Sydney Girls High School

MATHEMATICS EXTENSION 1

Year 12

HSC Assessment Task 3

2023

Reading time : 5 minutes

Writing time : 60 minutes

Total marks : 50

Topics: Mathematical Induction; Further Calculus; Motion, Forces and Projectiles

General Instructions:

- There are four (4) questions which **are not** of equal value.
- Attempt all questions.
- Start each question on a NEW page.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- A correct answer without working may be awarded a maximum of 1 mark.
- Write on one side of the paper only.
- Diagrams are NOT to scale, unless indicated otherwise.
- Board-approved calculators may be used.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.
- A Mathematics reference sheet is provided.

Student Name:__________________________ **Teacher Name:**______________

BLANK PAGE

Question 1 (13 marks)

a) Differentiate:

i) $y = \cos^{-1}(7x)$ 1

ii) $y = \tan^{-1}\left(\dfrac{x}{3}\right)$ 1

b) Find.

i) $\displaystyle\int \frac{1}{16+x^2}\,dx$ 1

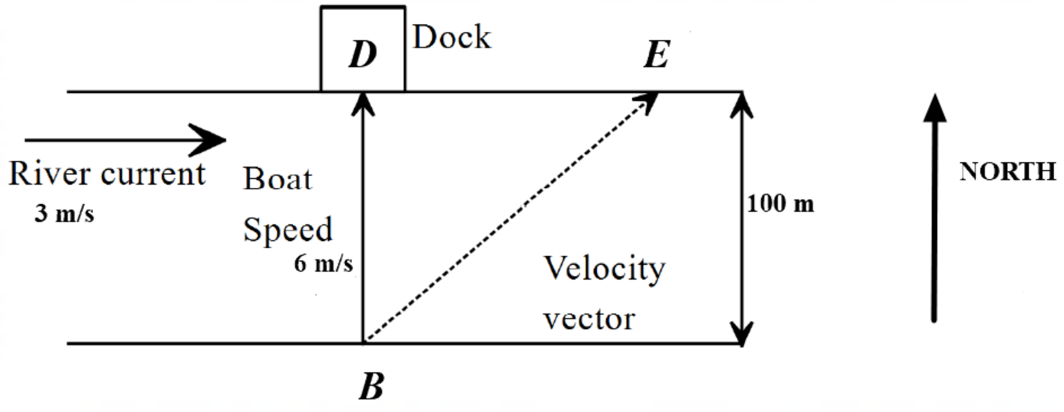
ii) $\displaystyle\int \frac{1}{\sqrt{5-2x^2}}\,dx$ 2

c) Evaluate the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$. 2

d) Find the equation of the tangent to the curve $y = 2\sin^{-1}(3x) + \dfrac{\pi}{2}$ at the point where $x = 0$. 2

Question 1 continues on page 4

e)



Breanna has a boat which moves at a top speed of 6 m/s in still water.
From point *B*, she wants to go due north to point *D* on the opposite side of the river, as shown in the diagram above.

Today the current in the river is flowing at 3 m/s.
Due to the current, she drifts from point *B* down the river and arrives at point *E*.

i) Taking *B* as the origin, write down Breanna's velocity vector in the form $x\underset{\sim}{i} + y\underset{\sim}{j}$ and find the magnitude of this vector. 2

ii) What is the bearing, to the nearest degree, of Breanna's velocity vector? 1

iii) How far does she travel from *B* to *E*? 1

End of Question 1

Question 2 (12 marks) Begin writing on a NEW page.

a) Prove by mathematical induction for all integers $n \geq 1$: 3

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \ldots + (2n-1)(2n) = \frac{n}{3}(n+1)(4n-1)$$

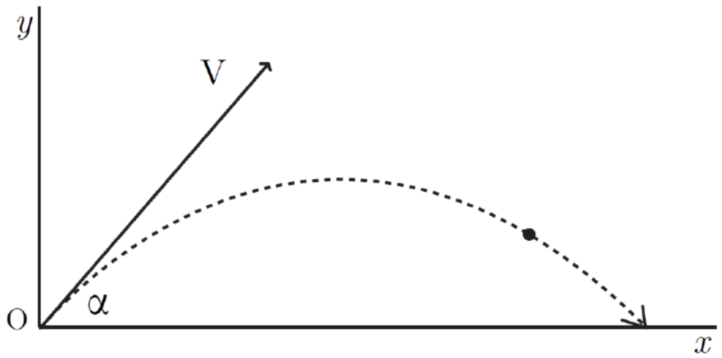
b) Use the substitution $u = e^x + 1$ to find $\int \frac{e^x}{\sqrt{e^x+1}}\,dx$. 2

c) Use the substitution $u = \tan x$ to find $\int \sec^2 x \tan^6 x \, dx$. 1

d) A projectile is fired from the origin O with limited speed V in metres per second and with an angle of projection α° . The time is measured in seconds.

The path of the projectile is given by the equations:

$x = Vt\cos\alpha$; $y = Vt\sin\alpha - \frac{1}{2}gt^2$, where g is the acceleration due to gravity.



Show all necessary working for the following:

i) In terms of V and α, find the maximum height h attained by the particle. 2

ii) Find the range R in terms of V and α. 2

iii) Show that: $\tan\alpha = \frac{4h}{R}$. 2

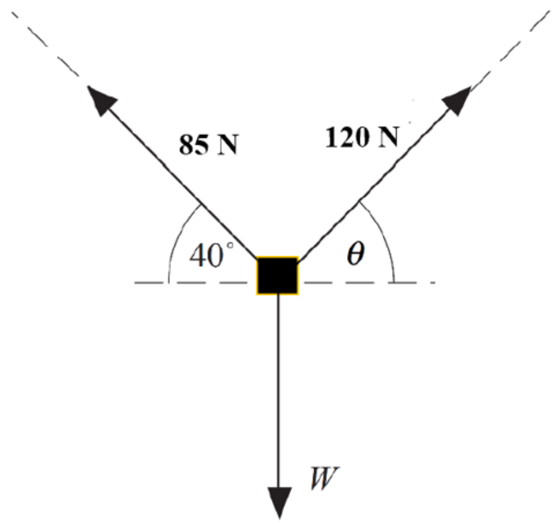
End of Question 2

Question 3 (13 marks) Begin writing on a NEW page.

a) The diagram shows the weight force of gravity, W, that pulls straight down on an object of mass m kg. Two forces of 85 N and 120 N are exerted by cables as shown. The resultant force acting on the object is zero.

i) Find the angle θ to the nearest degree. 2

ii) Using your answer from part (i), find the weight force W to the nearest Newton. 2



b)

i) Show that: $\frac{d}{dx}\left(x\tan^{-1}x\right)=\tan^{-1}(x)+\frac{x}{1+x^2}$. Show all working. 1

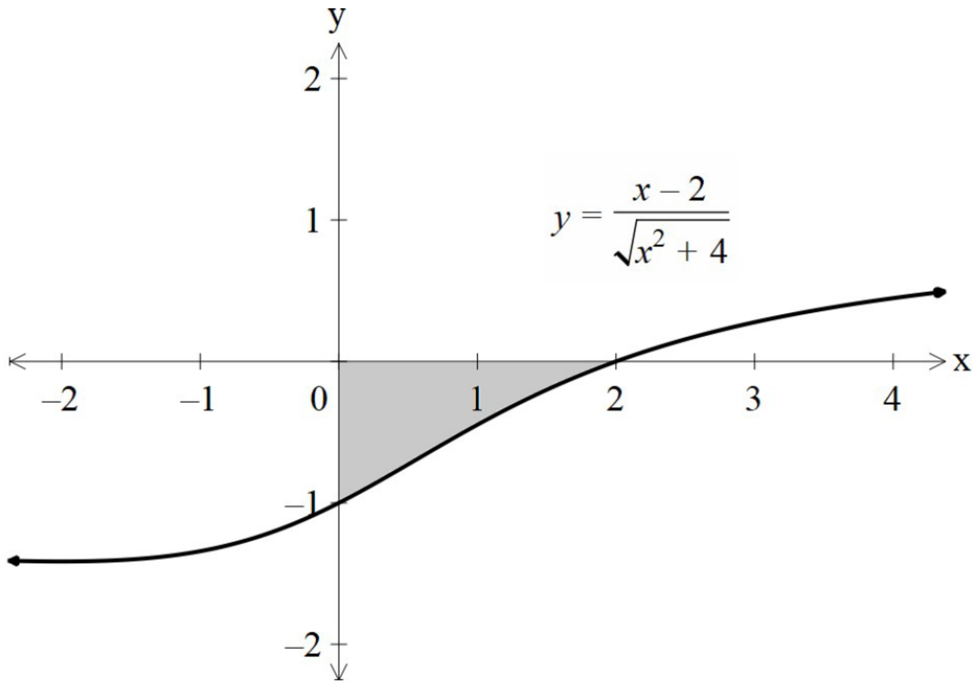
ii) Hence, evaluate find the exact value of $\int_0^{\sqrt{3}} \tan^{-1} x\,dx$. 2

Question 3 continues on page 7

c) Prove by mathematical induction that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n. 3

d) 3

The shaded region bounded by the curve $y = \frac{x-2}{\sqrt{x^2+4}}$ and $y = 0$, from $x = 0$ to $x = 2$ is shown in the diagram below.



End of Question 3

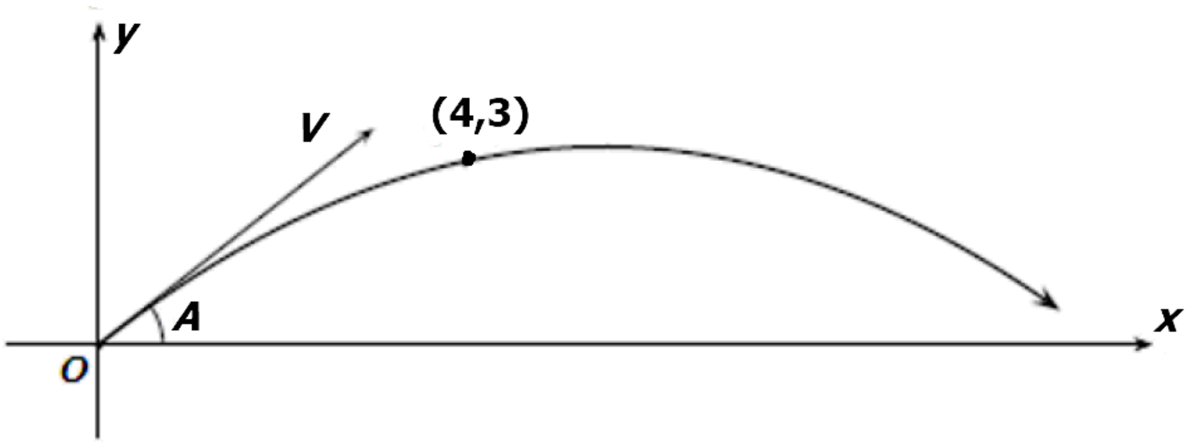
Question 4 (12 marks) Begin writing on a NEW page.

a) Prove by mathematical induction: $1\times 1! + 2\times 2! + 3\times 3! + \ldots + n\times n! = (n+1)! - 1$ for all integers $n \geq 1$. 3

b) A particle is projected from a point O with speed of V metres per second at an angle of A to the horizontal. Air resistance is negligible, and the acceleration due to gravity is $g\ ms^{-2}$.

The particle also passes through the point $(4,3)$.

The displacement-time equations for the projectile are: $\underset{\sim}{r}(t) = Vt\cos A\,\underset{\sim}{i} + \left(Vt\sin A - \frac{1}{2}gt^2\right)\underset{\sim}{j}$



i) Show that the Cartesian equation of the trajectory is given by:

$y = x\tan A - \frac{gx^2}{2V^2}\left(1+\tan^2 A\right)$. 2

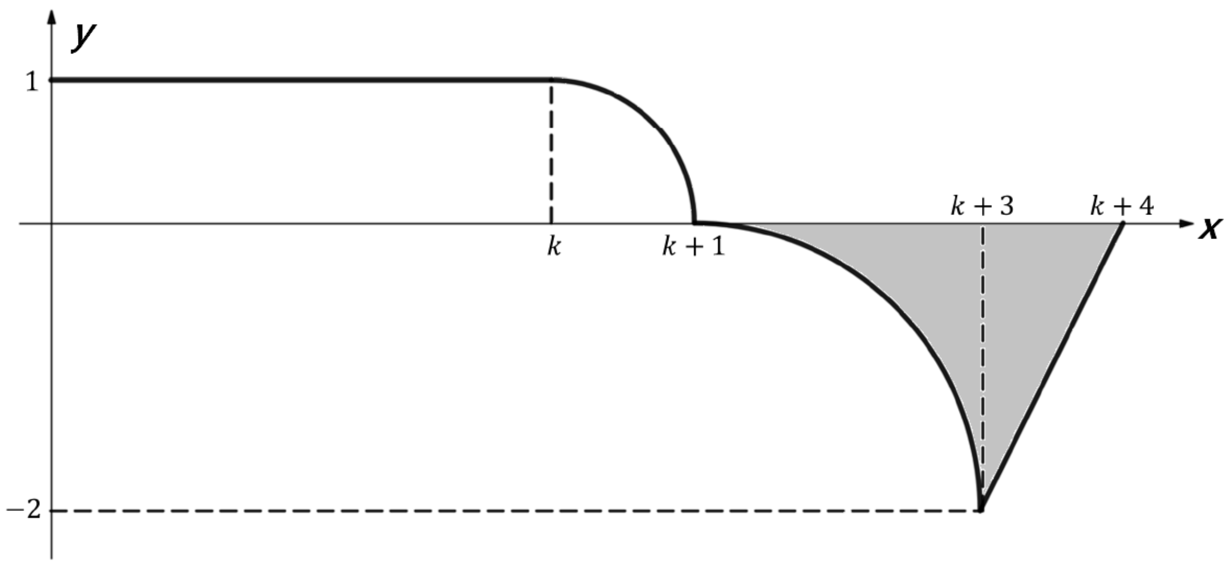
Assume $V^2 = 8g$ for part (ii) and part (iii).

ii) Show that the initial angle A of projection is given by: $A = \tan^{-1}(2)$. 2

iii) Show that the range R of the projectile is $\frac{32}{5}\ m$. 2

Question 4 continues on page 9

c) For the given graph it is given that $\int_{0}^{k+4} f(x)\,dx = 0$.



i) Find the value of k in the exact form. 1

ii) The shaded area is rotated about the x-axis. Find the volume of the solid formed, leave your answer in the exact form. 2

End of Task.

Question 1

a) i) $y = \cos^{-1}(7x)$ ✓

$$y' = \frac{-7}{\sqrt{1-49x^2}}$$

Don't forget to include the correct sign.

ii) $y = \tan^{-1}\left(\frac{x}{3}\right)$

$$y' = \frac{1}{3} \div 1 + \frac{x^2}{9}$$ ✓

$$= \frac{1}{3} \times \frac{9}{9+x^2}$$

$$= \frac{3}{9+x^2}$$

b) i) $\int \frac{1}{16+x^2}\,dx = \frac{1}{4}\tan^{-1}\frac{x}{4} + c$ ✓

ii) $\int \frac{1}{\sqrt{5-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}}{\sqrt{5-2x^2}}\,dx$ ✓

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}x}{\sqrt{5}} + c$$ ✓

c) $\int_0^{\pi/4} \sin^2 x\,dx = \int_0^{\pi/4}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$ ✓

$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{8} - \frac{1}{4}$$ ✓

d) $y = 2\sin^{-1}(3x) + \frac{\pi}{2}$

$y' = 2 \times \frac{3}{\sqrt{1-9x^2}}$

$= \frac{6}{\sqrt{1-9x^2}}$ ✓

when $x = 0$ $\quad y = 2\sin^{-1}(0) + \frac{\pi}{2}$ $\quad m_T = 6$

$= \frac{\pi}{2}$

∴ equation of tangent is:

$y - \frac{\pi}{2} = 6(x - 0)$

$y = 6x + \frac{\pi}{2}$ ✓

e) i) $\underset{\sim}{v} = 3\underset{\sim}{i} + 6\underset{\sim}{j}$ ✓ Some students mixed up the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components.

$|\underset{\sim}{v}| = \sqrt{3^2 + 6^2}$

$= \sqrt{45}$

$= 3\sqrt{5}$ m/s ✓ } both acceptable answers.

$\doteqdot 6.71$ m/s

ii) $\tan\theta = \frac{3}{6}$

$\tan\theta = \frac{1}{2}$

$\theta \doteqdot 27°$

∴ Bearing of Breanna's velocity vector is 027° T. ✓

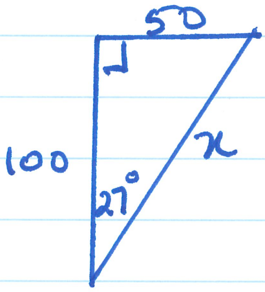
The answer needed to be given to the nearest degree, as specified in the question, to receive the mark.

iii) $\cos 27 = \dfrac{100}{x}$

$x = \dfrac{100}{\cos 27}$

$x \doteqdot 112.23\text{ m}$

alternate soln:



$x^2 = 50^2 + 100^2$

$x = 50\sqrt{5}$

$\doteqdot 111.8\text{ m}$

A range of slightly different answers accepted depending on approach.

Question 2 (12 marks) Begin writing on a NEW page.

a) Prove by mathematical induction for all integers $n \geq 1$: 3

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \ldots + (2n-1)(2n) = \frac{n}{3}(n+1)(4n-1)$$

Let $S(n) = (1 \times 2) + (3 \times 4) + (5 \times 6) + \ldots + (2n-1)(2n) = \frac{n}{3}(n+1)(4n-1)$

For $n=1$: $LHS = (2n-1)(2n)$ $\quad RHS = \frac{n}{3}(n+1)(4n-1)$

$= (2-1)(2)$ $\quad = \frac{1}{3}(2)(3)$

$= 2$ $\quad = 2$

Since $LHS = RHS$, then $S(n)$ is true for $n=1$. ✓

Assume $S(k)$ is true for integer $k \geq 1$.

That is:

$S(k) = (1 \times 2) + (3 \times 4) + (5 \times 6) + \ldots + (2k-1)(2k) = \frac{k}{3}(k+1)(4k-1)$

Prove that $S(k+1)$ is true.

That is show: $S(k+1) = \frac{(k+1)}{3}(k+1+1)(4(k+1)-1)$

$S(k+1) = \frac{1}{3}(k+1)(k+2)(4k+3)$

Proof: $LHS = S(k+1)$

$= S(k) + T(k+1)$ ✓

$= \frac{k}{3}(k+1)(4k-1) + [2(k+1)-1][2(k+1)]$ (from assumption)

$= \frac{k}{3}(k+1)(4k-1) + (2k+1)(2k+2)$

$= \frac{k}{3}(k+1)(4k-1) + (2k+1)2(k+1)$

Students need to write this at this step.

$= \frac{1}{3}(k+1)[k(4k-1) + 6(2k+1)]$

$= \frac{1}{3}(k+1)[4k^2 - k + 12k + 6]$ ✓

$= \frac{1}{3}(k+1)[4k^2 + 11k + 6]$

3

$= \frac{1}{3}(k+1)(k+2)(4k+3)$

$= RHS.$

$\therefore$ $S(k+1)$ is true.

Since $S(n)$ is true for $n=1$ and for $n=k+1$ when it is true for $n=k$, then by the Principle of Mathematical Induction, $S(n)$ is true for all integers $n \geq 1$.

* Students who skipped the factorisation step in the proof were deducted one mark.

b) Use the substitution $u = e^x + 1$ to find $\int \frac{e^x}{\sqrt{e^x+1}} dx$. 2

$u = e^x + 1$

$du = e^x dx$

$$\int \frac{e^x dx}{\sqrt{e^x+1}}$$

$$= \int \frac{du}{\sqrt{u}}$$

$$= \int u^{-1/2} du$$

$$= \frac{u^{1/2}}{1/2} + C$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{e^x+1} + C$$

2

* Students lost one mark if they did not return their answer in terms of x.

c) Use the substitution $u = \tan x$ to find $\int \sec^2 x \tan^6 x \, dx$.

$u = \tan x$

$du = \sec^2 x \, dx$

$\int \sec^2 x \tan^6 x \, dx$

$= \int \tan^6 x \sec^2 x \, dx$

$= \int u^6 . du$

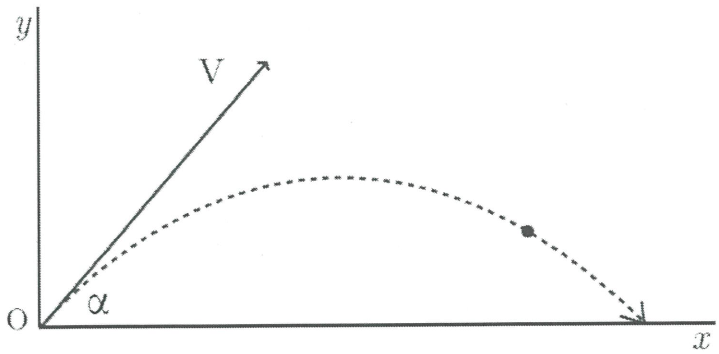
$= \dfrac{u^7}{7} + c$ ①

$= \dfrac{\tan^7 x}{7} + c.$ ✓ Correct answer only.

d) A projectile is fired from the origin O with limited speed V in metres per second and with an angle of projection α°. The time is measured in seconds.

The path of the projectile is given by the equations:

$x = Vt\cos\alpha$; $y = Vt\sin\alpha - \frac{1}{2}gt^2$, where g is the acceleration due to gravity.



Show all necessary working for the following:

i) In terms of V and α, find the maximum height h attained by the particle. 2

ii) Find the range R in terms of V and α. 2

iii) Show that: $\tan\alpha = \dfrac{4h}{R}$. 2

i) Given $y = Vt\sin\alpha - \frac{1}{2}gt^2$

max height occurs when $\dot{y} = 0$:

$$\dot{y} = V\sin\alpha - gt$$

$(\dot{y} = 0)$ $0 = V\sin\alpha - gt$

$$gt = V\sin\alpha$$

$$t = \frac{V\sin\alpha}{g} \checkmark$$

All formulae needed to be derived.

Hence max height, h:

$$y = V\sin\alpha\left(\frac{V\sin\alpha}{g}\right) - \frac{g}{2}\left(\frac{V\sin\alpha}{g}\right)^2$$

$$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$$

$$\therefore h = \frac{V^2\sin^2\alpha}{2g} \checkmark$$

2

ii) Range occurs when $y = 0$:

$$y = Vt\sin\alpha - \frac{gt^2}{2}$$

$(y=0)$: $0 = t\left(V\sin\alpha - \frac{gt}{2}\right)$

$\therefore$ either $t = 0$ or $V\sin\alpha - \frac{gt}{2} = 0$.

$$\frac{gt}{2} = V\sin\alpha$$

$$\therefore t = \frac{2V\sin\alpha}{g} \checkmark$$

Hence Range:

$$x = Vt\cos\alpha$$

$$x = V\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$$

$$\therefore \quad R = \frac{2V^2\sin\alpha\cos\alpha}{g}$$

$$R = \frac{V^2\sin 2\alpha}{g}. \checkmark \qquad 2.$$

iii) $RHS = \dfrac{4h}{R}$

$$= \frac{4V^2\sin^2\alpha}{2g} \times \frac{g}{2V^2\sin\alpha\cos\alpha} \checkmark$$

$$= \frac{4V^2 g\sin^2\alpha}{4V^2 g\sin\alpha\cos\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha} \checkmark$$

$$= \tan\alpha$$

$$= LHS$$

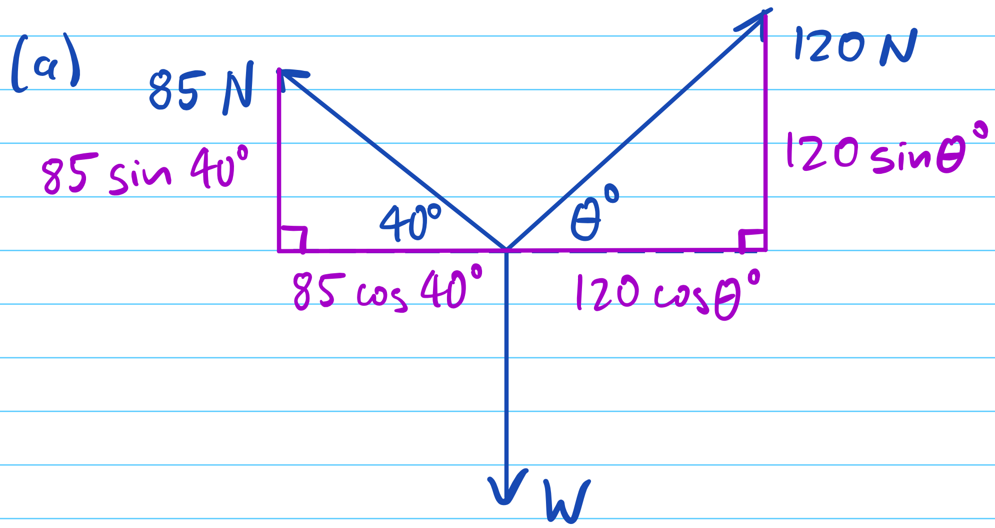
2

$\therefore \tan\alpha = \dfrac{4h}{R}$, as required.

* Most marks were lost by students in part iii).

Question 3

(a)



2 (i) Since the resultant force acting on the object is zero, the net horizontal force is zero. ✓

$$\therefore\ 120\cos\theta^\circ - 85\cos 40^\circ = 0$$

$$\cos\theta = \frac{85\cos 40^\circ}{120}$$

$$\theta = \cos^{-1}\left(\frac{85\cos 40^\circ}{120}\right)$$

$$= 57^\circ \text{ (nearest degree)}$$ ✓

2 (ii) Net vertical force is zero.

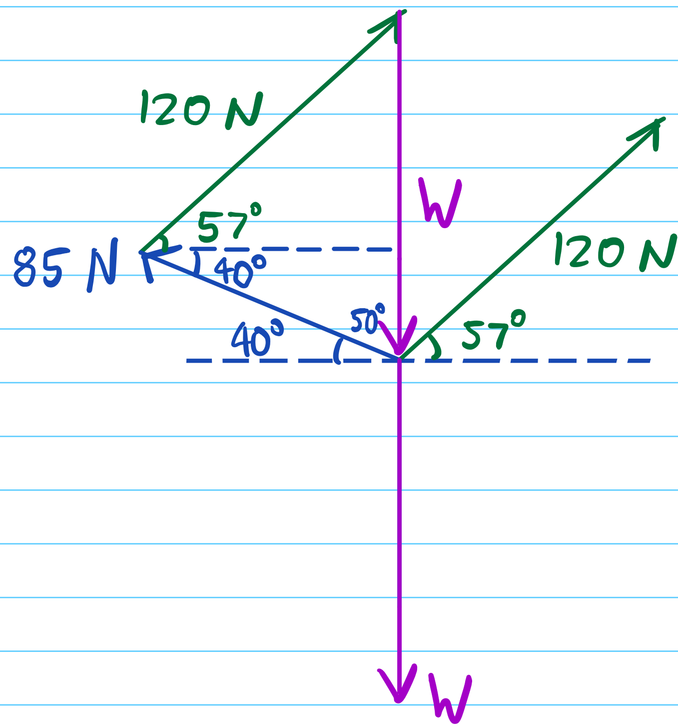
$$\therefore\ 85\sin 40^\circ + 120\sin\theta^\circ - W = 0$$ ✓

$$W \doteqdot 155 \text{ N (nearest newton)}$$ ✓

Alternative method (Not recommended)

Note: Most students who used this method made at least one mistake in finding angles. The previous method is much safer than this one!

(a) (ii) Take $\theta^\circ \doteqdot 57^\circ$ from (i).



Note: As the forces are in equilibrium the vectors can be rearranged to form a triangle, as shown.

$$\frac{W}{\sin 97^\circ} = \frac{120}{\sin 50^\circ} \quad \text{(sine rule)}$$

$$W = \frac{120 \sin 97^\circ}{\sin 50^\circ}$$

$$\doteqdot 155 \text{ N} \quad \text{(nearest newton)}$$

1 (b) (i) $\frac{d}{dx}\left(\underbrace{x}_{u}\,\underbrace{\tan^{-1}x}_{v}\right)$

$u = x$
$u' = 1$

$v = \tan^{-1}x$
$v' = \frac{1}{1+x^2}$

$= vu' + uv'$

$= \tan^{-1}x \times 1 + x \times \frac{1}{1+x^2}$

$= \tan^{-1}x + \frac{x}{1+x^2}$ ✓

2 (ii) $\int_0^{\sqrt{3}} \tan^{-1}x \, dx$

$= \int_0^{\sqrt{3}} \left(\frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}\right) dx$ ✓

$= \left[x\tan^{-1}x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{1+x^2}\,dx$

$= \sqrt{3}\tan^{-1}\sqrt{3} - \frac{1}{2}\left[\ln\left|1+x^2\right|\right]_0^{\sqrt{3}}$

$= \sqrt{3}\left(\frac{\pi}{3}\right) - \frac{1}{2}\ln 4$ ✓

$= \frac{\pi\sqrt{3}}{3} - \ln 2$

3 (c) Base case: For $n=1$,

$$9^{n+2} - 4^n = 9^{1+2} - 4^1$$
$$= 729 - 4$$
$$= 725$$
$$= 5 \times 145,$$

which is divisible by 5.

Assume true for $n=k$:

$$9^{k+2} - 4^k = 5P,$$

where P is a positive integer.

i.e. $9^{k+2} = 5P + 4^k$ $(*)$

Prove true for $n=k+1$:

$$9^{k+3} - 4^{k+1} = 9(\underline{9^{k+2}}) - 4(4^k)$$

From $(*)$ $= 9(\underline{5P + 4^k}) - 4(4^k)$

$$= 5(9P) + 9(4^k) - 4(4^k)$$
$$= 5(9P) + 5(4^k)$$
$$= 5(9P + 4^k)$$

which is divisible by 5, as required.

3 (d) Volume $= \pi \int_0^2 y^2 \, dx$ ✓

$$= \pi \int_0^2 \left(\frac{x-2}{\sqrt{x^2+4}} \right)^2 dx$$

$$= \pi \int_0^2 \frac{(x-2)^2}{x^2+4} \, dx$$

$$= \pi \int_0^2 \frac{x^2-4x+4}{x^2+4} \, dx$$

$$= \pi \int_0^2 \left(\frac{x^2+4}{x^2+4} - \frac{4x}{x^2+4} \right) dx$$ ✓

$$= \pi \int_0^2 \left(1 - \frac{4x}{x^2+4} \right) dx$$

$$= \pi \left[x - 2\ln\left|x^2+4\right| \right]_0^2$$

$$= \pi \left[2 - 2\ln 8 - (0 - 2\ln 4) \right]$$ ✓

$$= \pi\left(2 - 2\ln 8 + 2\ln 4\right)$$

$$= \pi\left(2 - 6\ln 2 + 4\ln 2\right)$$

$$= 2\pi\left(1 - \ln 2\right) \text{ cubic units}$$

Notes:

(c) The best method for the induction question was to sub $9^{k+2} = 5M + 4^k$ into $9^{k+3} = 9(9^{k+2})$. Students who tried to use manipulation often made a mistake with $\pm$ signs.

(d) No need to use substitution for $\int_0^2 \frac{4x}{x^2+4}\,dx$ Students are encouraged to recognise that the top is almost the same as the derivative of the bottom, which leads to $\left[2\ln|x^2+4|\right]_0^2$.

Q4

a) $1\times1! + 2\times2! + 3\times3! + \dots + n\times n! = (n+1)!-1$

prove it is true for $n = 1$

$LHS = 1\times1! = 1$

$RHS = (1+1)! - 1 = 1$

$LHS = RHS$

True for $n = 1$

· Assume it is true for $n = k$

$1\times1! + 2\times2! + 3\times3! + \dots + k\times k! = (k+1)! - 1$

prove it is true for $n = k+1$

$1\times1! + 2\times2! + 3\times3! + \dots k\times k! + (k+1)(k+1)! = (k+2)! - 1$

$LHS = (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,[k+1+1] - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1$

$=$ RHS. This is true for $n = k+1$

It is proven by Mathematical Induction.

$\boxed{Q4}$

b) $x = Vt\cos A$ ①

$y = -\frac{1}{2}gt^2 + Vt\sin A$ ②

i) from ① $t = \frac{x}{V\cos A}$ subs into ②

$$y = -\frac{1}{2}g\left(\frac{x^2}{V^2\cos^2 A}\right) + V\left(\frac{x}{V\cos A}\right)\sin A$$

$$y = x\tan A - \frac{gx^2}{2V^2} \cdot \sec^2 A$$

$$y = x\tan A - \frac{gx^2}{2V^2}(1+\tan^2 A) \quad (*)$$

ii) use $V^2 = 8g$ and $x = 4$, $y = 3$

subs into $(*)$

$$3 = 4\tan A - \frac{g \times 16}{2 \times 8g}(1+\tan^2 A)$$

$$3 = 4\tan A - 1 - \tan^2 A$$

$$\tan^2 A - 4\tan A + 4 = 0$$

$$(\tan A - 2)^2 = 0$$

$$\tan A = 2$$

$$A = 63^\circ 26'$$

Q4/b

iii) from (i)

$$y = x\tan A - \frac{gx^2}{2V^2}(1+\tan^2 A)$$

Subs $\tan A = 2$, $v^2 = 8g$, $y = 0$

Range = Max horizontal disp $\therefore$ $y = 0$

$$0 = 2x - \frac{gx^2}{16g}(1+4)$$

$$2x - \frac{5x^2}{16} = 0$$

$$32x - 5x^2 = 0$$

$$x(32 - 5x) = 0$$

$$x = 0$$

$$x = \frac{32}{5}$$

$\therefore$ Range $= \frac{32}{5} = 6.4\,m$

$\boxed{Q4}$ c/(i) $\int_0^{k+4} f(x)\,dx = 0$ means

top Areas = bottom Areas

Top Areas $= (k \times 1) + \frac{\pi \times 1^2}{4} = k + \frac{\pi}{4}$

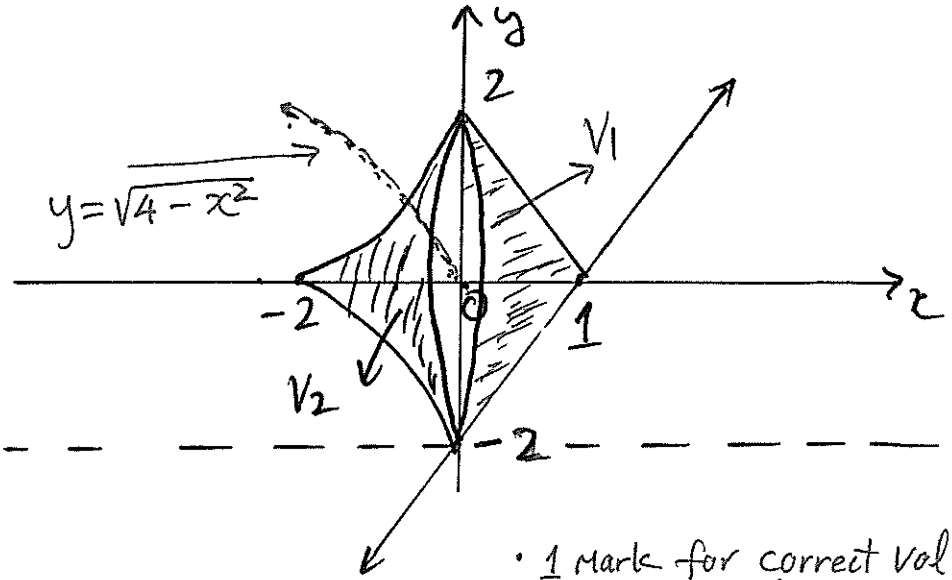
Bottle Areas $= \left(2\times 2 - \frac{\pi \times 2^2}{4}\right) + \left(\frac{1 \times 2}{2}\right)$

$= 5 - \pi$

$k + \frac{\pi}{4} = 5 - \pi$

$k = 5 - \frac{5\pi}{4}$

Q4 c(ii) The diagram can be drawn as



• 1 Mark for correct Vol of the Cone.
OR correct vol of V_2
OR correct Area under the x-axis.

$V = V_1 + V_2$

V_1 = Vol of a cone, radius = 2 u, height = 1 u

V_2 = Vol of the curve $\sqrt{4-x^2}-2$ rotated about the x-axis from -2 to 0.

$$V_1 = \frac{\pi r^2 h}{3} = \frac{\pi \times 2^2 \times 1}{3} = \frac{4\pi}{3}$$

$$V_2 = \pi \int_{-2}^{0} y^2\,dx = \pi \int_{-2}^{0} \left(\sqrt{4-x^2}-2\right)^2 dx$$

$$V_2 = \pi \int_{-2}^{0} \left(4 - x^2 - 4\sqrt{4-x^2} + 4\right)dx$$

$$= \pi \int_{-2}^{0} (8-x^2)\,dx - \pi\int_{-2}^{0} 4\sqrt{4-x^2}\,dx$$

$$= \pi\left[8x - \frac{x^3}{3}\right]_{-2}^{0} - 4\pi\left[\frac{1}{4}\pi \times 2^2\right] = \frac{40\pi}{3} - 4\pi^2$$

Total Vol: $V = V_1 + V_2 = \frac{4\pi}{3} + \frac{40\pi}{3} - 4\pi^2 = \frac{44\pi}{3} - 4\pi^2 \ u^3$