

Sydney Girls' High School



March - 2010 MATHEMATICS - Extension 1 YEAR 12 ASSESSMENT TASK 2

Time Allowed: 75 minutes (plus 5 mins reading time)

Name: _____

Teacher: _____

TOPICS: Exponential and Logarithmic Functions, the Trigonometric Functions and Trigonometric Function II.

Directions to Candidates

- There are five (5) questions.
- Attempt ALL questions.
- Questions are of equal value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- Diagrams are NOT drawn to scale.
- Board-approved calculators may be used.

Total: 75 marks

QUESTION 1 (15 marks)

Marks

(a) Differentiate the following with respect to x

(i) $\log_e (3x + 1)$ 1

(ii) $x \sin x$ 2

(iii) $x \log_e \cos x + \sin^2 x$ 3

(b) Solve $e^{3t} = 5$ 2

(c) Find the value of a if $\int_a^e \frac{1}{x} dx = 5$ 3

(d) Find the equation of the normal to the curve $y = \log_e x$ at the point where $x = e^2$. 4

QUESTION 2 (15 marks)

(a) Find $\lim_{x \rightarrow 0} \frac{\sin 2x}{6x}$ 2

(b) Find the primitive of: (i) $\sqrt{x} - e^x$ 1

(ii) $\sin x + \sec^2 x$ 1

(c) Evaluate: (i) $\int_0^{\frac{\pi}{2}} \sin 2x dx$ 2

(ii) $\int_3^4 \frac{x-1}{x^2-2x} dx$ 3

(d) Prove $\frac{\sin 2x}{1 + \cos 2x} = \tan x$ 2

(e) If $\tan \alpha = \frac{3}{4}$ and α is acute and $\cos \beta = \frac{-2}{5}$ and β is obtuse, find the exact values of:

(i) $\sin 2\alpha$ 2

(ii) $\cos (\alpha + \beta)$ 2

QUESTION 3 (15 marks)

Marks

- (a) For the curve $y = f(x)$ it is given that $f'(x) = \sin^2 x$.

The curve passes through the point $(\frac{\pi}{4}, \frac{\pi}{8})$.

Find the equation of the curve.

4

- (b) Find the acute angle between $x - 2y = 0$ and $3x - y - 15 = 0$

3

- (c)

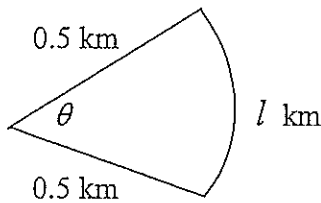


Figure not to scale

A car travels at 40 km/h on a circular curve whose radius is 0.5 km.

- (i) Find the distance, l km, that the car travels in one minute. 2
- (ii) Calculate the size of the angle θ through which the car turns in one minute. 2
- Give your answer to the nearest degree.

- (d) Express $\sqrt{3} \sin x - \cos x$ in the form $R \sin(x - \alpha)$, $R > 0$.

4

Hence or otherwise, find the solution for $\sqrt{3} \sin x - \cos x = 1$ in the domain $0 \leq x \leq 2\pi$.

QUESTION 4 (15 marks)

Marks

(a) Differentiate xe^x and hence, evaluate $\int_0^1 xe^x dx$. 3

(b) Given $y = \sin x$ and $y = \frac{2}{\pi}x$ for $0 \leq x \leq \pi$.

(i) Sketch $y = \sin x$ and $y = \frac{2}{\pi}x$ on the same diagram for $0 \leq x \leq \pi$. 2

(ii) Find their point(s) of intersection. 2

(iii) Find the area enclosed between the two curves. 2

(c) (i) If $0 \leq \theta \leq \frac{\pi}{2}$, prove that $\tan \theta = \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}}$ 3

(ii) Hence, show that the exact value of $\tan \frac{\pi}{8}$ is $\sqrt{2} - 1$. 3

QUESTION 5 (15 marks)

(a) Solve $3t^2 + 4t - 4 = 0$ for $0 \leq x \leq 2\pi$ where $t = \tan \frac{x}{2}$. 4
Give answer in radians, correct to two decimal places.

(b) Find the smallest positive value of θ if $\cos \theta - \sin 2\theta = 0$. 3

(c) Calculate the volume of the solid generated when the region in the first quadrant bounded by the curve $y = e^x$, the line $x = \log_e 2$ and the co-ordinate axes is rotated about the x-axis. 3

(d) (i) Show that $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$ 2

(ii) Hence, solve the equation $\cos 3\theta - \cos \theta = 0$, $0^\circ \leq \theta \leq 360^\circ$. 3

THE END

Question 1 (15 marks)

i) let $y = \log_e(3x+1)$

$$\frac{dy}{dx} = \frac{3}{3x+1}$$

(1)

ii) let $y = x \sin x$

$u = x, v = \sin x$

$$\frac{du}{dx} = 1, \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

$$\frac{dy}{dx} = \sin x + x \cos x$$

(2)

iii) let $y = x \cdot \log_e \cos x + \sin^2 x$

$$y = x \cdot \log_e \cos x + (\sin x)^2$$

$$\frac{dy}{dx} = x \cdot \frac{-\sin x}{\cos x} + 1 \cdot \ln \cos x$$

$$2 \sin x \cos x$$

$$\frac{dy}{dx} = \ln \cos x - x \tan x + 2 \sin x \cos x$$

(3)

$$e^{3t} = 5$$

$$\log_e e^{3t} = \log_e 5$$

$$3t \cdot \log_e e = \log_e 5$$

$$3t = \log_e 5$$

$$t = \frac{\log_e 5}{3}$$

(2)

$$c) \int_a^e \frac{1}{x} dx = 5$$

$$\ln e - \ln a = 5$$

$$1 - \ln a = 5$$

$$\ln a = -4$$

$$\therefore a = e^{-4} = \frac{1}{e^4}$$

(3)

d) $y = \log_e x \quad x = e^2$

$$\frac{dy}{dx} = \frac{1}{x} \quad \text{at } x = e^2$$

$$\frac{dy}{dx} = \frac{1}{e^2} \quad \text{and} \quad y = \log_e e^2$$

$$y = 2 \log_e e$$

$$\text{since } m_1 = -\frac{1}{m_2}$$

$$y = 2$$

$$\therefore m_2 = -e^2 \quad \text{and}$$

$$P(e^2, 2)$$

$\therefore$ Equation of normal

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -e^2(x - e^2)$$

$$y - 2 = -e^2 x + e^4$$

$$\therefore y = -e^2 x + e^4 + 2$$

(4)

Question 2 (15 marks) - Ext 1

a) $\lim_{x \rightarrow 0} \frac{\sin 2x}{6x} = \frac{2}{6} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$

$$= \frac{1}{3}$$

(2)

b) i) $\int x^{3/2} - e^{-x} dx$

$$= \frac{2}{3} x^{5/2} + e^{-x} + C$$

(1)

ii) $\int \sin x + \sec^2 x dx$

$$= -\cos x + \tan x + C$$

(1)

c) i) $\int_0^{\pi/2} \sin 2x dx$

$$= \left[-\frac{\cos 2x}{2} \right]_0^{\pi/2}$$

$$= -\frac{1}{2} [\cos \pi - \cos 0]$$

$$= -\frac{1}{2} x - 2$$

$$= 1$$

(2)

ii) $\int_3^4 \frac{x-1}{x^2-2x} dx$

$$= \frac{1}{2} \log_e(x^2 - 2x) \Big|_3^4$$

$$= \frac{1}{2} [\log_e(16-8) - \log_e(9-6)]$$

$$= \frac{1}{2} [\log_e 8 - \log_e 3]$$

$$= \frac{1}{2} \log_e \frac{8}{3}$$

(3)

d) Prove $\frac{\sin 2x}{1 + \cos 2x} = \tan x$

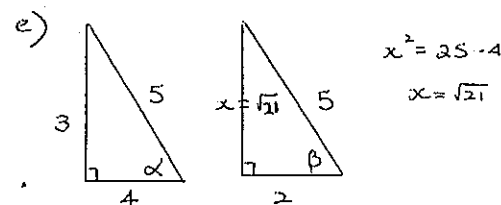
$$\text{L.H.S} = \frac{2 \sin x \cos x}{2 \cos^2 x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$$= \text{R.H.S.}$$

(2)



i) $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$$= 2 \times \frac{3}{5} \times \frac{4}{5}$$

$$= \frac{24}{25}$$

(2)

ii) $\cos(\alpha + \beta) =$

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{4}{5} \times \frac{2}{5} - \frac{3}{5} \times \frac{\sqrt{21}}{5}$$

$$= \frac{-8}{25} - \frac{3\sqrt{21}}{25}$$

$$= \frac{-8 - 3\sqrt{21}}{25}$$

(2)

Question 3 (15 marks) - Ex + ①

$$f'(x) = \sin^2 x$$

$$= \frac{1}{2} (1 - \cos 2x)$$

$$f(x) = \int \frac{1}{2} (1 - \cos 2x) dx$$

$$= \frac{1}{2} (x - \frac{1}{2} \sin 2x) + C$$

$$f(\frac{\pi}{4}) = \frac{\pi}{8}$$

$$\frac{\pi}{8} = \frac{1}{2} (\frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2}) + C$$

$$\frac{\pi}{8} = \frac{\pi}{8} - \frac{1}{4} + C$$

$$\therefore C = \frac{1}{4}$$

$$f(x) = \frac{1}{2} (x - \frac{1}{2} \sin 2x) + \frac{1}{4}$$

$$f(x) = \frac{1}{2} x - \frac{1}{4} \sin 2x + \frac{1}{4}$$

④

$$x - 2y = 0 \quad 3x - y - 15 = 0$$

$$-2y = -x \quad -y = -3x + 15$$

$$y = \frac{1}{2}x \quad y = 3x - 15$$

$$m_1 = \frac{1}{2} \quad m_2 = 3$$

$$\therefore \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\tan \theta = \left| \frac{\frac{1}{2} - 3}{1 + \frac{1}{2} \cdot 3} \right|$$

$$\tan \theta = 1$$

$$\therefore \theta = 45^\circ$$

③

c) i) $d = \frac{3}{t}$

$$d = \frac{40 \text{ km/h}}{60}$$

$$= \frac{2}{3} \text{ km in 1 minute}$$

$$\therefore d = \frac{2}{3} \text{ km} \quad \text{②}$$

ii) $\theta = \frac{l}{r}$

$$= \frac{\frac{2}{3}}{0.5}$$

$$\theta = \frac{4}{3} \text{ radians}$$

$$\therefore \theta = \frac{4}{3} \times \frac{180}{\pi}$$

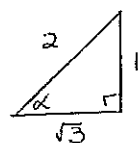
$$= 76^\circ$$

②

d) $R = \sqrt{(\sqrt{3})^2 + 1} = 2$

$$\frac{\sqrt{3} \sin x - \cos x}{2} = \frac{\cos \alpha \sin x - \sin \alpha \cos x}{2}$$

$$\text{where } \cos \alpha = \frac{\sqrt{3}}{2} \text{ and}$$



$$\sin \alpha = \frac{1}{2}$$

$$\text{and } \alpha \text{ is acute}$$

$$\alpha = \frac{\pi}{6}$$

$$\text{Now, } \frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x = \frac{1}{2}$$

$$\sin(x - \alpha) = \frac{1}{2}$$

$$x - \alpha = \frac{\pi}{6}, \pi - \frac{\pi}{6}$$

$$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore x = \frac{\pi}{3} \text{ and } \pi$$

④

Question 4 (15 marks) - Ex + ①

a) let $y = x e^x$

$$\frac{dy}{dx} = e^x + x e^x$$

$$\therefore \int_0^1 x e^x dx = \int_0^1 (e^x + x e^x) dx -$$

$$\int_0^1 e^x dx$$

$$= \left[x e^x - e^x \right]_0^1$$

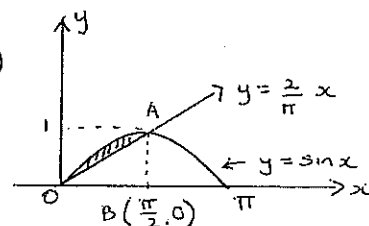
$$= \left[(1 \cdot e^1 - e^1) - (0 - e^0) \right]$$

$$= [0 - -1]$$

$$= 1$$

③

b) i)



②

ii) Points of intersection

$$P_1(0,0)$$

$$P_2(\frac{\pi}{2}, 0)$$

②

iii) $\int_0^{\pi/2} \sin x dx -$
Shaded

$$\text{Area} = \text{area of } \Delta AOB$$

$$= \left[-\cos x \right]_0^{\pi/2} - \frac{1}{2} \cdot \frac{\pi}{2} \cdot 1$$

$$= -\left[\cos \frac{\pi}{2} - \cos 0 \right] - \frac{\pi}{4}$$

$$= (1 - \frac{\pi}{4}) \text{ units}^2$$

②

①

c) i) $2 \cos^2 \theta = \cos 2\theta + 1$

$$2 \sin^2 \theta = 1 - \cos 2\theta$$

$$\tan \theta = \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}}$$

$$\text{R.H.S} = \sqrt{\frac{2 \sin^2 \theta}{2 \cos^2 \theta}}$$

$$= \sqrt{\tan^2 \theta}$$

$$= \tan \theta \text{ since } \tan \theta > 0$$

$$= \text{L.H.S} \text{ for } 0 \leq \theta \leq \frac{\pi}{2}$$

③

ii) Substitute $\theta = \frac{\pi}{8}$

$$\therefore \tan \frac{\pi}{8} = \sqrt{\frac{1 - \cos \frac{\pi}{4}}{1 + \cos \frac{\pi}{4}}}$$

$$= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}}$$

$$= \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1}}$$

$$= \sqrt{\frac{(\sqrt{2} - 1)^2}{2 - 1}}$$

$$= \sqrt{(\sqrt{2} - 1)^2}$$

$$= \sqrt{2} - 1$$

$$\therefore \tan \frac{\pi}{8} = \sqrt{2} - 1$$

③

Question 5 (15 marks) - Ext 1

a) $t = \tan \frac{x}{2}$

$$3t^2 + 4t - 4 = 0$$

$$(3t-2)(t+2) = 0$$

$$3t = 2 \quad t = -2$$

$$t = \frac{2}{3}$$

$$\therefore \tan \frac{x}{2} = \frac{2}{3} \quad \text{or} \quad \tan \frac{x}{2} = -2$$

$$\frac{x}{2} = \tan^{-1} \frac{2}{3} \quad \frac{x}{2} = \tan^{-1} -2$$

$$\frac{x}{2} = 0.588 \quad \frac{x}{2} = \pi - 1.107$$

$$x = 1.18 \quad x = 2\pi - 2.2$$

$$x = 4.07$$

$$\therefore x = 1.18 \text{ or } 4.07$$

(4)

b) $\cos \theta - \sin 2\theta = 0$

$$\sin 2\theta = \cos \theta$$

$$2\sin \theta \cos \theta = \cos \theta$$

$$2\sin \theta \cos \theta - \cos \theta = 0$$

$$\cos \theta (2\sin \theta - 1) = 0$$

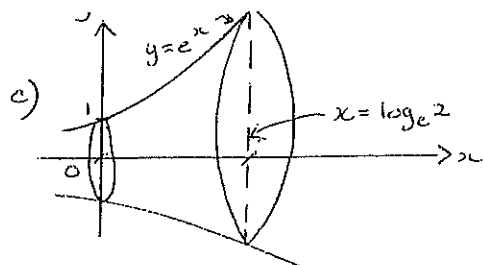
$$\cos \theta = 0 \quad \text{or} \quad 2\sin \theta = 1$$

$$\sin \theta = \frac{1}{2}$$

$$\therefore \theta_1 = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \quad \theta_2 = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$\therefore$ smallest positive value of θ is $\frac{\pi}{6}$.

(3)



Volume of solid generated

$$= \pi \int_0^{\log_e 2} y^2 dx \quad \text{where } y = e^x$$

$$= \pi \int_0^{\log_e 2} e^{2x} dx$$

$$= \frac{\pi}{2} \left[e^{2x} \right]_0^{\log_e 2}$$

$$= \frac{\pi}{2} \left[e^{2\log_e 2} - e^0 \right]$$

$$= \frac{\pi}{2} [4 - 1]$$

$$= \frac{3\pi}{2} \text{ units}^3 \quad (3)$$

d) i) $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$
L.H.S. $\cos(3\theta) = \cos(2\theta + \theta)$

$$= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$$

$$= (2\cos^2 \theta - 1)\cos \theta -$$

$$2\cos \theta \sin \theta \sin \theta$$

$$= 2\cos^3 \theta - \cos \theta - 2\cos \theta \sin^2 \theta$$

$$= 2\cos^3 \theta - \cos \theta - 2\cos \theta (1 - \cos^2 \theta)$$

$$= 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta$$

$$= 4\cos^3 \theta - 3\cos \theta$$

$$= \text{R.H.S.} \quad (2)$$

Question 5 (cont)

ii) Solve

$$\cos 3\theta - \cos \theta = 0$$

$$4\cos^3 \theta - 3\cos \theta - \cos \theta = 0$$

$$4\cos^3 \theta - 4\cos \theta = 0$$

$$4\cos \theta (\cos^2 \theta - 1) = 0$$

$$4\cos \theta = 0 \quad \text{or} \quad \cos^2 \theta - 1 = 0$$

$$\cos \theta = 0$$

$$\cos^2 \theta = 1$$

$$\cos \theta = \pm 1$$

$$\therefore \theta = 90^\circ \text{ or } 270^\circ$$

$$\cos \theta = 1$$

$$\therefore \theta = 0, 360^\circ$$

$$\text{or } \cos \theta = -1$$

$$\theta = 180^\circ$$

$$\therefore \theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ \text{ or } 360^\circ.$$

(3)