SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

2023

YEAR 12
TERM 1
HSC TASK 2

Maths Class:
Circle your class
12MX A B 1 2 P L U S

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- **In Questions 11 – 13, show ALL relevant mathematical reasoning and/or calculations**

Total Marks: 61

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 51 marks (pages 6 – 11)

- Attempt Questions 11 – 13
- Allow about 1 hour 15 minutes for this section

Examiner:

External Examiner

NOT TO BE REMOVED FROM EXAMINATION ROOM

Section I

10 marks

Attempt all questions.
Allow about 15 minutes for this section.
Use the multiple choice answer sheet for Questions 1 – 10.

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1 Jack starts at the origin and walks along the vector $2\underset{\sim}{i} + 3\underset{\sim}{j}$. He then turns and walks along the vector $4\underset{\sim}{i} - 2\underset{\sim}{j}$.

How far is Jack from the origin?

A. $\sqrt{11}$

B. 5

C. $\sqrt{37}$

D. $\sqrt{61}$

2 Oscar made an error proving that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. The proof is shown below.

Step 1: To prove true for $n = 1$

$2^1 + (-1)^{1+1} = 2 + 1 = 3 \times 1$ Line 1

Result is true for $n = 1$

Step 2: Assume true for $n = k$

$2^k + (-1)^{k+1} = 3M$ (M an integer) Line 2

Step 3: To prove true for $n = k + 1$

$2^{k+1} + (-1)^{k+1+1} = 2(2^k) + (-1)^{k+2}$ Line 3

$= 2[3M + (-1)^{k+1}] + (-1)^{k+2}$ Line 4

$= 2 \times 3M + 2 \times (-1)^{k+2} + (-1)^{k+2}$

$= 3[2M + (-1)^{k+2}]$

Which is a multiple of 3 since M and k are integers.

Step 4: True by induction

In which line did Oscar make an error?

A. Line 1

B. Line 2

C. Line 3

D. Line 4

3 Let $P(x) = qx^3 + x^2 + rx + q$, where q and r are constants and $q \neq 0$.
One of the zeros of $P(x)$ is -1.
Given that α is a zero of $P(x)$, $\alpha \neq -1$, which one of the following is also a zero?

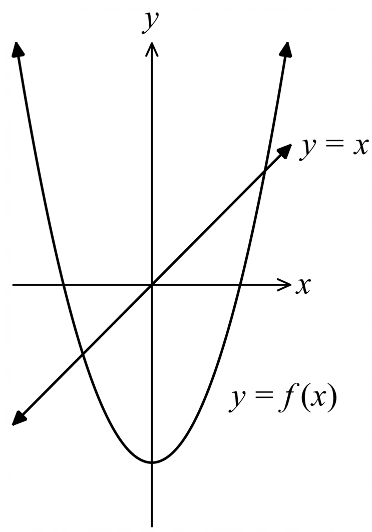
A. $-\dfrac{1}{\alpha}$

B. $-\dfrac{q}{\alpha}$

C. $\dfrac{1}{\alpha}$

D. $\dfrac{q}{\alpha}$

4



The graph of $y = f(x)$ and the graph of $y = x$ are shown above.

How many real solutions does the equation $[f(x)]^2 = x$ have?

A. 1

B. 2

C. 3

D. 4

5 The graph of the even polynomial $P(x)$ passes through the point (1, 2).

What is the remainder when $P(x)$ is divided by $(x + 1)$?

A. –2

B. –1

C. 1

D. 2

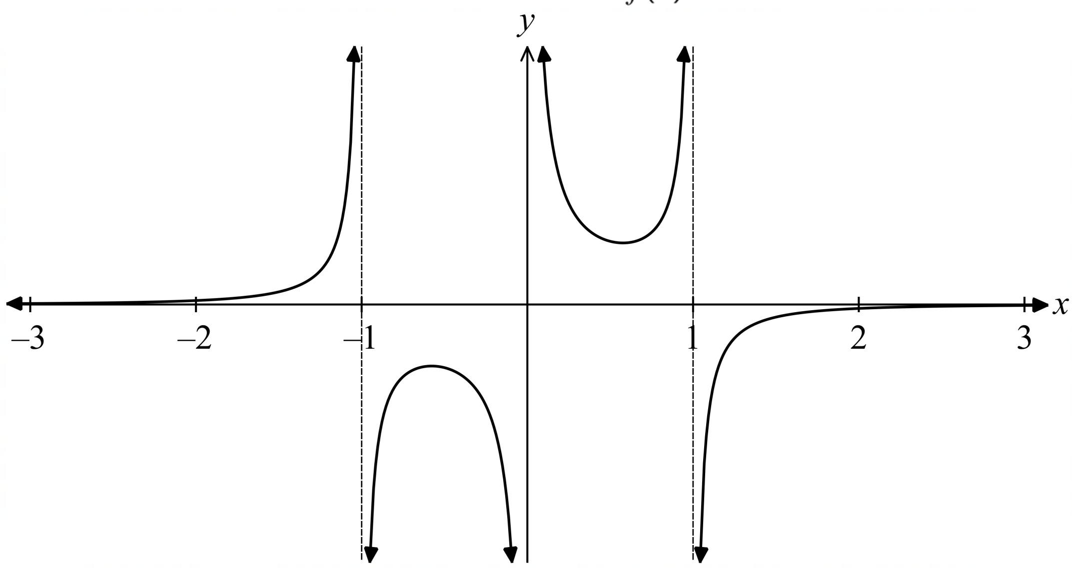
6 The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

What is the value of $\frac{dy}{dx}$ at the point (1, 2)?

A. $\frac{1}{3}$ B. $\frac{1}{2}$

C. $\frac{3}{4}$ D. 1

7 The graph below shows the graph of the function $y = \frac{1}{f(x)}$.



Which of the following best represents the equation of $f(x)$?

A. $f(x) = x(x^2 - 1)$ B. $f(x) = x(1 - x^2)$

C. $f(x) = x^2(x^2 - 1)$ D. $f(x) = x^2(1 - x^2)$

8 Consider the vectors

$$\underset{\sim}{a} = e^P \underset{\sim}{i} + 3\underset{\sim}{j},\ \underset{\sim}{b} = 2\underset{\sim}{i} + 2e^P \underset{\sim}{j}, \text{ and } \underset{\sim}{c} = e^P \underset{\sim}{i} + \underset{\sim}{j}$$

Given that the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{c}$ is equal to the vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{c}$, what is the value(s) of P?

A. $P = 0, P = \ln 3$ B. $P = \ln 3$

C. $P = 0, P = \ln 2$ D. $P = \ln 2$

9 Consider the distinct vectors $\vec{p} = \begin{pmatrix} t-8 \\ 6 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} 3 \\ 2t \end{pmatrix}$.

What are the possible values of t so that $\vec{p}$ and $\vec{q}$ are parallel?

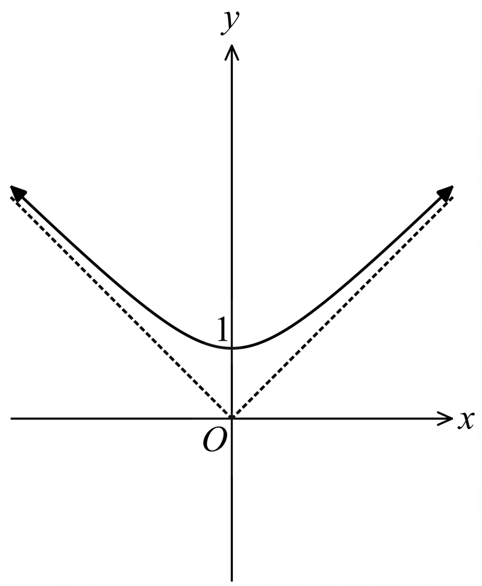
A. $-3, -11$

B. $-1, 9$

C. $1, -9$

D. $3, 11$

10



The diagram above shows the top branch of the hyperbola $y^2 - x^2 = 1$.
Three pairs of parametric equations are listed below:

1. $\begin{cases} x = \tan t \\ y = \sec t \end{cases}$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

2. $\begin{cases} x = t \\ y = \sqrt{1+t^2} \end{cases}$ for $-\infty < t < \infty$

3. $\begin{cases} x = \sqrt{t^2-1} \\ y = t \end{cases}$ for $1 < t < \infty$

How many of these parametric equations give a correct representation of the curve shown in the diagram?

A. 0

B. 1

C. 2

D. 3

End of Section I

Section II

51 marks

Attempt Questions 11–13
Allow about 1 hour 15 minutes for this section
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (18 marks) Start a NEW Answer Booklet

(a) The vectors $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -x \end{pmatrix}$ are perpendicular, where $x \in \mathbb{R}$. **2**
What is the value of x?

(b) A real polynomial equation has roots α, β, and γ. **3**
Write down a possible equation where

$$\alpha + \beta + \gamma = 2$$
$$\alpha^2 + \beta^2 + \gamma^2 = -6$$
$$\alpha\beta\gamma = -2$$

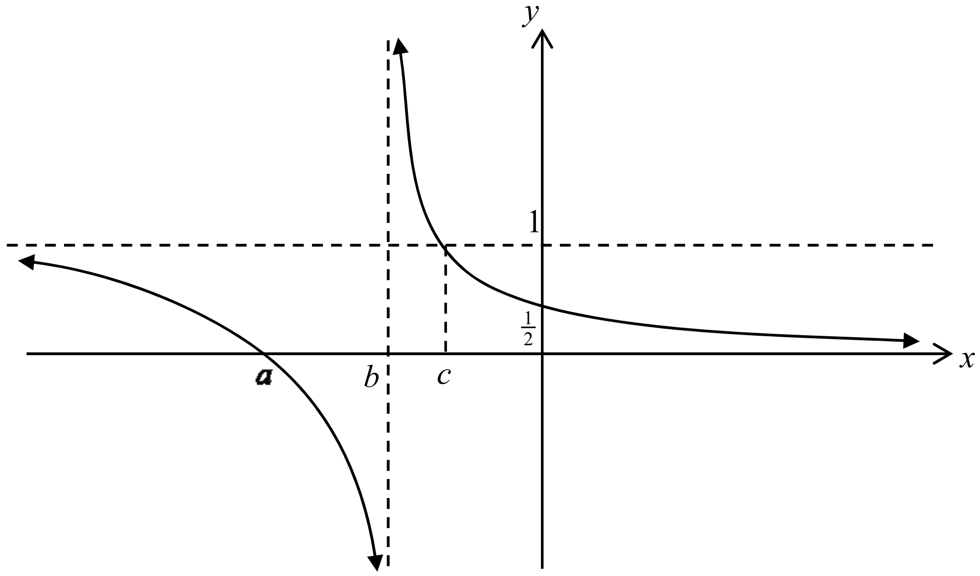
(c) The polynomial $P(x) = 8x^4 - 38x^3 + 9x^2 + ax + b$ has a double root at $x = 3$. **3**
Find the values of a and b, where a and b are real numbers.

(d) Use mathematical induction to prove that $7^{2n+1} + 11^{2n+1}$ is divisible by 3, **3**
for $n = 0, 1, 2, 3, \ldots$

Question 11 continues on page 7

Question 11 (continued)

(e) Below is the graph of $y = f(x)$.
The line $y = 1$ is a horizontal asymptote and $x = b$ is a vertical asymptote.
The x-intercept is at $x = a$ and the y-intercept is at $y = \frac{1}{2}$.

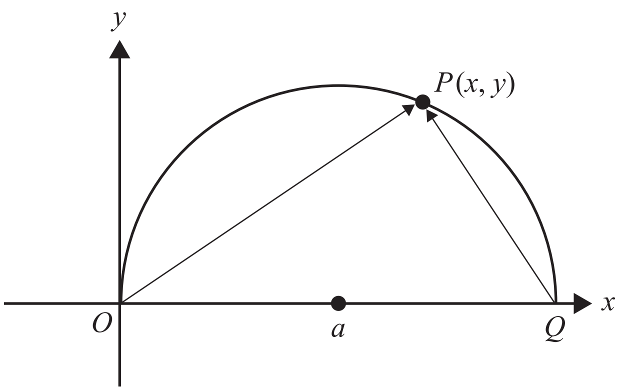


Neatly sketch the graphs of the following showing all important information, including the location of a, b, and c, where necessary.

(i) $y^2 = f(x)$. **2**

(ii) $y = f(-|x|)$. **2**

(f) OPQ is a semicircle of radius a with equation $y = \sqrt{a^2 - (x-a)^2}$.
$P(x, y)$ is a point on the semicircle OPQ, as shown below.



(i) Express the vectors $\overrightarrow{OP}$ and $\overrightarrow{QP}$ in terms of a, x, y, $\underset{\sim}{i}$ and $\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$, are unit vectors in the direction of the positive x and y axes respectively. **1**

(ii) Show that $\angle OPQ = 90°$. **2**

End of Question 11

Question 12 (17 marks) Start a NEW Answer Booklet

(a) Three coplanar forces of magnitudes 5 newtons, 7 newtons, and 10 newtons maintain a particle in equilibrium. **3**

Find the angle θ between the forces of magnitudes 5 newtons and 7 newtons. Leave your answer to the nearest degree.

(b) The polynomial $f(x)$ is given by **5**

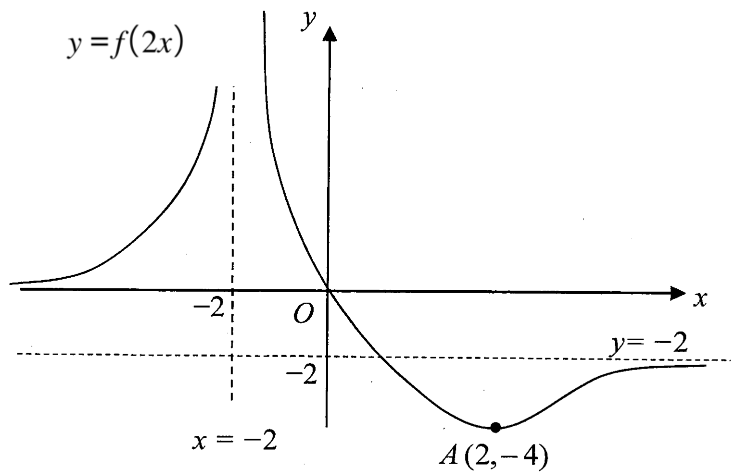
$$f(x) = x^3 + kx^2 - 7x - 15,$$

where k is a constant.
When $f(x)$ is divided by $(x + 1)$ the remainder is r, where r is a constant.

When $f(x)$ is divided by $(x - 3)$ the remainder is $3r$.

Show that $f(x) = 0$ has only one real root, and find that root.

(c) The diagram shows the graph of $y = f(2x)$. **3**



The lines $x = -2$ and $y = -2$ are asymptotes to the curve.
The minimum point A has coordinates $(2, -4)$ and the curve passes through the origin $(0, 0)$.

Sketch the graph of

$$y = \frac{1}{f(x)},$$

indicating clearly the equations of any asymptotes, intercepts with the axes and turning points.

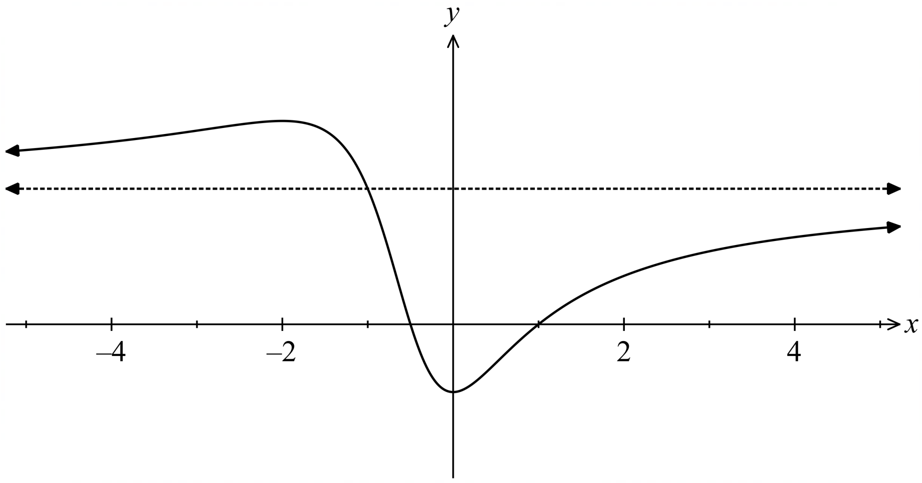
Question 12 continues on page 9

(d) Quadrilateral $OPQR$ has vertices $O(0, 0)$, $P(10, 5)$, $Q(12, 16)$, and $R(2, 11)$.

(i) Using vector methods prove that $\frac{1}{2}\overrightarrow{OQ} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PR}$. **1**

(ii) Hence, show that $OPQR$ is a rhombus. **3**

(e) The graph of $y = \dfrac{2x^2 - x - 1}{x^2 + x + 1}$ is drawn below. **2**



Note: There are two stationary points at $x = 0$ and $x = -2$.

By sketching $y = \left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right|$, find the value(s) of k, $k \in \mathbb{R}$, where

$$\left|\frac{2x^2 - x - 1}{x^2 + x + 1}\right| = k$$

has four distinct solutions.

End of Question 12

Question 13 (16 marks) Start a NEW Answer Booklet

(a) (i) Prove by mathematical induction that, for all positive integers n, **3**

$$1 \times 2 \times 3 + 2 \times 3 \times 5 + \ldots + n(n+1)(2n+1) = \frac{1}{2}n(n+1)^2(n+2)$$

(ii) Hence, show that, for all positive integers n, **2**

$$n(n+1)(2n+1) + (n+1)(n+2)(2n+3) + \ldots + 2n(2n+1)(4n+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$$

where a, b, c, and d are integers to be determined.

(b) (i) Show that $(x + \tan\theta)$ and $(x - \cot\theta)$ are factors of $x^2 - 2(\cot 2\theta)x - 1 = 0$. **3**

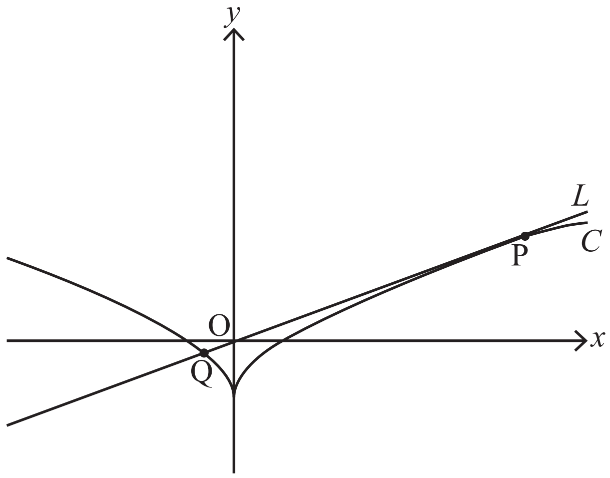
(ii) Hence, show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$. **2**

(iii) Hence, find the exact value of $\cot\frac{\pi}{16} - \tan\frac{\pi}{16}$. **1**

Question 13 continues on page 11

(c) The following diagram shows part of the curve C with parametric equations

$$\begin{cases} x = t^3 \\ y = t^2 - 1 \end{cases} \quad t \in \mathbb{R}$$



The line L passes through the origin O and is tangential to C at the point $P\left(p^3, p^2 - 1\right)$, where $p > 0$. The line L intersects C again at Q.

(i) Show that L has equation $y = \dfrac{2}{3\sqrt{3}}x$. **3**

(ii) Determine the exact coordinates of Q. **2**

End of paper

SYDNEY BOYS HIGH SCHOOL

2023 | YEAR 12 | TASK 2

Mathematics Extension 1

Sample Solutions

NOTE: Some of you may be disappointed with your mark.

This process of 'checking your mark' is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.
Just because you have shown 'working' does not justify that what you have written is worth any marks.

MC Answers

1	C	**6**	D
2	D	**7**	B
3	C	**8**	A
4	B	**9**	B
5	D	**10**	C

Section I Multiple Choice Solutions

1 Jack starts at the origin and walks along the vector $2\underset{\sim}{i}+3\underset{\sim}{j}$. He then turns and walks along the vector $4\underset{\sim}{i}-2\underset{\sim}{j}$.

How far is Jack from the origin?

$$\left|\left(2\underset{\sim}{i}+3\underset{\sim}{j}\right)+\left(4\underset{\sim}{i}-2\underset{\sim}{j}\right)\right| = \left|6\underset{\sim}{i}+\underset{\sim}{j}\right|$$
$$= \sqrt{37}$$

A. $\sqrt{11}$

B. 5

C. $\sqrt{37}$

D. $\sqrt{61}$

2 Oscar made an error proving that $2^n+(-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. The proof is shown below.

Step 1: To prove true for $n = 1$

$2^1+(-1)^{1+1}=2+1=3\times 1$ Line 1

Result is true for $n = 1$

Step 2: Assume true for $n = k$

$2^k+(-1)^{k+1}=3M$ (M an integer) Line 2

Step 3: To prove true for $n = k + 1$

$2^{k+1}+(-1)^{k+1+1}=2\left(2^k\right)+(-1)^{k+2}$ Line 3

$=2\left[3M+(-1)^{k+1}\right]+(-1)^{k+2}$ Line 4

$=2\times 3M+2\times(-1)^{k+2}+(-1)^{k+2}$

$=3\left[2M+(-1)^{k+2}\right]$

Which is a multiple of 3 since M and k are integers.

Step 4: True by induction

In which line did Oscar make an error?

A. Line 1

B. Line 2

C. Line 3

D. Line 4

$2^k+(-1)^{k+1}=3M \Rightarrow 2^k=3M-(-1)^{k+1}$

Now:

$$3M-(-1)^{k+1}=3M+(-1)(-1)^{k+1}$$
$$=3M+(-1)^{k+2}$$

So the line after 'Line 4' is correct, meaning that 'Line 4' is the only incorrect statement.

3 Let $P(x) = qx^3 + x^2 + rx + q$, where q and r are constants and $q \neq 0$.
One of the zeros of $P(x)$ is -1.
Given that α is a zero of $P(x)$, $\alpha \neq -1$, which one of the following is also a zero?

Let β be the other zero.

Consider the product of the roots: $\alpha \times \beta \times (-1) = -\dfrac{q}{q} = -1$

$\therefore \beta = \dfrac{1}{\alpha}$

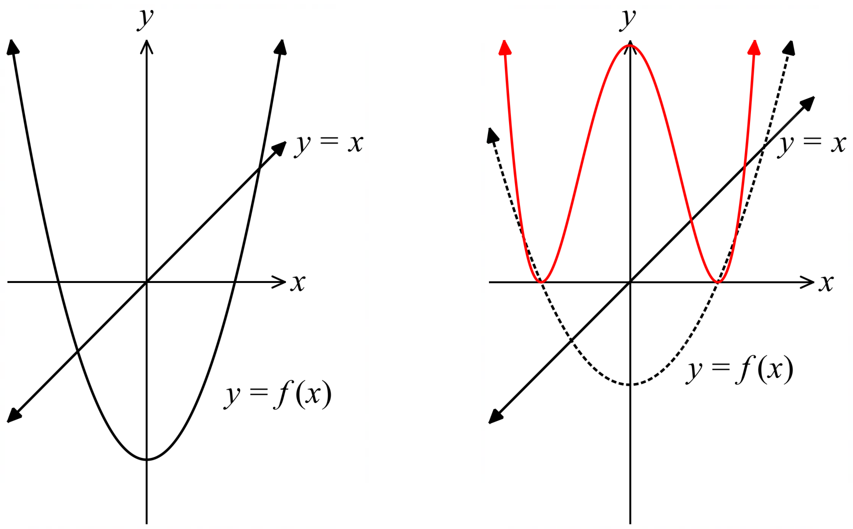
A. $-\dfrac{1}{\alpha}$

B. $-\dfrac{q}{\alpha}$

C. $\dfrac{1}{\alpha}$

D. $\dfrac{q}{\alpha}$

4



The graph of $y = f(x)$ and the graph of $y = x$ are shown above.

How many real solutions does the equation $[f(x)]^2 = x$ have?

A. 1

B. 2

C. 3

D. 4

5 The graph of the even polynomial $P(x)$ passes through the point (1, 2) i.e. $P(1)=2$

What is the remainder when $P(x)$ is divided by $(x+1)$? i.e. $P(-1)=?$

Being even: $P(-1)=P(1)=2$

A. –2

B. –1

C. 1

D. 2

6 The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

What is the value of $\frac{dy}{dx}$ at the point (1, 2)?

$$\therefore\ 3x^2 - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3x \times \frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2 - 3x\right) = 3x^2 + 3y$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 + 3y}{3y^2 - 3x}$$

At (1, 2): $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\times 1 + 3\times 2}{3\times 2^2 - 3\times 1} = \frac{9}{9} = 1$$

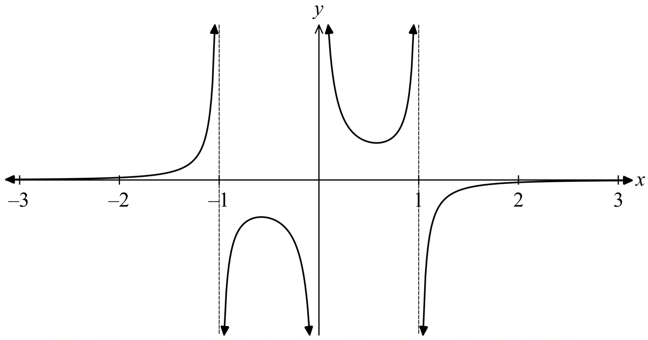
A. $\frac{1}{3}$

B. $\frac{1}{2}$

C. $\frac{3}{4}$

D. 1

7 The graph below shows the graph of the function $y = \dfrac{1}{f(x)}$.



Which of the following best represents the equation of $f(x)$?

A. $f(x) = x(x^2 - 1)$

B. $f(x) = x(1 - x^2)$

C. $f(x) = x^2(x^2 - 1)$

D. $f(x) = x^2(1 - x^2)$

The reciprocal of an odd function is an odd function. So C and D are out.

As $x \to \infty, \dfrac{1}{f(x)} \to 0^-$, so as $x \to \infty, f(x) \to -\infty$

8 Consider the vectors

$$\underset{\sim}{a} = e^P \underset{\sim}{i} + 3\underset{\sim}{j},\ \underset{\sim}{b} = 2\underset{\sim}{i} + 2e^P \underset{\sim}{j}, \text{ and } \underset{\sim}{c} = e^P \underset{\sim}{i} + \underset{\sim}{j}$$

Given that the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{c}$ is equal to the vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{c}$, what is the value(s) of P?

A. $P = 0, P = \ln 3$

B. $P = \ln 3$

C. $P = 0, P = \ln 2$

D. $P = \ln 2$

$\text{proj}_{\underset{\sim}{c}}\underset{\sim}{b} = (\underset{\sim}{b} \cdot \hat{\underset{\sim}{c}})\hat{\underset{\sim}{c}}$

$\text{proj}_{\underset{\sim}{c}}\underset{\sim}{a} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{c}})\hat{\underset{\sim}{c}}$

$\text{proj}_{\underset{\sim}{c}}\underset{\sim}{a} = \text{proj}_{\underset{\sim}{c}}\underset{\sim}{b} \Rightarrow \underset{\sim}{a} \cdot \hat{\underset{\sim}{c}} = \underset{\sim}{b} \cdot \hat{\underset{\sim}{c}} \Rightarrow \underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{c}$

$\underset{\sim}{a} \cdot \underset{\sim}{c} = e^{2P} + 3$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = 2e^P + 2e^P = 4e^P$

$P = 0$ is an obvious solution here, so A or C

$\therefore e^{2P} + 3 = 4e^P \Rightarrow (e^P - 3)(e^P - 1) = 0$

$\therefore e^P = 1, 3$

$\therefore P = 0, \ln 3$

9 Consider the distinct vectors $\vec{p} = \begin{pmatrix} t-8 \\ 6 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} 3 \\ 2t \end{pmatrix}$.

What are the possible values of t so that $\vec{p}$ and $\vec{q}$ are parallel?

A. $-3, -11$

B. $-1, 9$

C. $1, -9$

D. $3, 11$

D is the obvious red herring.

We need $\vec{p} = \lambda\vec{q}, \lambda \in {}^{\sim}$

$$\therefore \begin{pmatrix} t-8 \\ 6 \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 2t \end{pmatrix}$$

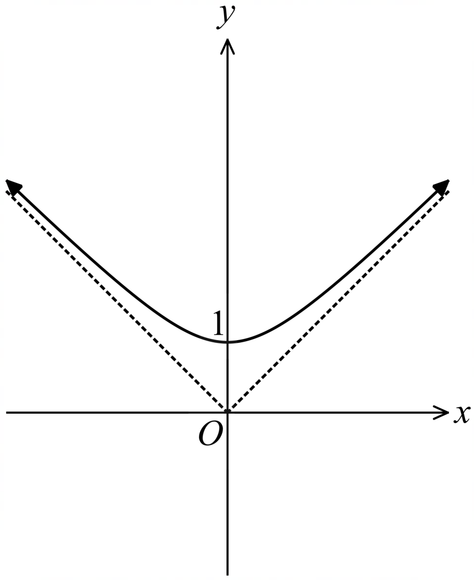
$$\therefore \begin{cases} t-8 = 3\lambda \\ 3 = \lambda t \end{cases}$$

$$\therefore \frac{3}{t} = \frac{t-8}{3} \Rightarrow t^2 - 8t = 9$$

$$\therefore (t-9)(t+1) = 0$$

$$\therefore t = 9, -1$$

10



The diagram above shows the top branch of the hyperbola $y^2 - x^2 = 1$.
Three pairs of parametric equations are listed below:

1. $\begin{cases} x = \tan t \\ y = \sec t \end{cases}$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

2. $\begin{cases} x = t \\ y = \sqrt{1+t^2} \end{cases}$ for $-\infty < t < \infty$

3. $\begin{cases} x = \sqrt{t^2 - 1} \\ y = t \end{cases}$ for $1 < t < \infty$

How many of these parametric equations give a correct representation of the curve shown in the diagram?

Eqn (3) only has $x \geqslant 0$

A. 0

B. 1

C. 2

D. 3

2023 Year 12 HSC Task 1 Ext 1 Q11 Solutions

Q11a

$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -x \end{pmatrix}$ are perpendicular.

$$\underset{\sim}{a} \bullet \underset{\sim}{b} = 0 \quad \therefore \quad x_1 x_2 + y_1 y_2 = 0$$
$$5 \times 4 + 2(-x) = 0$$
$$20 - 2x = 0$$
$$\therefore \quad x = 10$$

Criteria	Marks
Provide correct solution	**2**
Observe that the dot product is equal to zero, or equivalent merit	**1**

<u>Marker's Comment:</u> Very well done.

Only a handful of students did not know how to find the dot product and that if it is equal to zero, then the vectors are perpendicular.

Q11b

$$\alpha+\beta+\gamma=-\frac{b}{a}=2$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}$$

$$\alpha\beta\gamma=-\frac{d}{a}=-2$$

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)=-6$$

$$=2^2-2\left(\frac{c}{a}\right)=-6$$

$$-2\left(\frac{c}{a}\right)=-6-4=-10$$

$$\therefore \quad \frac{c}{a}=5$$

$$\therefore P(x)=x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}$$

$$=a\left(x^3-2x^2+5x+2\right)$$

$\therefore P(x)=x^3-2x^2+5x+2$ or equivalent.

Criteria	Marks
Provide correct solution	**3**
Use $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)=-6$ to evaluate $\frac{c}{a}=5$	**2**
Show the sum and the product of roots relationship to the expression	**1**

Marker's comments: Most students did very well on this question.

Some did not know how to find the sum of roots two at a time using $\alpha^2+\beta^2+\gamma^2$.

A few did not see the relevance between the sums and product of roots to the polynomial.

Q11c

$P(x) = 8x^4 - 38x^3 + 9x^2 + ax + b$

$P'(x) = 32x^3 - 114x^2 + 18x + a$

$P(3) = 8(3)^4 - 38(3)^3 + 9(3)^2 + 3a + b = 0$

$\therefore \quad 3a + b = 297 \quad *$

$P'(3) = 32(3)^3 - 114(3)^2 + 18(3) + a = 0$

$\therefore \quad -108 + a = 0$

$\therefore \quad a = 108$

Sub into * $\therefore b = 297 - 3 \times 108$

$\therefore b = -27$

$\therefore a = 108$ and $b = -27$

Criteria	Marks
Provide correct solution	**3**
Correctly differentiate $P(x)$ and simplify the equation of $P(3)$.	**2**
Correctly differentiate $P(x)$ or simplify the equation of $P(3)$.	**1**

Marker's comments: Mostly well done.

Few candidates used the sums and product of roots and polynomial division to answer this question, but few succeeded.

There were some calculation errors; students must be careful when using the calculator.

Some students need to revise the relationship between multiplicity and derivatives.

Q11d

$7^{2n+1}+11^{2n+1}$ is divisible by 3 for n = 0, 1, 2, 3, …

① Test true for $n = 0$

$7^1+11^1=18$ divisible by $3 \therefore$ true for n = 0

② Assume true for $n = k$

$7^{2k+1}+11^{2k+1}=3M$, for some integer, M.

③ Prove true for $n = k + 1$

$7^{2(k+1)+1}+11^{2(k+1)+1}=3N$, for some integer, N.

$7^{2k+2+1}+11^{2k+2+1}$

$7^{2k+1}\times 7^2+11^{2k+1}\times 11^2$

From ②

$7^{2k+1}=3M-11^{2k+1}$

$\therefore\left(3M-11^{2k+1}\right)49+\left(11^{2k+1}\right)121$

$=49(3M)-49\left(11^{2k+1}\right)+121\left(11^{2k+1}\right)$

$=49(3M)+72\left(11^{2k+1}\right)$

$=3\left(49M+24\left(11^{2k+1}\right)\right) \therefore$ divisible by 3

$\therefore$ By Mathematical Induction, the statement is true for n = 0, 1, 2, 3, …

Criteria	Marks
Provide correct proof by logically showing 3 as a factor	**3**
Proves the inductive step by using the assumption.	**2**
Establishes the base case, n = 0	**1**

Marker's comment: Several students did not start the proof with the base value of n = 0.

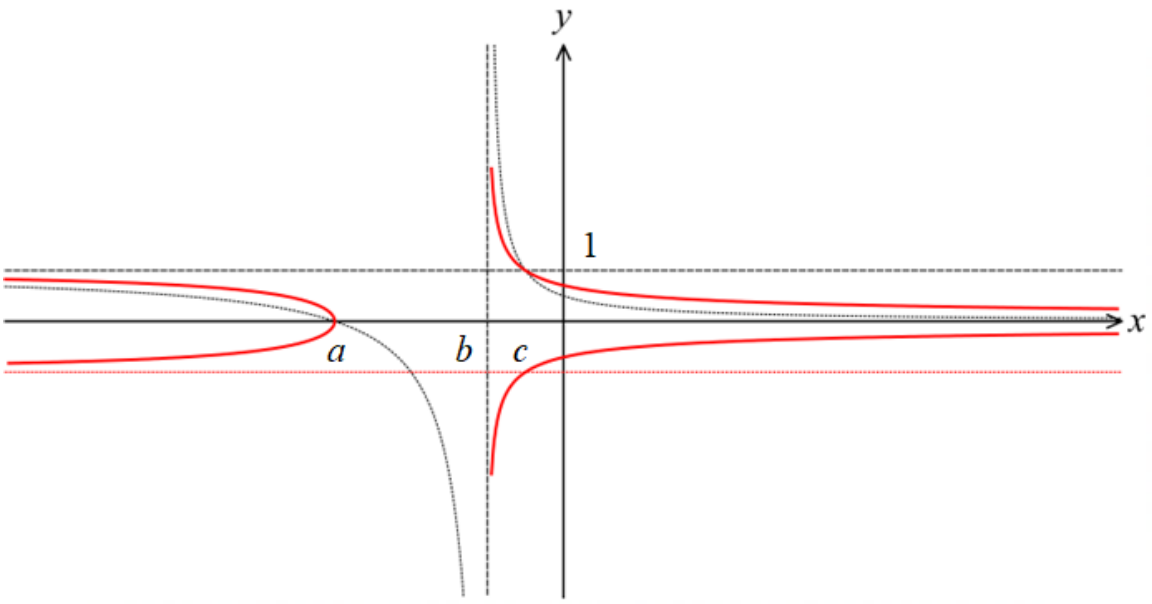
Most students understood the concept of MI and used the assumption to prove true for $n = k + 1$, but they shouldn't try to eliminate both 7^{2k+1} and 11^{2k+1} i.e. they substituted both $7^{2k+1}=3M-11^{2k+1}$ and $11^{2k+1}=3M-7^{2k+1}$ for the proof, this is meaningless.

Some students did not receive full marks even though they showed a common factor of 3 due to incorrect algebraic errors in their working.

There were a few students who made the error for $n = k + 1$, i.e., $2(k + 1) +1$, not simply $2k + 2$.

Q11e

(i) $y^2 = f(x)$



Criteria	Marks
Provides correct sketch	**2**
In addition to below, show the new y-intercept above the x-axis is greater than 0.5. In fact, the new y-intercepts are $\pm\dfrac{1}{\sqrt{2}}$. This is important information and must be included in the solution	**1.5**
Correctly show the reflection from the original graph about the x-axis $x \le a$ and $x > b$	**1**
The horizontal asymptote is also reflected $\therefore$ horizontal asymptotes at $y = 1$ and $y = -1$.	**0.5**

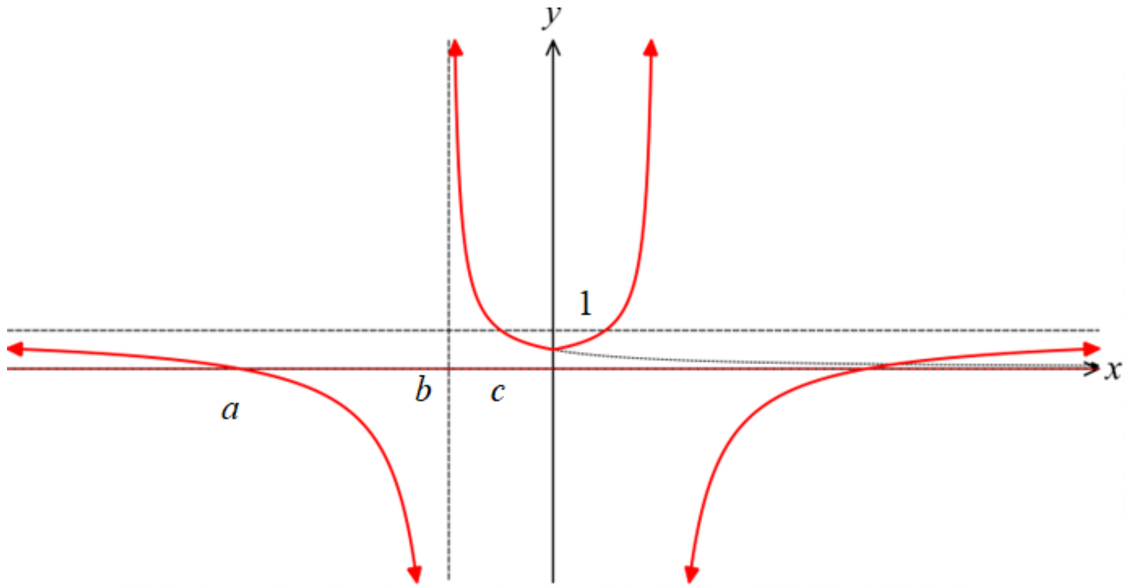
Marker's comments: Not very well done.

The graph should be narrower for $x \le a$; as indicated in the solution, some students drew a very wide-based graph (for the sideway parabola-shaped section).

Students in this situation will not be penalised if they draw the original diagram and show that they drew a wider graph to indicate that $y^2 > y$.

Q11e (continued)

(ii) $y = f(-|x|)$



Criteria	Marks
Provides correct sketch	**2**
Clearly indicate the vertical asymptotes $y = \pm b$ and the x-intercepts of $\pm a$	**1.5**
Correctly show the reflection from the left-hand side of the original graph about the y-axis $x < b$ or $b < x \leq 0$	**1**
Any reflection about the y-axis.	**0.5**

Marker's comment: Several students sketched the absolute value of x graph i.e. $f(\ |x|\ )$, which has shown some reflection about the y-axis; but does not indicate the other criteria of the correct solution.

It is worth noting that the point (0, 0.5) is not differentiable.
It is a point where the graph is reflected from left to right.

Q11f

(i)

$y=\sqrt{a^2-(x-a)^2}$ semi-circle, centra at $(a, 0)$ with radius equal to a units.

$\overrightarrow{OP}=x\underset{\sim}{i}+y\underset{\sim}{j}$

$\overrightarrow{OQ}=2a\underset{\sim}{i}$

$\overrightarrow{OQ}+\overrightarrow{QP}=\overrightarrow{OP}$

$\therefore \overrightarrow{QP}=x\underset{\sim}{i}+y\underset{\sim}{j}+2a\underset{\sim}{i}$

$\quad =(x-2a)\underset{\sim}{i}+y\underset{\sim}{j}$

Criteria	Marks
Provides correct both solutions $\overrightarrow{OP}$ and $\overrightarrow{QP}$	**1**
Provides one of the correct solutions $\overrightarrow{OP}$ or $\overrightarrow{QP}$	**0.5**

Marker's comment: Some students used $y=\sqrt{a^2-(x-a)^2}$ for $x\underset{\sim}{i}+y\underset{\sim}{j}$ but y is the simplest form, and the question did mention to use y.

(ii)

For $\angle OPQ=90°$ then:

$x(x-2a)+y^2=x^2-2ax+y^2$

$\quad =x^2-2ax+\left(a^2-(x-a)^2\right)$

Since $y=\sqrt{a^2-(x-a)^2}$

$=x^2-2ax+\left(a^2-\left(x^2-2ax+a^2\right)\right)$

$=x^2-2ax+\left(a^2-x^2+2ax-a^2\right)$

$=x^2-2ax-x^2+2ax$

$=0$

$\therefore \overrightarrow{OP}\bullet\overrightarrow{QP}=0$ hence, $\angle OPQ=90°$.

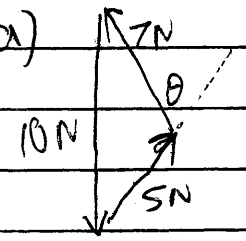
Criteria	Marks
Provide correct solution	**2**
Attempts to find the dot product of $\overrightarrow{OP}$ and $\overrightarrow{QP}$ using the equation of the semi-circle.	**1**

Marker's comment: Mostly well done.

Students must fully show how $\angle OPQ=90°$.

Some simply stated the circle geometry property and that is not showing the required angle is equal to the right angle.

Question 12

(a) 

These vectors form a triangle. Notice that the angle between the vectors is the exterior angle (not the interior angle).

Cosine rule: $2\times5\times7\cos(180°-\theta) = 5^2+7^2-10^2$

$180°-\theta = \cos^{-1}\left(\frac{-26}{35}\right)$

$= 112°$ to nearest degree

So $\theta = 180° - 112°$

$= 68°$ to nearest degree.

Forming a closed triangle was an easier way then equating components, which many students did. Also remember the definition of the angle between vectors (you place them tail to tail).

Question 12

(b) $f(x) = x^3 + kx^2 - 7x - 15$

$f(-1) = -1 + k + 7 - 15$ (is the remainder when dividing $f(x)$ by $x+1$, by Remainder Theorem)

$= k - 9$

So $r = k - 9$ ①

$f(3) = 27 + 9k - 21 - 15$ (remainder of $f(x) \div (x-3)$)

$= 9k - 9$

So $3r = 9k - 9$ ②

Then ① $= \frac{②}{3}$, ie. $k - 9 = 3k - 3$

$2k = -6$

$\therefore k = -3$

So $f(x) = x^3 - 3x^2 - 7x - 15$

Check factors of -15:

$f(5) = 125 - 75 - 35 - 15$

$= 0$

$\therefore x = 5$ is a zero of $f(x)$.

Factorise $f(x)$:

$$
\begin{array}{r}
x^2 + 2x + 3 \\
x - 5 \overline{) x^3 - 3x^2 - 7x - 15} \\
\underline{x^3 - 5x^2} \\
2x^2 - 7x \\
\underline{2x^2 - 10x} \\
3x - 15 \\
\underline{3x - 15} \\
0
\end{array}
$$

So $f(x) = (x-5)(x^2+2x+3)$

Δ of x^2+2x+3: $\Delta = 4 - 12 = -8 < 0$

$\therefore$ No real zeroes of x^2+2x+3)

$\therefore f(x) = 0$ has only one real root: $x = 5$.

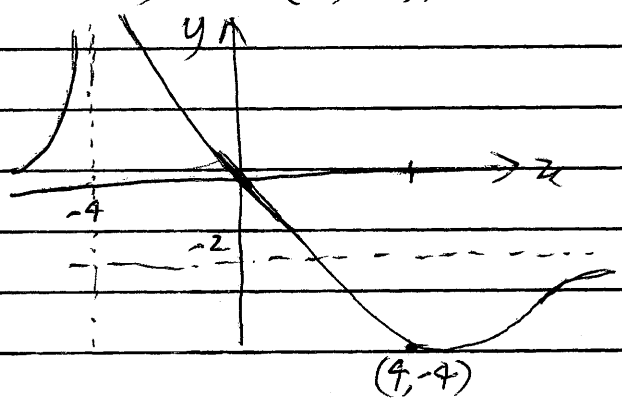
Make sure you find k first then subsitute its value into the function definition. This is where some students fell down.

Also beware the when $f(x) = (x+1)Q_1(x) + r$ and $f(x) = (x-3)Q_2(x) + 3r$ that $Q_1(x) \neq Q_2(x)$,

Question 12

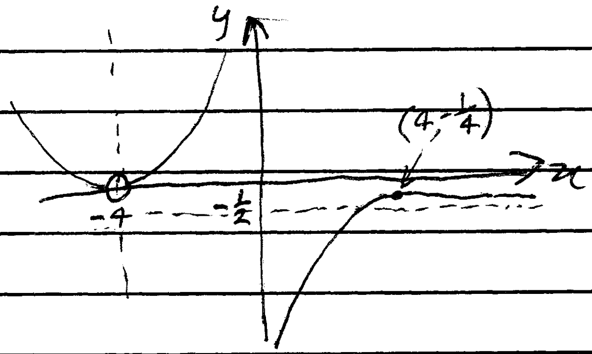
(c) As a first step draw $y = f(x)$. This will expand the graph horizontally about the y axis. To check this, see $f(2(2)) = -4$, ie $f(4) = -4$. So point A moves from $(2, -4)$ to $(4, -4)$, for $f(2x)$ to $f(x)$

~~Next~~



Then for $y = \frac{1}{f(x)}$ rememember $\lim_{x \to a} f(x) = \infty$ goes to $\lim_{x \to a} \frac{1}{f(x)} = 0$ and $\lim_{x \to a} f(x) = 0$ goes to $\lim_{x \to a} \frac{1}{f(x)} = \infty$.

The horizontal asymptote will also be reciprocated. $f(-4)$ is not defined so neither will $\frac{1}{f(-4)}$ be.



asymptotes: $x = -4$

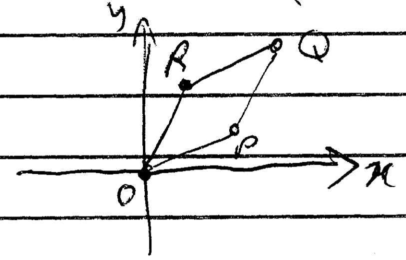
$y = -\frac{1}{2}$

local max at $(4, -\frac{1}{4})$.

There are no intercepts with the axes.

Question 12

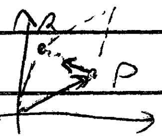
(d)(i)



$$RHS = \vec{OP} + \tfrac{1}{2}\vec{PR}$$
$$= \langle 10,5\rangle + \tfrac{1}{2}\langle 2-10, 11-5\rangle$$
$$= \langle 10,5\rangle + \langle -4, +3\rangle$$
$$= \langle 6, 8\rangle$$
$$= \tfrac{1}{2}\langle 12, 16\rangle$$
$$= \tfrac{1}{2}\vec{OQ}$$
$$= LHS.$$

Starting with the complicated side makes things easier. Please explicitly equate both sides otherwise you might not get full marks.

(ii) From (i), obviously $\frac{1}{2}\vec{OQ}$ goes to the midpoint of $\vec{OQ}$ and $\vec{OP}+\frac{1}{2}\vec{PR}$ goes to the midpoint of the diagonal $\vec{PR}$.



$\therefore$ The midpoints coincide, and thus the diagonals bisect each other.
So the quadrilateral is a parallelogram.
To then prove it is a rhombus, we have two paths.
The most elegant path is to prove the diagonals are perpendicular

$$\vec{OQ}\cdot\vec{PR} = \langle 12,16\rangle \cdot \langle -8, 6\rangle$$
$$= -96 + 96$$
$$= 0$$

$\therefore \vec{OQ}$ and $\vec{PR}$ are $\perp$

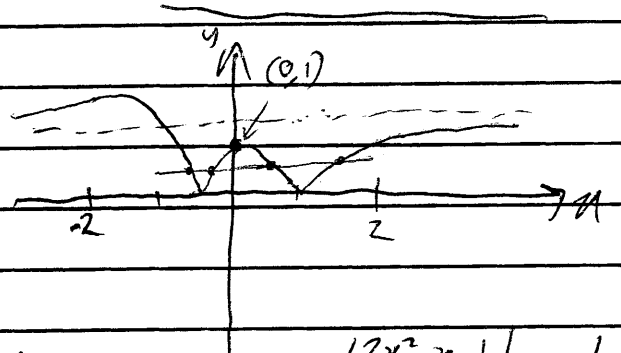
$\therefore$ OPQR is a rhombus.

Other way could have shown that a pair of adjacent sides are equal in length
e.g. $|\vec{OP}| = \sqrt{125} = |\vec{PQ}|$

Note: This was a "hence" not "hence or otherwise" question.

Question 12

(e) local max now at (0,1).



At $x=0$, $y=\left|\frac{2x^2-x-1}{x^2+x+1}\right|=\left|\frac{-1}{1}\right|=1$.

We get 4 distinct solutions when a horizontal line intercepts the graph in 4 places. obviously from the graph this only occurs when $0<k<1$.

Question 13 Solutions

General comments

Factorise before expanding if possible.

Simplify at each step before proceeding.

Don't use $\frac{b}{a}$ for an abbreviation, instead write 'sum of roots'.

If you get an equation with $P^2 = Q$ then either $P = \pm\sqrt{Q}$ or you have to justify what P has to be.

Don't assume that the marker will just fill in the gaps of your proofs when you omit something that you think is too obvious.

(a) (i) Prove by mathematical induction that, for all positive integers n, **3**

$$1\times2\times3+2\times3\times5+\ldots+n(n+1)(2n+1)=\frac{1}{2}n(n+1)^2(n+2)$$

Test $n = 1$:

$$\text{LHS}=1\times2\times3=6$$

$$\text{RHS}=\frac{1}{2}(1)(1+1)^2(1+2)=6$$

NOTE: You have to show this.

$\therefore$ true for $n = 1$

Assume true for $n = k$:

$$1\times2\times3+2\times3\times5+\ldots+k(k+1)(2k+1)=\frac{1}{2}k(k+1)^2(k+2)$$

Need to prove true for $n = k + 1$:

$$1\times2\times3+\ldots+k(k+1)(2k+1)+(k+1)(k+2)(2k+3)=\frac{1}{2}(k+1)(k+2)^2(k+3)$$

$$\text{LHS}=1\times2\times3+\ldots+k(k+1)(2k+1)+(k+1)(k+2)(2k+3)$$

$$=\frac{1}{2}k(k+1)^2(k+2)+(k+1)(k+2)(2k+3)$$

$$=\frac{1}{2}(k+1)(k+2)[k(k+1)+2(2k+3)]$$

$$=\frac{1}{2}(k+1)(k+2)\left(k^2+5k+6\right)$$

NOTE: You cannot jump from something like $k^2 + k + 4k + 6$ to the final answer.

$$=\frac{1}{2}(k+1)(k+2)(k+2)(k+3)$$

$$=\frac{1}{2}(k+1)(k+2)^2(k+3)$$

$$=\text{RHS}$$

By the principle of mathematical induction, the formula is true for all positive integers n.

Comments No half marks except where the setting out was bad enough to deduct 0.5 marks.

This question was not a hard induction question.

Induction proofs by their very nature are "Show that"-proofs as you have the answer/result you intend on proving. Students need to realise this.

What is obvious is that many students have lost their Y9 skills of factorising.

Students' setting out of induction proofs has to improve, including using LHS & RHS properly.

(a) (ii) Hence, show that, for all positive integers n, **2**

$$n(n+1)(2n+1)+(n+1)(n+2)(2n+3)+\ldots+2n(2n+1)(4n+1)=\frac{1}{2}n(n+1)(an+b)(cn+d)$$

where a, b, c, and d are integers to be determined.

Let $f(n)=\sum_{k=1}^{n} k(k+1)(2k+1)$

$\therefore\ f(n)=\frac{1}{2}n(n+1)^2(n+2)$

$$n(n+1)(2n+1)+(n+1)(n+2)(2n+3)+\ldots+2n(2n+1)(4n+1)=f(2n)-f(n-1)$$

$$\begin{aligned}
f(2n)-f(n-1) &= \frac{1}{2}(2n)(2n+1)^2(2n+2)-\frac{1}{2}(n-1)(n)(n+1) \\
&= 2(n)(2n+1)^2(n+1)-\frac{1}{2}(n-1)(n)(n+1) \\
&= \frac{1}{2}n(n+1)\left[4(2n+1)^2-n(n-1)\right] \\
&= \frac{1}{2}n(n+1)\left(15n^2+17n+4\right) \\
&= \frac{1}{2}n(n+1)(3n+1)(5n+4)
\end{aligned}$$

This is in the required form

Comments No half marks

This question starts with “Hence, …”. This does not mean you now start another mathematical induction, this is also confirmed by the mark value.

Many students got confused by the phrase “ *where a, b, c, and d are integers to be determined*”. Students have to determine these values!

Also, some confused this for a True or False question.

(b) (i) Show that $(x+\tan\theta)$ and $(x-\cot\theta)$ are factors of $x^2-2(\cot 2\theta)x-1$. **3**

Method 1a:

$$(x+\tan\theta)(x-\cot\theta)=x^2+(\tan\theta-\cot\theta)x-\tan\theta\times\cot\theta$$

$$=x^2+\left(\tan\theta-\frac{1}{\tan\theta}\right)x-1$$

$$=x^2+2\left(\frac{\tan^2\theta-1}{2\tan\theta}\right)x-1$$

$$=x^2-2\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)x-1$$

$$=x^2-2\left(\frac{1}{\tan 2\theta}\right)x-1$$

$$=x^2-2(\cot 2\theta)\,x-1$$

Method 1a:

$$x^2-2(\cot 2\theta)x-1=x^2-2\left(\frac{1}{\tan 2\theta}\right)x-1$$

$$=x^2-\cancel{2}\left(\frac{1-\tan^2\theta}{\cancel{2}\tan\theta}\right)x-1$$

$$=x^2-\left(\frac{1}{\tan\theta}-\tan\theta\right)x-\cot\theta\tan\theta$$

$$=x^2-(\cot\theta-\tan\theta)x-\cot\theta\tan\theta$$

$$=(x-\cot\theta)(x+\tan\theta)$$

Comments Students did not need to write $\cot 2\theta$ in terms of $\cot\theta$, but they certainly needed to have written down $\cot 2\theta$ in terms of $\tan\theta$.
Either students already knew the formula for $\tan 2\theta$ or they could have used their Reference sheet to help.

A lot of students do not know the difference between roots and factors.
If $x=a$ is a root of $f(x)=0$ then $f(a)=0$ and so $(x-a)$ is a factor of $f(x)$.
This time it wasn't penalised.

Some students couldn't get full marks as they only considered the sum of the roots and did not take in to account the product of the roots. Don't assume that this is obvious. It may be trivial, but you must show your whole argument.

(b) (i) **Method 2:** If $f(a)=0$ then $(x-a)$ is a factor of $f(x)$.

Let $f(x)=x^2-2(\cot 2\theta)x-1$

Substitute $x=-\tan\theta$:

$$(-\tan\theta)^2-2(\cot 2\theta)(-\tan\theta)-1=\tan^2\theta+\frac{2\tan\theta}{\tan 2\theta}-1$$

$$=\tan^2\theta+2\tan\theta\times\frac{1-\tan^2\theta}{2\tan\theta}-1$$

$$=\tan^2\theta+1-\tan^2\theta-1$$

$$=0$$

Substitute $x=\cot\theta$:

$$(\cot\theta)^2-2(\cot 2\theta)(\cot\theta)-1=\cot^2\theta-\frac{2}{\tan\theta\tan 2\theta}-1$$

$$=\cot^2\theta-\frac{2}{\tan\theta}\times\frac{1-\tan^2\theta}{2\tan\theta}-1$$

$$=\cot^2\theta-\frac{1-\tan^2\theta}{\tan^2\theta}-1$$

$$=\cot^2\theta-\cot^2\theta+1-1$$

$$=0$$

So $(x+\tan\theta)$ and $(x-\cot\theta)$ are factors of $x^2-2(\cot 2\theta)x-1$.

Comments Students who chose to do it this way, found it harder to get all the marks as there was a lot more work for each part with not many marks.

Students should be referencing the factor theorem or equivalent.
Some students just showed e.g. $f(-\tan\theta)=0$ and no discussion of what they did.

They are relying on the fact that the marker knows their logic. You can't do this!

(b) (ii) Hence, show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$. **2**

Method 1: Let $\theta = \frac{\pi}{8}$

From (i), $\left(x + \tan\frac{\pi}{8}\right)$ is a factor of $x^2 - 2\left(\cot 2 \times \frac{\pi}{8}\right)x - 1$

i.e. $x^2 - 2\left(\cot\frac{\pi}{4}\right)x - 1 = x^2 - 2x - 1$

$\therefore\ -\tan\frac{\pi}{8}$ is a root of $x^2 - 2x - 1 = 0$

The roots of $x^2 - 2x - 1 = 0$ are $\dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

As $-\tan\frac{\pi}{8} < 0$ then $-\tan\frac{\pi}{8} = 1 - \sqrt{2}$.

$\therefore \tan\frac{\pi}{8} = \sqrt{2} - 1$

Method 2: Let $x = -\tan\frac{\pi}{8}$

As $x = -\tan\frac{\pi}{8}$ is a root (from part (i)) then

$$\left(-\tan\frac{\pi}{8}\right)^2 - 2\left(\cot\frac{2\pi}{8}\right)\left(-\tan\frac{\pi}{8}\right) - 1 = 0$$

$\therefore \tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$

$\therefore \tan\frac{\pi}{8} = \dfrac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$

$\therefore \tan\frac{\pi}{8} = -1 + \sqrt{2}$ $\left[\frac{\pi}{8} \text{ is in the 1st quad.}\right]$

Comments Students who chose method 2 generally forgot to change the sign of the coefficient of x or wrote something like $\left(-\tan\frac{\pi}{8}\right)^2 = -\tan^2\frac{\pi}{8}$.

This was a "Hence…" question, as a result students who didn't may links could not get full marks.

(b) (iii) Hence, find the exact value of $\cot\dfrac{\pi}{16} - \tan\dfrac{\pi}{16}$. **1**

If α and β are the roots of $x^2 - 2(\cot 2\theta)x - 1$ then $\alpha + \beta = 2\cot 2\theta$.

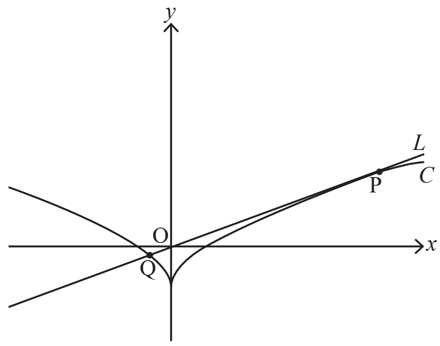
From (i), $\cot\dfrac{\pi}{16} - \tan\dfrac{\pi}{16}$ is the sum of the roots of $x^2 - 2\left(\cot 2 \times \dfrac{\pi}{16}\right)x - 1 = 0$.

$\therefore \cot\dfrac{\pi}{16} - \tan\dfrac{\pi}{16} = 2\cot\dfrac{\pi}{8} = \dfrac{2}{\sqrt{2}-1}$ or $2\left(\sqrt{2}+1\right)$

Comments This was a "Hence " question as a result students who didn't may links could not get full marks.
This question was worth only 1 mark, so if students can look for the links then they are avoiding a lot of work.

(c) The following diagram shows part of the curve C with parametric equations

$$\begin{cases} x = t^3 \\ y = t^2 - 1 \end{cases} \quad t \in \mathbb{R}$$



The line L passes through the origin O and is tangential to C at the point $P\left(p^3, p^2 - 1\right)$, where $p > 0$. The line L intersects C again at Q.

(i) Show that L has equation $y = \dfrac{2}{3\sqrt{3}}x$. **3**

$\dfrac{dx}{dt} = 3t^2$ and $\dfrac{dy}{dt} = 2t$

$\therefore \dfrac{dy}{dx} = \dfrac{2t}{3t^2} = \dfrac{2}{3t}$

At $P(t = p)$: $\dfrac{dy}{dx} = \dfrac{2}{3p}$

So the tangent at P is $y - \left(p^2 - 1\right) = \dfrac{2}{3p}\left(x - p^3\right)$

But the line passes through the origin.

$$0 - \left(p^2 - 1\right) = \frac{2}{3p}\left(0 - p^3\right) \Rightarrow 3p - 3p^3 = -2p^3$$

$$\therefore 3p = p^3$$

$$\therefore 3 = p^2 \quad (p \neq 0)$$

$$\therefore p = \sqrt{3} \quad (p > 0)$$

Passing through the origin, the tangent has equation $y = \dfrac{2}{3p}x = \dfrac{2}{3\sqrt{3}}x$

(c) (ii) Determine the exact coordinates of Q. **2**

The equation to find P and Q is found by substituting $\begin{cases} x = t^3 \\ y = t^2 - 1 \end{cases}$ into $y = \dfrac{2}{3\sqrt{3}}x$.

$$\therefore\ t^2 - 1 = \frac{2}{3\sqrt{3}}t^3$$

$$\therefore\ t^3 - \frac{3\sqrt{3}}{2}t^2 + \frac{3\sqrt{3}}{2} = 0$$

Let $t = q$ be the parameter for Q.

Method 1: As this is a cubic which has three roots, and P is a point of tangency, then $t = p$ must be a double root.

Using sum of roots $2p + q = \dfrac{3\sqrt{3}}{2}$:
As $p = \sqrt{3}$ then:

$$2\sqrt{3} + q = \frac{3\sqrt{3}}{2} \Rightarrow q = \frac{3\sqrt{3}}{2} - 2\sqrt{3}$$

$$\therefore\ q = -\frac{\sqrt{3}}{2}$$

With $Q\left(q^3,\, q^2 - 1\right)$ then $Q\left(-\dfrac{3\sqrt{3}}{8},\ -\dfrac{1}{4}\right)$.

(c) (ii) **Method 2:** $t = p$ must be a root of $t^3 - \frac{3\sqrt{3}}{2}t^2 + \frac{3\sqrt{3}}{2} = 0$

$$t^3 - \frac{3\sqrt{3}}{2}t^2 + \frac{3\sqrt{3}}{2} = \left(t-\sqrt{3}\right)\left(t^2 - Bt - \frac{3}{2}\right) = \left(t-\sqrt{3}\right)\left(t^2 - \frac{\sqrt{3}}{2}t - \frac{3}{2}\right)$$

$$= \left(t-\sqrt{3}\right)\left(t-\sqrt{3}\right)\left(t + \frac{\sqrt{3}}{2}\right)$$

Note: The quadratic formula could be used instead of factorising

$$t^2 - \frac{\sqrt{3}}{2}t - \frac{3}{2} = 0 \Rightarrow 2t^2 - \sqrt{3}\,t - 3 = 0$$

$$\therefore t = \frac{\sqrt{3} \pm \sqrt{3+24}}{4} = \frac{\sqrt{3} \pm 3\sqrt{3}}{4}$$

$$= \sqrt{3},\ -\frac{\sqrt{3}}{2}$$

$$\therefore\ q = -\frac{\sqrt{3}}{2}$$

The solution continues as in Method 1.

Comments For (i) and (ii)

This is a ‘parametrics’ question and yet a lot of students just treated it as a ‘Cartesian’ question, they couldn’t wait to convert it over.
These students then found it harder to get their marks.

In (i), too much tie was wasted proving that $t = p$. The parametric variable, t, is a general variable and from the diagram at P, $t = p$. It’s like the variable x in Cartesian equations, where at a point $\left(x_0, y_0\right)$ then $x = x_0$.
Students who didn’t realise this had their parametric tangent in terms of t and p.

Many students realised that, from the equation of the tangent, $p = \sqrt{3}$ but they couldn’t justify it. They ‘deduced’ this from the look of the equation of the tangent in parametric and Cartesian forms.

In (ii), some students either saw that they already had a root of their equation or in some cases knew that it was a double root.

End of solutions