File

SYDNEY GIRLS HIGH SCHOOL

LABOR OMNIA VINCIT

MATHEMATICS EXTENSION I

Assessment Task 2 for 2008 HSC

March 6th, 2008

Reading Time 5 minutes
Time allowed: 75 minutes

Topics: Trigonometry

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are of equal value
- There are 4 questions with part marks shown
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Answers may be left in either exact form or correct to three significant figures
- Write on one side of the paper only
- Note the Integrals: $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

  $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

Question 1 **(15 marks)**

Marks

a) Evaluate :

i. $3\lim\limits_{x\to 0}\dfrac{\tan x}{x}$ 1

ii. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{4x}$ 2

b) Find, correct to the nearest degree, the acute angle between the lines 4

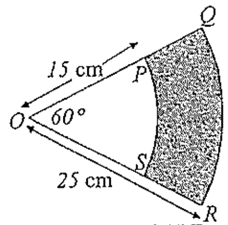
$x + y - 4 = 0$ and $y = 2x + 1$

c) Differentiate with respect to θ

i. $y = 4\cos^3\theta - 3\cos\theta$ 2

ii. $y = \sec\theta - \cot\theta$ 2

d)



NOT TO SCALE

PS and QR are arcs of concentric circles with O as centre. 4

Calculate in terms of π, the perimeter of the shaded region PQRS.

Question 2 (15 marks) (Start a new page)

Marks

a) Find the exact value of $\tan\frac{5\pi}{6}$, expressing your answer in surd form with a rational denominator. 2

b)

i. Write out the expansion of $\sin(x+y)$ 1

ii. Hence show that $\sin(x+\frac{\pi}{2}) = \cos x$ 1

c) Find:

i. $\int_0^{\frac{\pi}{4}} \cos 3x dx$ 2

ii. $\int_0^{\frac{\pi}{2}} \sin^2 2x.dx$ 2

d)

i. Express $\sqrt{3}\sin x + \cos x$ in the form $C\sin(x+\alpha)$ where $C > 0$ and $0 \le \alpha \le \frac{\pi}{2}$ 2

ii. Hence or otherwise, solve the equation $\sqrt{3}\sin x + \cos x = 1$ for $0 \le x \le 2\pi$ 2

e) Prove the identity $\frac{\cos 2\theta}{\sin\theta + \cos\theta} = \cos\theta - \sin\theta$. 3

Question 3 (15 marks) (Start a new page)

Marks

a) (i) Express $\tan(\alpha - \beta)$ in terms of $\tan\alpha$ and $\tan\beta$. 1

(ii) Use the result from (i) to find the exact value of $\tan 15°$ 3

b) Let $t = \tan\frac{\theta}{2}$ and starting with $\tan\theta = \tan(\frac{\theta}{2}+\frac{\theta}{2})$, show that $\tan\theta = \frac{2t}{1-t^2}$. 3

c) Use the 't' results to solve $3\cos\theta + 4\sin\theta = -3$ for $0° \le \theta \le 360°$ giving your answer correct to the nearest minute. 4

d) Find the exact value of the volume of solid of revolution formed when the region bounded by curve $y = \sin x$, the x-axis and the line $x = \frac{\pi}{4}$ is rotated about the x-axis. 4

Question 4 (15 marks) (Start a new page) Marks

a) Find the equation of the tangent to the curve $y = \frac{1}{2}\cos 4x$ at the point where $x = \frac{\pi}{8}$. 3

b) A sector of a circle has an area of 24 cm^2. Find the radius if the angle at the centre of the circle is 3 radians. 2

c) Solve the equation $\tan^2 2x = 1$ for $0 \le x \le \pi$. 3

d) Solve the equation $2\cos^2 x - \sin x - 1 = 0$ for $0 \le x \le 2\pi$. 3

e) (i) Draw a neat sketch of $y = \frac{1}{2}\sin 2x + 2$ for $0 \le x \le \pi$ showing all important features. 2

(ii) On the same graph sketch $y = 1$. 1

(iii) How many solution of $\frac{1}{2}\sin 2x + 1 = 0$ are there for $0 \le x \le \pi$? 1

End of Paper

S.G.H.S. Ext 1 - Task 5 March 2008

Question 1 (15 marks)

A ✓ E ✓
B ✓ F ✓
C ✓ G ✓
D ✓ H ✓

a) i) $3 \lim_{x \to 0} \frac{\tan x}{x} = 3 \times 1$

$= 3$ 1

(ii) $\lim_{x \to 0} \frac{\sin 5x}{4x} = \frac{5}{4} \lim_{x \to 0} \frac{\sin 5x}{5x}$

$= \frac{5}{4}$ 2

b) $x + y - 4 = 0$ $\quad y = 2x + 1$

$y = -x + 4$ $\quad m_2 = 2$ 1

$m_1 = -1$ 1

$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$= \left| \frac{-1 - 2}{1 + (-1 \times 2)} \right|$ 1

$= \left| \frac{-3}{1 - 2} \right|$

$\tan\theta = 3$ 4

$\therefore \theta = 71°34'$

$\doteqdot 72°$ 1

c) i) $y = 4\cos^3\theta - 3\cos\theta$

$\frac{dy}{dx} = 4 \cdot 3\cos^2\theta \cdot -\sin\theta + 3\sin\theta$

$= 3\sin\theta - 12\sin\theta\cos^2\theta$ 2

(ii) $y = \sec\theta - \cot\theta$

$= \frac{1}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$

c) (ii) $y = (\cos\theta)^{-1} - (\tan\theta)^{-1}$

$\frac{dy}{dx} = -1(\cos\theta)^{-2} \cdot -\sin\theta + 1(\tan\theta)^{-2} \times \sec^2\theta$

$= \frac{\sin\theta}{\cos^2\theta} + \frac{\sec^2\theta}{\tan^2\theta}$

$= \frac{\sin\theta}{\cos^2\theta} + \frac{\cancel{\cos^2\theta}}{\sin^2\theta \cdot \cos^2\theta}$

$= \frac{\sin\theta}{\cos^2\theta} + \frac{1}{\sin^2\theta}$ 2

$= \tan\theta \cdot \sec\theta + \operatorname{cosec}^2\theta$

d) $\theta = 60°$

$= \frac{\pi}{3}$

length of arc $PS = r\theta$

$= 15 \times \frac{\pi}{3}$

$= 5\pi$ cm 1

length of arc $QR = r\theta$

$= 25 \times \frac{\pi}{3}$

$= \frac{25\pi}{3}$ 1

$PQ = SR = 25 - 15$

$= 10$ cm 1

$\therefore$ Perimeter of Shaded Region $= 5\pi + \frac{25\pi}{3} +$

$= \frac{40\pi}{3} + 20$ cm. 1

4

Q_2 a) $\tan\left(\frac{5\pi}{6}\right) = \tan\left(\pi - \frac{\pi}{6}\right)$

$= -\tan\frac{\pi}{6}$

$= \frac{-1}{\sqrt{3}}$ ($1\frac{1}{2}$ marks)

$= -\frac{\sqrt{3}}{3}$ (2 marks)

b) i) $\sin(x+y) = \sin x \cos y + \cos x \sin y$ (1 mark)

ii) $\sin\left(x+\frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}$

$= \sin x \times 0 + \cos x \times 1$

$= \cos x$ (1 mark)

c) i) $\int_0^{\pi/4} \cos 3x \, dx = \left[\frac{1}{3}\sin 3x\right]_0^{\pi/4}$

$= \frac{1}{3}\sin\frac{3\pi}{4} - 0$

$= \frac{1}{3} \times \frac{1}{\sqrt{2}}$

$= \frac{1}{3\sqrt{2}}$ ($1\frac{1}{2}$ marks)

$= \frac{\sqrt{2}}{6}$ (2 marks)

ii) $\int_0^{\pi/2} \sin^2 2x \, dx = \frac{1}{2}\int_0^{\pi/2} (1-\cos 4x)\, dx$

$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2}$ (1 mark)

$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] - [0]$

$= \frac{\pi}{4}$ (2 marks)

d) I) $C\sin(x+\alpha) = C(\sin x \cos\alpha + \cos x \sin\alpha)$

$\sqrt{3}\sin x + \cos x = C \cos\alpha \sin x + C\sin\alpha \cos x$

$\therefore C\cos\alpha = \sqrt{3}$ ----- ①

$C\sin\alpha = 1$ ----- ②

$C^2(\sin^2\alpha + \cos^2\alpha) = 3+1$

$C^2 = 4$

$C = 2$ $\quad C > 0$

$\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$

$\sqrt{3}\sin x + \cos x = 2\sin\left(x+\frac{\pi}{6}\right)$ (2 marks)

II) $2\sin\left(x+\frac{\pi}{6}\right) = 1$

$\sin\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$ (1 mark)

$x + \frac{\pi}{6} = \frac{\pi}{6}, \pi - \frac{\pi}{6}, \frac{13\pi}{6}$

$x = 0, \frac{2\pi}{3}, 2\pi$ (2 mark)

e) $\frac{\cos 2\theta}{\sin\theta + \cos\theta}$

$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta + \cos\theta}$ ← (1 mark)

$= \frac{\cancel{(\cos\theta + \sin\theta)}(\cos\theta - \sin\theta)}{\cancel{(\sin\theta + \cos\theta)}}$ (2 marks)

$= \cos\theta - \sin\theta$ (3 marks)

Q3 a) i) $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

ii) $\tan 15^\circ = \tan(45^\circ - 30^\circ)$

$= \dfrac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}$

OR. Use $(60^\circ - 45^\circ)$ etc.

$= \dfrac{1-\frac{1}{\sqrt3}}{1+1\times\frac{1}{\sqrt3}} \times \dfrac{\sqrt3}{\sqrt3}$

$= \dfrac{(\sqrt3-1)}{\sqrt3+1} \times \dfrac{(\sqrt3-1)}{(\sqrt3-1)}$

$= \dfrac{4-2\sqrt3}{2}$

$= 2-\sqrt3$

b) $\tan\theta = \tan\left(\frac{\theta}{2}+\frac{\theta}{2}\right)$

$= \dfrac{\tan\frac{\theta}{2}+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}\cdot\tan\frac{\theta}{2}}$

$= \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$= \dfrac{2t}{1-t^2}$

c) $3\cos\theta + 4\sin\theta = -3$ for $0 \le \theta \le 360^\circ$

$\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$

$\therefore 3\dfrac{(1-t^2)}{1+t^2} + \dfrac{4(2t)}{1+t^2} = -3$

$3(1-t^2) + 8t = -3(1+t^2)$

$3 - \cancel{3t^2} + 8t = -3 - \cancel{3t^2}$

$8t = -6$

$t = -\frac{3}{4}$

$\therefore \tan\frac{\theta}{2} = -\frac{3}{4}$ for $0^\circ \le \frac{\theta}{2} \le 180^\circ$

$\therefore \frac{\theta}{2}$ is an obtuse angle

$\frac{\theta}{2} = \tan^{-1}\left(-\frac{3}{4}\right)$

$= -36^\circ 52'$

but test $\theta = 180^\circ$ as 't' method does not allow for such a solution.

When $\theta = 180^\circ$

$LHS = 3\cos 180^\circ + 4\sin 180^\circ$

$= 3\times -1 + 0$

$= -3$ $\therefore$ (True).

$= RHS$

$\therefore$ Solutions are 180°, $286^\circ 16'$

d) $V = \pi\int_a^b y^2\,dx$

where $y = \sin x$

$\therefore y^2 = \sin^2 x$

$\therefore V = \pi\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$

but $\sin^2 x = \frac{1}{2}(\cos 2x + 1)$

$\therefore V = \pi \times \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos 2x + 1)\,dx$

$= \frac{\pi}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\times\left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{\pi}{4}\right)$

$= \frac{\pi}{2}\left(\frac{1}{2} - \frac{\pi}{4}\right)$

$= -\frac{\pi}{4} + \frac{\pi^2}{8}$ (u^3)

$= \dfrac{\pi^2}{8} - \dfrac{\pi}{4}$

Question 4

a) $y = \frac{1}{2}\cos 4x$

$\frac{dy}{dx} = -2\sin 4x$

at $x = \frac{\pi}{8}$, $m = -2\sin\frac{\pi}{2}$
$= -2$

$y = \frac{1}{2}\cos\frac{\pi}{2}$
$= 0$

Eqn of tangent

$y - 0 = -2(x - \frac{\pi}{8})$

$y = -2x + \frac{\pi}{4}$ #

$\boxed{3}$

b) $A = \frac{1}{2}r^2\theta$, $A = 24$, $\theta = 3$

$24 = \frac{3}{2}r^2$

$r^2 = 16$

$r = 4$ cm #

$\boxed{2}$

c) $\tan^2 2x = 1$ $\quad 0 \le x \le \pi$

$\tan 2x = \pm 1$ $\quad 0 \le 2x \le 2\pi$

$2x = \frac{\pi}{4}, (\pi + \frac{\pi}{4}), (\pi - \frac{\pi}{4}), (2\pi - \frac{\pi}{4})$

$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ #

$\boxed{3}$

d) $2\cos^2 x - \sin x - 1 = 0$ $\quad 0 \le x \le 2\pi$

$2(1 - \sin^2 x) - \sin x - 1 = 0$

$2 - 2\sin^2 x - \sin x - 1 = 0$

$-2\sin^2 x - \sin x + 1 = 0$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

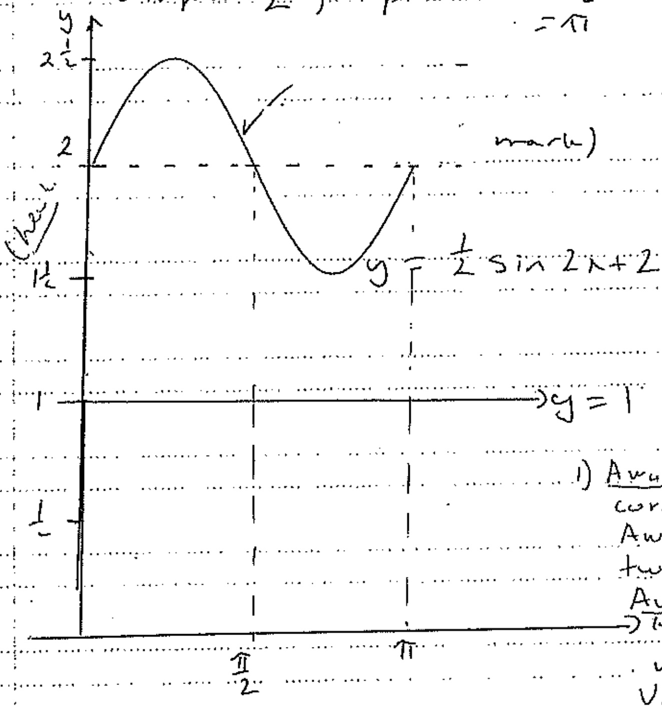
$\sin x = \frac{1}{2}$ or $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ #

$\boxed{3}$

e) i) $y = \frac{1}{2}\sin 2x + 2$

amp $= \frac{1}{2}$, period $= \frac{2\pi}{2} = \pi$, Vertical Shift +



mark)

$\boxed{2}$

$\boxed{1}$

1) Award 1st mark for correct amp, period, Vshift. Award 1 mark for any two of above correct. Award second mark for correct sketch with correct amp, period, V. Shift. OR 1 mark for incorrect sketch (wrong period, amp) etc of equivalent difficulty

ii) as above

iii) $\frac{1}{2}\sin 2x + 1 = 0$

ie $\frac{1}{2}\sin 2x + 2 = 1$

this is solved by finding the intersection of $y = \frac{1}{2}\sin 2x + 2$ and $y = 1$.

As the two graphs do not intersect there are NO solutions

[No carry through here as it changes the intent of the question]

$\boxed{1}$

Total $\boxed{15}$

A B C D E F G H