Sydney Girls High School

Mathematics Extension 1

YEAR 12

2021 HSC Assessment Task 3

Reading time : 5 minutes

Writing time : 60 minutes

Total marks : 50

Topics: Further Calculus, Motion, Forces and Projectiles, Differential Equations and Mathematical Induction

General Instructions:

- There are four (4) questions which are not of equal value.
- Attempt all questions.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale, unless stated.
- Board-approved calculators may be used.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.
- A Mathematics reference sheet is provided.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- A correct answer without necessary working will be awarded a maximum of 1 mark.

Student Name:__________________________ **Teacher Name:**____________________

Question One (12 marks)

a) Differentiate the following with respect to x:

i) $y = \sin^{-1}(x^2)$ (2)

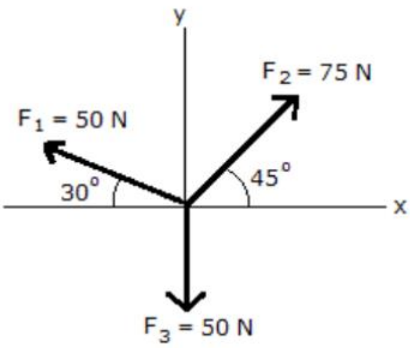
ii) $y = 2x\tan^{-1}(x)$ (2)

b) Find $\int_0^{0.125} \frac{2}{\sqrt{1-4x^2}}\,dx$. Write your answer correct to 3 decimal places. (2)

c) Solve $f'(x) = e^x + 6x - x^{\frac{1}{2}}$, given that $f(0) = 8$. (2)

d) Use the substitution $u = x^2$ to find $\int \frac{4x}{1+x^4}\,dx$. (2)

e) There are only three forces acting on an object as shown in the diagram.
Determine the magnitude of the resultant force.
(answer correct to 2 decimal places) (2)



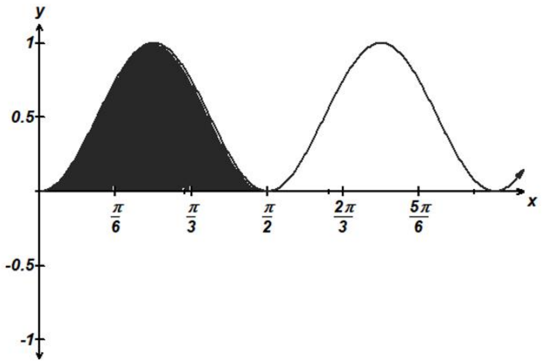
Question Two (13 marks)

a) The graph of $y = 5\sqrt{1 - \frac{x^2}{4^2}}$ from $x = -2$ to $x = 2$ is rotated about the x-axis.

Evaluate the exact volume generated. (3)

b) Find the general solutions of : $y' = x - xy$. (2)

c) The diagram below shows the graph of $y = \sin^2 2x$. Find the shaded area. (3)



d) Use the substitution $u = \frac{\pi}{x}$ to find $\int \frac{4\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$. (2)

e) Prove by induction for all positive integers of n.

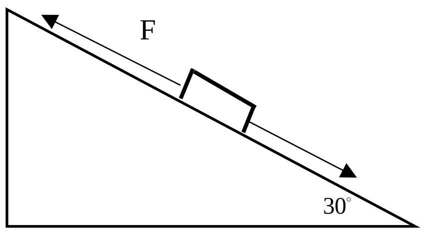
$$\frac{1}{1\times 5} + \frac{1}{5\times 9} + \frac{1}{9\times 13} + \ldots\ldots\ldots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$ (3)

Question Three (13 marks)

a) Evaluate $\int_1^6 x\sqrt{x+3}\,dx$ using the substitution $u^2 = x+3$. (3)

b) A 12 kg block is sliding down a 30° incline with an acceleration of 1.5 ms^{-2}.

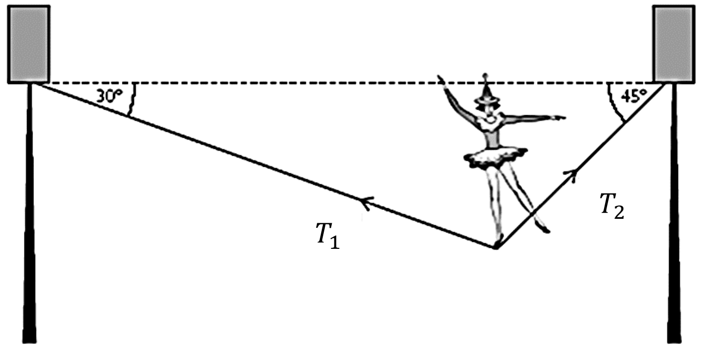
Calculate the magnitude of the frictional force, F. (Use $g = 10$ ms^{-2}) (2)



c) Prove by mathematical induction that $9^{n+2} - 4^n$ is divisible by 5, for all positive integers n. (3)

d) Given that $x\cos x$ is an odd function from $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, evaluate $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x-\cos x)^2\,dx$. (2)

e) The diagram below shows a circus performer in a high wire act. The total mass of the performer is 76.5 kg. Acceleration due to gravity is given as 10 ms^{-2}. Find the tensions T_1 and T_2 in the cable. (Answer correct to nearest whole number) (3)



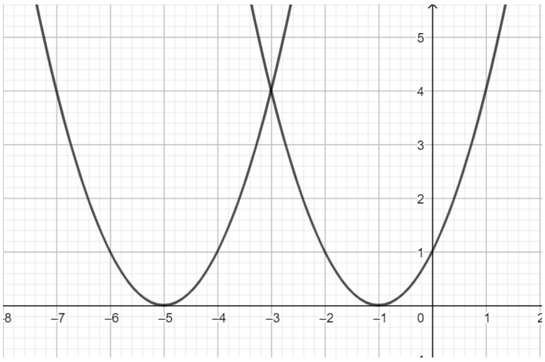
Question Four (12 marks)

a) The population, P, of animals in an environment in which there are scarce resources is increasing such that $\frac{dP}{dt} = P(100-P)$, where t is time. When $t=0, P=10$.

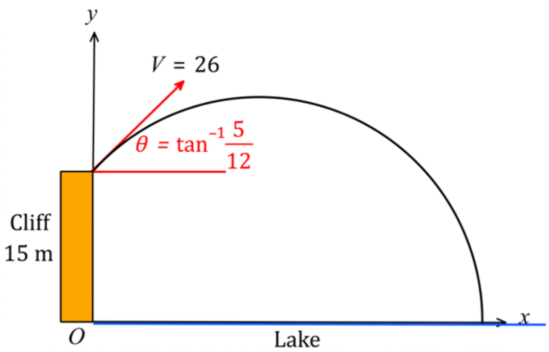
i) Show that $\frac{1}{100}\left(\frac{1}{P}+\frac{1}{100-P}\right)=\frac{1}{P(100-P)}$. (1)

ii) Find an expression for P in terms of t. (3)

b) In the diagram shown below, the area enclosed between two curves and the x –axis is rotated about the y -axis. Find the volume of the solid of revolution correct to one decimal place. (Diagram is to scale) (3)



c) A rock is thrown from the top of a 15 m cliff with an initial velocity of 26 m/s and angle of projection of $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. The cliff overlooks a lake and the acceleration due to gravity is 10 ms^{-2}.



i) Derive the equations of $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$ in component form. (3)

ii) Calculate the distance of the impact from the base of the cliff. (2)

THE END

Question One

(a)(i) $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^4}} \times 2x$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

(ii) $$\frac{dy}{dx} = \frac{2x}{1+x^2} + 2\tan^{-1} x$$

(b) $$\int_0^{0.125} \frac{2}{\sqrt{1-4x^2}}\,dx = \left[\sin^{-1}(2x)\right]_0^{0.125}$$

$$= \sin^{-1}(0.25)$$

$$\approx 0.253$$

The answer should have been in radians. No marks were lost for answer in degrees, this time.

(c) $$f(x) = e^x + 3x^2 - \frac{2}{3}x^{\frac{3}{2}} + C$$

$$f(0) = e^0 + 3(0)^2 - \frac{2}{3}(0)^{\frac{3}{2}} + C$$

$$8 = 1 + C$$

$$C = 7$$

$$f(x) = e^x + 3x^2 - \frac{2}{3}x^{\frac{3}{2}} + 7$$

(d) $$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$du = 2xdx$$

$$\int \frac{4x}{1+x^4}\,dx = \int \frac{2}{1+u^2}\,du$$

$$= 2\tan^{-1}(u) + C$$

$$= 2\tan^{-1}(x^2) + C$$

(e) $F_1 = -50\cos 30°\,\underset{\sim}{i} + 50\sin 30°\,\underset{\sim}{j}$

$= -25\sqrt{3}\underset{\sim}{i} + 25\underset{\sim}{j}$

$F_2 = 75\cos 45°\,\underset{\sim}{i} + 75\sin 45°\,\underset{\sim}{j}$

$= \frac{75}{\sqrt{2}}\underset{\sim}{i} + \frac{75}{\sqrt{2}}\underset{\sim}{j}$

$F_3 = -50\underset{\sim}{j}$

$$F_1 + F_2 + F_3 = \left(\frac{75}{\sqrt{2}} - 25\sqrt{3}\right)\underset{\sim}{i} + \left(25 + \frac{75}{\sqrt{2}} - 50\right)\underset{\sim}{j}$$

$$|F_1 + F_2 + F_3|^2 = \left(\frac{75}{\sqrt{2}} - 25\sqrt{3}\right)^2 + \left(\frac{75}{\sqrt{2}} - 25\right)^2$$

$$|F_1 + F_2 + F_3| = 29.67\ N$$

Question Two

(a) $$2\pi\int_0^2 25\left(1-\frac{x^2}{4^2}\right)dx = 50\pi\left[x-\frac{x^3}{3\times 4^2}\right]_0^2$$

$$= 50\pi\left(2-\frac{2\times 4}{3\times 4^2}\right)$$

$$= 50\pi\left(2-\frac{1}{6}\right)$$

$$= \frac{275\pi}{3}\text{ units}^3$$

(b) $$\frac{dy}{dx} = -x(y-1)$$

$$\int\frac{1}{y-1}\,dy = \int -x\,dx$$

Some student didn't not integrate the x side of the equation. Students should take care in the definition of the arbitrary constant.

$$\ln|y-1| = -\frac{x^2}{2}+C$$

$$|\pm(y-1)| = Ae^{-\frac{x^2}{2}}$$ where $A = e^C > 0$

$$y-1 = Ae^{-\frac{x^2}{2}}$$ allow $A \neq 0$

$$y = Ae^{-\frac{x^2}{2}}+1$$ By removing all restrictions on A this will include the constant solution $y = 1$.

(c) $$\int_0^{\frac{\pi}{2}} \sin^2(2x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos(4x))\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(4x)}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right)$$

$$= \frac{\pi}{4}\ units^2$$

Extension 1 students all should be able the integrate $\sin^2 x$ and $\cos^2 x$.

(d) $$u = \frac{\pi}{x}$$

$$\frac{du}{dx} = -\frac{\pi}{x^2}$$

$$-\frac{1}{\pi}du = \frac{1}{x^2}dx$$

$$\int \frac{4\cos\left(\frac{\pi}{x}\right)}{x^2}dx = -\frac{4}{\pi}\int \cos(u)\,du$$

$$= -\frac{4}{\pi}\sin(u) + C$$

$$= -\frac{4}{\pi}\sin\left(\frac{\pi}{x}\right) + C$$

Some students integrated u, to get $\pi \ln x$, instead of differentiating.

(e) Prove for $n = 1$

$$LHS = \frac{1}{1 \times 5} = \frac{1}{5} \qquad RHS = \frac{1}{4(1)+1} = \frac{1}{5} = LHS$$

Assume for $n = k$

$$\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \cdots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

Prove for $n = k + 1$

$$\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \cdots + \frac{1}{(4k-3)(4k+1)} + \frac{1}{(4(k+1)-3)(4(k+1)+1)}$$

$$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$$

$$= \frac{k(4k+5)}{(4k+1)(4k+5)} + \frac{1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2 + 5k + 1}{(4k+1)(4k+5)}$$

$$= \frac{(4k+1)(k+1)}{(4k+1)(4k+5)}$$

$$= \frac{k+1}{4(k+1)+1}$$

Therefore by the principles of mathematical induction the statement is true for all positive integers of n.

$\boxed{Q3}$

a) $\displaystyle\int_1^6 x\sqrt{x+3}\,dx$

Let $u^2 = x+3 \therefore u = \sqrt{x+3}$

$x = u^2 - 3$

$\dfrac{dx}{du} = 2u \therefore dx = 2u\,du$

$x = 6 \longrightarrow u = 3$

$x = 1 \longrightarrow u = 2$

$$I = \int_2^3 (u^2-3)u \cdot 2u\,du$$

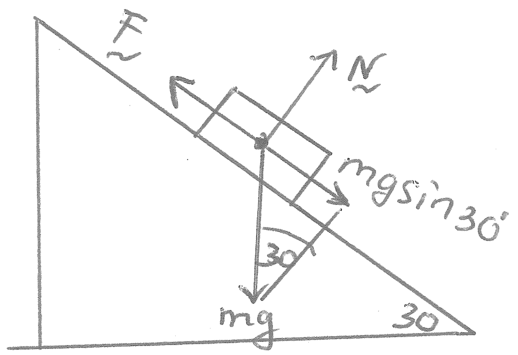
$$I = \int_2^3 (2u^4 - 6u^2)\,du$$

$$I = \left[\frac{2u^5}{5} - 2u^3\right]_2^3$$

$$I = \frac{486}{5} - 54 - \left(\frac{64}{5} - 16\right)$$

$$I = \frac{232}{5}$$

b)



$\underset{\sim}{ma} = \underset{\sim}{mg} \sin 30° - \underset{\sim}{F}$

$\underset{\sim}{F} = \underset{\sim}{mg} \sin 30° - \underset{\sim}{ma}$

$\underset{\sim}{F} = 12 \times 10 \times \frac{1}{2} - 12 \times 1.5$

$|\underset{\sim}{F}| = 42\ N$

Q3

c) prove it is true for $n = 1$

$9^3 - 4 = 725$ which is divisible by 5

$\therefore$ It is true for $n = 1$

Assume it is true for $n = k$

$$9^{k+2} - 4^k = 5M$$

prove it is true for $n = k+1$

$$9^{k+3} - 4^{k+1} = 9\left(9^{k+2} - 4^k\right) + 5.4^k$$

$$= 9(5M) + 5.4^k$$

$$= 5\left(9M + 4^k\right)$$

This is divisible by 5. Hence it is true for $n = k+1$. Proven by Mathematical Induction.

d) $$\int_{-\frac{\pi}{2}}^{\pi/2} (x - \cos x)^2 dx = 2\int_0^{\pi/2} (x - \cos x)^2 dx$$

$$I = 2\int_0^{\pi/2} \left(x^2 - 2x\cos x + \cos^2 x\right) dx$$

Note $\int_{-\pi/2}^{\pi/2} x\cos x\, dx = 0$ ($x\cos x$ is odd fn)

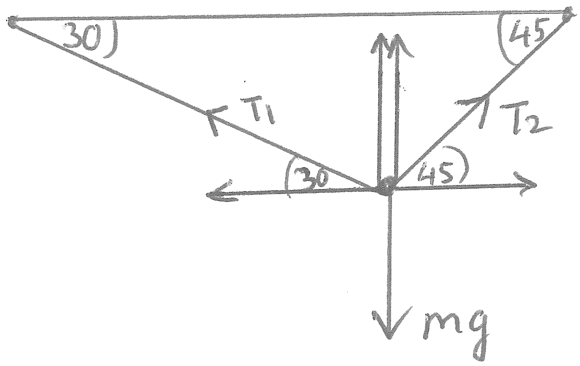
$$I = 2\int_0^{\pi/2} \left(x^2 + \tfrac{1}{2}(\cos 2x + 1)\right) dx$$

$$I = \left[\frac{2x^3}{3} + \frac{1}{2}\sin 2x + x\right]_0^{\pi/2}$$

$$I = \frac{2}{3} \times \frac{\pi^3}{8} + 0 + \frac{\pi}{2} - 0 = \frac{\pi^3}{12} + \frac{\pi}{2}$$

Q4

e)



Horizontal: $T_1 \cos 30° = T_2 \cos 45°$ (1)

vertical: $T_1 \sin 30° + T_2 \sin 45° = mg$ (2)

from (1) $T_1 = \dfrac{T_2 \cos 45}{\cos 30} = \dfrac{\sqrt{6}}{3} T_2$

subs into (2)

$$\frac{\sqrt{6}}{6} T_2 + \frac{\sqrt{2}}{2} T_2 = 76.5 \times 10$$

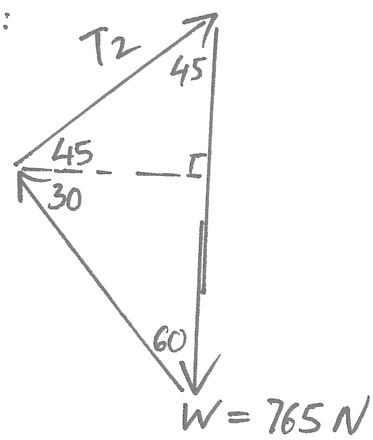
$$T_2 \left(\frac{\sqrt{6}}{6} + \frac{\sqrt{2}}{2} \right) = 765$$

$$T_2 = 685.9 \text{ (N)} = 686 \text{ N}$$

$$T_1 = \frac{\sqrt{6}}{3} \times 685.9 = 560 \text{ (N)}$$

Alternate Method:

Sine Rule



$$\frac{T_1}{\sin 45°} = \frac{765}{\sin 75} \quad (1)$$

$$\frac{T_2}{\sin 60} = \frac{765}{\sin 75} \quad (2)$$

Q4. Task 3 Ext 1 2021

ai) $LHS = \frac{1}{100}\left(\frac{100-P+P}{P(100-P)}\right)$

$= \frac{1}{\cancel{100}}\left(\frac{\cancel{100}}{P(100-P)}\right)$

$= \frac{1}{P(100-P)}$

$= RHS$

ii) $\frac{dP}{dt} = P(100-P)$

$\frac{dP}{P(100-P)} = dt$

$\int \frac{1}{100}\left(\frac{1}{P} + \frac{1}{100-P}\right) dP = \int dt$

$\int \frac{1}{P} + \frac{1}{100-P} \, dP = \int 100 \, dt$

$\ln|P| - \ln|100-P| = 100t + c$

$\ln\left|\frac{P}{100-P}\right| = 100t + c$

$\left|\frac{P}{100-P}\right| = e^{100t+c}$

$\left|\frac{P}{100-P}\right| = e^{c} \cdot e^{100t}$

$\frac{P}{100-P} = A e^{100t} \quad A \in R$

* Many students didn't find the value of A

at $t=0, P=10$

$\frac{10}{100-10} = A e^{0} \quad \therefore A = \frac{1}{9}$

$$\frac{P}{100-P} = \frac{1}{9}e^{100t}$$

$$\frac{9P}{100-P} = e^{100t}$$

$$9P = (100-P)e^{100t}$$

$$9P = 100e^{100t} - Pe^{100t}$$

$$9P + Pe^{100t} = 100e^{100t}$$

$$P(9 + e^{100t}) = 100e^{100t}$$

$$P = \frac{100e^{100t}}{9 + e^{100t}}$$

$$= \frac{\frac{100e^{100t}}{\cancel{e^{100t}}}}{\frac{9}{e^{100t}} + \frac{e^{100t}}{e^{100t}}}$$

$$\boxed{P = \frac{100}{9e^{-100t} + 1}}$$

b) $y = (x+1)^2$ $\qquad$ $y = (x+5)^2$

$y = (x+5)^2$ $\qquad$ $x + 5 = \pm\sqrt{y}$

$x + 1 = \pm\sqrt{y}$ $\qquad$ $x = -5 \pm \sqrt{y}$

$x = -1 \pm \sqrt{y}$

$$V = \pi\int_0^4 (-5+\sqrt{y})^2 - (-1-\sqrt{y})^2 \, dy$$

$$= \pi\int_0^4 25 - 10\sqrt{y} + y - (1 + 2\sqrt{y} + y)\, dy$$

$$\pi\int_0^4 25 - 10\sqrt{y} + y - 1 - 2\sqrt{y} - y \, dy$$

$$\pi\int_0^4 24 - 12\sqrt{y} \, dy$$

$$12\pi\left[2y - \frac{2y^{\frac{3}{2}}}{3}\right]_0^4$$

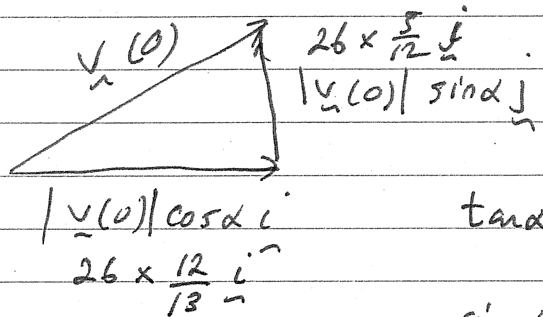
$$= 12\pi\left[8 - \frac{16}{3}\right]$$

$$= 32\pi \ u^3$$

* Many students couldn't do this question

They needed to take the correct branch of the parabola

c i) $\underset{\sim}{a}(t) = -10\underset{\sim}{j}$



$\tan\alpha = \frac{5}{12}$

$\sin\alpha = \frac{5}{13}$

$\cos\alpha = \frac{12}{13}$

$$\underset{\sim}{v}(t) = \int -10\underset{\sim}{j} \, dt$$

$$= -10t\underset{\sim}{j} + c$$

at $t = 0$ $\quad \underset{\sim}{v}(t) = \underset{\sim}{v}(0) \quad \therefore c = \underset{\sim}{v}(0)$

$$\underset{\sim}{v}(t) = |\underset{\sim}{v}(0)| \cos\alpha \underset{\sim}{i} + (|\underset{\sim}{v}(0)| \sin\alpha - 10t)\underset{\sim}{j}$$

$$= 26 \times \frac{12}{13}\underset{\sim}{i} + \left(26 \times \frac{5}{12} - 10t\right)\underset{\sim}{j}$$

$$= 24\underset{\sim}{i} + (10 - 10t)\underset{\sim}{j}$$

Many students didn't show the correct steps. Many stated the formulas rather than deriving.

$$\underset{\sim}{r}(t) = 24t\underset{\sim}{i} + (10t - 5t^2 + c)\underset{\sim}{j}$$

$t = 0 \quad \underset{\sim}{r}(t) = 15 \quad \therefore c = 15$

$$\underset{\sim}{r}(t) = 24t\underset{\sim}{i} + (10t - 5t^2 + 15)\underset{\sim}{j}$$

ii) $y = 0 \quad 10t - 5t^2 + 15 = 0 \quad \therefore (t-3)(t+1) = 0$

$t = 3$ as $t > 0$ $\quad \therefore x = 24 \times 3$

$= 72$ m