QUESTION ONE (Start a new answer booklet)

(a) Express 225° in radians in terms of π.

(b) An object is projected at an angle of 30° to the horizontal and with speed 40 m/s. What is its initial vertical speed?

(c) Differentiate with respect to x:

(i) $y = e^{4-x}$,

(ii) $y = \sin^{-1}\sqrt{x}$.

(d) Find primitives of:

(i) $\dfrac{1}{2x}$,

(ii) $\dfrac{1}{9+x^2}$.

(e) Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin x}{2x}$.

(f) Let $P(x) = 2x^3 + x^2 - 7x - 6$.

(i) Find the remainder when $P(x)$ is divided by $(x+2)$.

(ii) Suppose $P(x)$ has roots α, β and γ. Find the value of $\alpha\beta\gamma$ without finding the roots.

(g) A polynomial of degree 5 is divided by another polynomial of degree 2. What is the maximum degree of the remainder?

QUESTION TWO (Start a new answer booklet)

(a) Given $t = \tan\frac{\theta}{2}$, write down an expression in terms of t for $\cos\theta$.

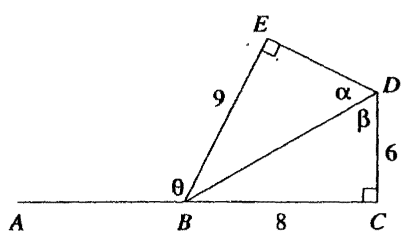
(b) The polynomial $f(x) = x^3 + ax^2 + b$ is exactly divisible by $(x-2)$, and has a remainder of 5 when divided by $(x-1)$. Find the values of a and b.

(c) The area under $y = \dfrac{5}{\sqrt{25+x^2}}$ between $x = 0$ and $x = 5$ is rotated about the x-axis.

(i) Write down the integral for the volume of this solid.

(ii) Hence find the volume.

(d)



In the diagram above, $BC = 8$, $CD = 6$ and $BE = 9$, and the angles at C and E are right angles as shown.

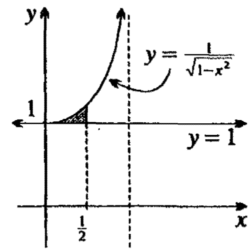
(i) Find the lengths of BD and ED.

(ii) Show that $\theta = \alpha + \beta$.

(iii) Hence or otherwise, show that $\sin\theta = \dfrac{27 + 4\sqrt{19}}{50}$.

QUESTION THREE (Start a new answer booklet)

(a)



(i) Write down the area of the shaded region in terms of an integral.

(ii) Hence show that this area is $\dfrac{\pi - 3}{6}$.

(b) The velocity of a particle is given by $v = t^2 - 2t$, and it starts at $x = 3$.

(i) When is the particle moving to the left?

(ii) Find the displacement x as a function of time.

(iii) What is the closest the particle gets to the origin?

(c) The monic polynomial $g(x)$ of degree 6 has a zero at $x = -3$, a double zero at $x = -1$, and a triple zero at $x = 1$.

(i) Write down $g(x)$ in factored form.

(ii) Find the y-intercept of the graph of $g(x)$.

(iii) Sketch $y = g(x)$, without using calculus, clearly showing the behaviour at each x-intercept.

QUESTION FOUR (Start a new answer booklet)

(a) Write $\cos 2t + \sin 2t$ in the form $A\cos(2t - \epsilon)$.

(b) (i) Sketch $y = e^x$ and $y = 4 - x^2$ on the same number plane, showing all intercepts with the axes.

(ii) Deduce from your answer to part (i) the number of solutions to the equation $e^x = 4 - x^2$.

(iii) Let $g(x) = e^x + x^2 - 4$. Given the initial estimate for a zero of this function is $x_0 = 1$, apply Newton's method once to find a better approximation x_1. Give your answer to three significant figures.

QUESTION FIVE (Start a new answer booklet)

(a) When an object has displacement x, its acceleration is given by

$$\ddot{x} = -3(2-x)^2 .$$

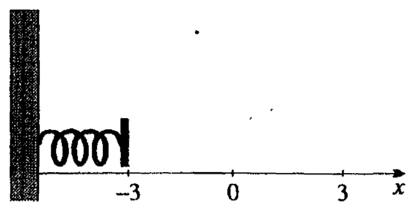
As the object passes through the origin, it has a velocity of 4 m/s.

(i) Show that $v^2 = 2(2-x)^3$.

(ii) Explain why the object can never be to the right of $x = 2$.

(iii) Show that the object's velocity can never be negative.

(b)



A plate attached to a spring mounted on a wall has mean position $x = 0$. The spring is compressed so that the plate is at $x = -3$, as shown above. The plate is then released from rest, and its displacement x metres at time t seconds satisfies

$$\ddot{x} = -\tfrac{16}{9}x .$$

(i) Given $x = -A\cos nt$, find the values of A and n.

(ii) What is the period of this motion?

(iii) What is the speed of the plate as it passes through the origin?

(iv) The experiment is repeated with a light ball placed in front of the plate. Explain why the ball separates from the plate as it passes through the origin.

QUESTION SIX (Start a new answer booklet)

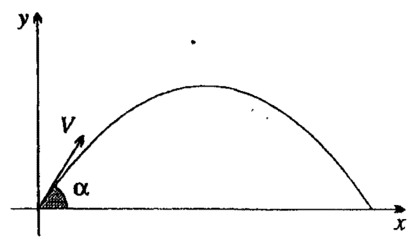
(a) Consider the cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$, which has roots α, β and γ such that $\gamma = \alpha + \beta$.

(i) Show $\alpha + \beta = -\dfrac{b}{2a}$.

(ii) Also show $\alpha\beta = \dfrac{2d}{b}$.

(iii) Hence show that α and β are also roots of $x^2 + \frac{b}{2a}x + \frac{2d}{b} = 0$.

(b)



A ball is thrown on level ground at an initial speed of V m/s and at an angle of projection α as in the diagram above. You may assume that t seconds after release, the horizontal and vertical displacements are:

$$x = Vt\cos\alpha$$
$$y = Vt\sin\alpha - \tfrac{1}{2}gt^2 .$$

(i) Find the Cartesian equation of the trajectory.

(ii) Hence show that the range is $\dfrac{V^2 \sin 2\alpha}{g}$.

(iii) When the ball is thrown with speed $V = 30$ m/s, it lands 45 metres away.

(α) Given $g = 10\,\text{m/s}^2$, find α.

(β) A 2 metre high fence is placed 40 metres away from the thrower. Examine each trajectory to see whether it is still possible.

SGS.

VI 3 Unit Mid-Yr. 1999

Solutions

1/ (a) $\frac{5\pi}{4}$

(b) $y = 40\sin 30°$

$= 20$ m/s.

(c) (i) $y = e^{4-x}$

$y' = -e^{4-x}$

(ii) $y = \sin^{-1}\sqrt{x}$

$y' = \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}}$

$= \frac{1}{2\sqrt{x-x^2}}$

(d) (i) $y' = \frac{1}{2x}$

$y = \frac{1}{2}\log x \quad (+c)$

(ii) $y' = \frac{1}{9+x^2}$

$y = \frac{1}{3}\tan^{-1}\frac{x}{3} \quad (+c)$

(e) $\lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}$

$= \frac{1}{2}$

(f) (i) $P(-2) = 2\times(-8) + 4 - 7\times(-2) - 6$

$= -4$

(ii) $\alpha\beta\gamma = \frac{-d}{a} = 3$

g) degree 1.

2/ (a) $\cos\theta = \frac{1-t^2}{1+t^2}$

(b) $f(2) = 8 + 4a + b = 0 \quad (1)$

$f(1) = 1 + a + b = 5 \quad (2)$

(1) - (2) yields

$7 + 3a = -5$

$a = -4$

so $b = 8$

and $f(x) = x^3 - 4x^2 + 8$.

(c) (i) $V = \pi\int_0^5 \frac{25}{25+x^2}\,dx$

(ii) so $V = 25\pi\left[\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)\right]_0^5$

$= 5\pi\left[\tan^{-1}1 - 0\right]$

$= \frac{5\pi^2}{4}$ cubic units.

(d) (i) $BD = 10$ (by Pyth)

$ED = \sqrt{19}$ (by Pyth)

(ii) $\angle EBD = 90 - \alpha$ ($\angle$ sum of ΔEBD)

$\angle CBD = 90 - \beta$ ($\angle$ sum of ΔCBD)

so $\angle EBC = 180° - (\alpha+\beta)$

$= 180 - \theta$

thus $\theta = \alpha + \beta$.

[or use the fact that EBCD is a cyclic quad.]

(iii) $\sin\theta = \sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{9}{10}\cdot\frac{6}{10} + \frac{\sqrt{19}}{10}\cdot\frac{8}{10}$

$= \frac{27 + 4\sqrt{19}}{50}$

3/ (a) (i) $A = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} - 1\,dx$

(ii) $A = \left[\sin^{-1}x - x\right]_0^{1/2}$

$= \frac{\pi}{6} - \frac{1}{2}$

$= \frac{\pi - 3}{6}$

(b) (i) ie $v < 0$ so $t^2 - 2t < 0$

thus $0 < t < 2$

(ii) $x = \int v\,dt$

$= \frac{t^3}{3} - t^2 + c$

at $t = 0$, $x = c = 3$

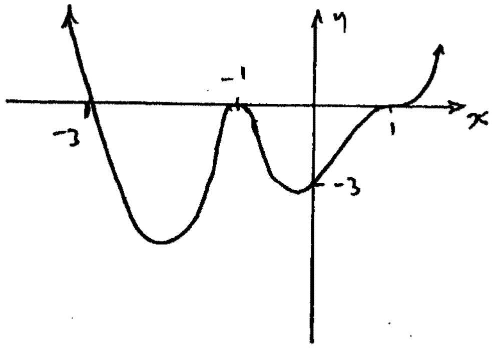
thus $x = \frac{t^3}{3} - t^2 + 3$

(iii) at $t = 2$; $x = \frac{5}{3}$.

$g(x) = (x+3)(x+1)^2(x-1)^3$

(ii) $g(0) = -3$

(iii)



4 (a) $\cos 2t + \sin 2t = A\cos(2t - \varepsilon)$
$= A\cos 2t \cos\varepsilon + A\sin 2t \sin\varepsilon$

equate coeffs. of $\sin 2t$ and $\cos 2t$:

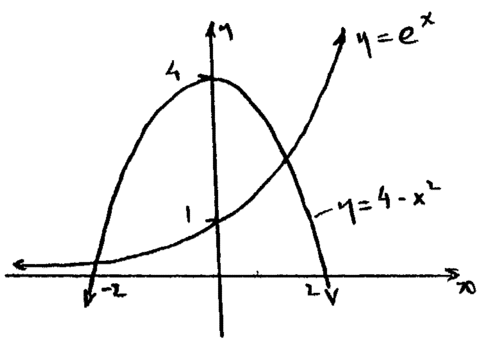
$A\cos\varepsilon = 1$

$A\sin\varepsilon = 1$

hence $A = \sqrt{2}$ and $\varepsilon = \pi/4$ since its in 1st quad.

so $\cos 2t + \sin 2t = \sqrt{2}\cos(2t - \frac{\pi}{4})$

(b)(i)



(ii) They intersect twice

(iii) $g(x) = e^x + x^2 - 4$ so $g(1) \doteqdot 0.2817$

$g'(x) = e^x + 2x$ so $g'(1) \doteqdot 4.7182$

$x_1 = 1 - \dfrac{g(1)}{g'(1)}$

$\doteqdot 1.0597\ldots$

$\doteqdot 1.06$ to 3 sig fig.

5(a)(i) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = -3(2-x)^2$

so $\frac{1}{2}v^2 = (2-x)^3 + \frac{c}{2}$ for some constant c.

at $x = 0$, $v = 4$ so $c = 0$

and $v^2 = 2(2-x)^3$

(ii) require $v^2 \geq 0$ so $2(2-x)^3 \geq 0$

hence $x \leq 2$.

(iii) for v to become -ve, it must first be zero.

when $v = 0$, $x = 2$

when $x = 2$, $\ddot{x} = 0$

Thus the particle is at rest and remains so [since $\ddot{x}$ is strictly a function of x]. Hence v is never -ve.

or

Let $v = \sqrt{2}(2-x)^{3/2}$ which is true at least once. Then

$t = \dfrac{\sqrt{2}}{\sqrt{2-x}} + k$ by integration.

From this we see x never reaches 2. So $v \neq 0$ and hence v never changes sign.

(b) (i) $n^2 = \frac{16}{9}$ so $n = \frac{4}{3}$

at $t = 0$ $x = -A = -3$

so $A = 3$

and $x = -3\cos\left(\frac{4t}{3}\right)$

(ii) period $= \dfrac{2\pi}{(4/3)} = \dfrac{3\pi}{2}$

(iii) $\dot{x} = 4\sin\left(\frac{4t}{3}\right)$

so at the origin speed $= 4$ m/s.

(iv) The plate begins to decelerate but the ball keeps going at the same rate.

[It may be tempting for 4u boys to introduce arguments about friction and the like. Given the information in the question, the above still stands - ie no friction etc....]

$\alpha+\beta+\gamma = -\frac{b}{a}$

so $2(\alpha+\beta) = -\frac{b}{a}$

and $\alpha+\beta = \frac{-b}{2a}$ (1)

(ii) $\alpha\beta\gamma = -\frac{d}{a}$

so $\alpha\beta(\alpha+\beta) = -\frac{d}{a}$

$\alpha\beta\left(\frac{-b}{2a}\right) = -\frac{d}{a}$

Hence $\alpha\beta = \frac{2d}{b}$. (2)

(iii) For the given equation, the sum of the roots is $-\frac{b}{2a}$ and $\alpha+\beta = -\frac{b}{2a}$; also the product of the roots is $\frac{2d}{b}$ and $\alpha\beta = \frac{2d}{b}$.

Hence α and β are the roots.

or

solve equations (1) and (2) simultaneously.

(b)(i) $t = \frac{x}{V\cos\alpha}$

so $y = V\sin\alpha \cdot \frac{x}{V\cos\alpha} - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2$

$= x\tan\alpha - \frac{g}{2V^2}x^2\sec^2\alpha$

(ii) $y = x\sec^2\alpha\left[\sin\alpha\cos\alpha - \frac{g}{2V^2}x\right]$

So the range is given by

$\frac{gx}{2V^2} = \sin\alpha\cos\alpha$

or $x = \frac{V^2\sin 2\alpha}{g}$

(iii)(α) $45 = \frac{900\sin 2\alpha}{10}$

so $\sin 2\alpha = \frac{1}{2}$

thus $\alpha = 15°$ or $75°$

(β) at $x = 40$

if $\alpha = 15°$

$y \doteqdot 1.19 < 2\text{ m}$

if $\alpha = 75°$

$y \doteqdot 16.59 > 2\text{m}$

thus $\alpha = 15°$ is not possible and hence $\alpha = 75°$.

7/(a)(i) in the Δ $C = \pi - (A+B)$

so the sine rule with angle C gives

$\frac{\sin[\pi-(A+B)]}{c} = \frac{\sin B}{b}$

but $\sin(\pi-\theta) = \sin\theta$

Hence $\frac{\sin(A+B)}{c} = \frac{\sin B}{b}$

(ii) expand LHS to get

$\frac{\sin A\cos B + \cos A\sin B}{c} = \frac{\sin B}{b}$

$\times \frac{cb}{\cos B}$ to get

$b\sin A + b\cos A\tan B = c\tan B$

so $\tan B(c - b\cos A) = b\sin A$

$\tan B = \frac{b\sin A}{c - b\cos A}$

$= \frac{\sin A}{\frac{c}{b} - \cos A}$.

(iii) The angle A is the included angle between sides b and c, and hence ΔABC is unique by the SAS congruence test.

(b)(i) in ΔABD $a^2 = b^2+1-2b\cos\theta$ (1)

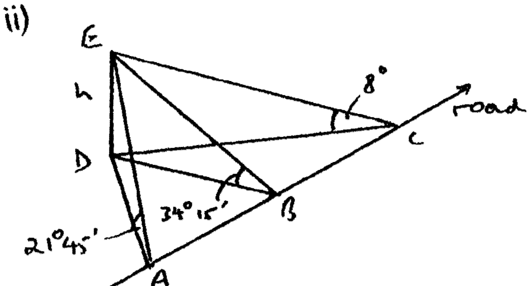
in ΔCBD $c^2 = b^2+1-2b\cos(\pi-\theta)$

or $c^2 = b^2+1+2b\cos\theta$ (2)

(1)+(2) yields

$a^2+c^2 = 2b^2+2$

(ii)



$DC = h\cot 8° = h\gamma$

$DB = h\cot 34°15' = h\beta$

$DA = h\cot 21°45' = h\alpha$

--- ctd

part (i)

$(h\alpha)^2 + (h\gamma)^2 = 2(h\beta)^2 + 2$

rearranging

$h^2(\alpha^2 + \gamma^2 - 2\beta^2) = 2$.

$h = \sqrt{\dfrac{2}{\alpha^2 + \gamma^2 - 2\beta^2}}$

$\doteqdot 0.195$ km

$= 195$ m to nearest metre.

or $EC = h\,\text{cosec}\,8° = h\gamma$

$EB = h\,\text{cosec}\,34°15' = h\beta$

$EA = h\,\text{cosec}\,21°45' = h\alpha$

and then proceed as above.

(c) Let the sides be

$b-d,\ b,\ b+d$

then by (b) (i)

$(b-d)^2 + (b+d)^2 = 2b^2 + 2$

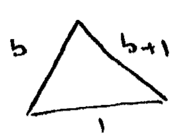
so $2b^2 + 2d^2 = 2b^2 + 2$

so $d^2 = 1$

and $d = 1$ (the -ve soln is redundant)

The sides are then $b-1, b, b+1$

so ΔBCD has dimensions



which is not possible for a Δ.

or

Let $b = \frac{a+c}{2}$

so $a^2 + c^2 = 2\left(\frac{a+c}{2}\right)^2 + 2$

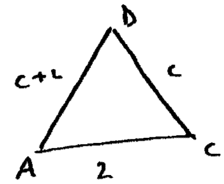
$2a^2 + 2c^2 = (a+c)^2 + 4$

thus $(a-c)^2 = 4$

and $a - c = 2$ (the -ve soln is redundant)

so $a = c + 2$

ΔADC then has the dimensions



which is not possible for a Δ.

[There are other variations on the same basic proof.]

DNW.
20/5/99.