c) Find a rule for $f^{-1}(x)$. 2

$$y = \sin^{-1}\left(\frac{x^2-1}{x^2+1}\right).$$

Interchange x and y and make y the subject ($\because$ it satisfy the existence of the inverse function.)

$$\sin y = \frac{x^2-1}{x^2+1} = \frac{x^2+1-2}{x^2+1}$$

$$\therefore \sin y = 1 - \frac{2}{x^2+1}$$

$$x^2+1 = \frac{2}{1-\sin y}$$

$$x^2 = \frac{2}{1-\sin y} - 1$$

$$= \frac{1+\sin y}{1-\sin y}.$$

Swap x and y

$$\therefore y^2 = \frac{1+\sin x}{1-\sin x},$$

$$y = -\sqrt{\frac{1+\sin x}{1-\sin x}} \quad \because [y \le 0].$$

(d) State the domain of $f^{-1}(x)$. 1

f is decreasing

For $x \le 0$ $-\frac{\pi}{2} \le y < \frac{\pi}{2}$

$\therefore$ $y = f^{-1}(x)$ interchange x, y

$$\boxed{-\frac{\pi}{2} \le x < \frac{\pi}{2} \text{ and } y \le 0}$$

(Q 18)

(a)

This part was too difficult for half of the student

Very few appreciated the algebraic manipulation of $\frac{x^2-1}{x^2+1} = \frac{(x^2+1)-2}{x^2+1}$

$= 1 - \frac{2}{x^2+1}$

Hence it is difficult to see

$\lim_{x \to -\infty} \sin^{-1}\left(\frac{x^2-1}{x^2+1}\right) = \sin^{-1}(1)$

$= \frac{\pi}{2}$.

Some Confusion was evident in stating the equation of the "horizontal" asymptote as $x = \frac{\pi}{2}$ rather than $y = \pi/2$

(b) Suprisingly well done in

(i) general.

- Some students tried to consider $\sin\left(\frac{x^2-1}{x^2+1}\right)$ as a many-valued function of x with disastrous result.
- Some students lack differentiate skills, especially in chain rule. that is, Errors in differentiation were common. Quite a few students mentioned than $f(x)$ was an even function of x.
- Many could not recognised $\sqrt{x^2} = |x|$ and therefore could not collapsed into the appropriate form required.

(ii) Most student could not properly the existence of $f^{-1}(x)$

- Most student simply "stating that it passes the horizontal line test" but $f^{-1}(x)$ exist

either: f is one-to-one

or f is increasing or (decreasing; proven in (a), when $x<0$)

(c) Most know the technique of interchange x and y and make y the subject and stop.

No consideration to the domain and range of $y=f(x)$ to get the correct form of $y=f^{-1}(x)$

(d) Poorly done.

There's confusion in thinking that $f^{-1}(x)$ as a many valued function with disastrous results again.