TRINITY GRAMMAR SCHOOL

Mathematics Department

2012

HALF YEARLY EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 – 14
- Write your Board of Studies Student Number **and** Class Teacher on the writing booklet(s) **or** sheet(s) submitted
- **Weighting**: 30%

Total marks – 70

Section I Pages 3 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 14

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

Section I **10 marks**

- Circle the correct response on the answer sheet provided
- Each question is worth 1 mark

---

1 The exact value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is:

(A) $\frac{\pi}{3}$

(B) $-\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$

(D) $-\frac{2\pi}{3}$

2 State the domain for which the function $y = 4x - x^2$ is an increasing, one-to-one function.

(A) $x \leq 2$

(B) $x < 2$

(C) $x > 0$

(D) $x < 4$

3 What is the value of $\lim_{x \to 0} \frac{\sin 3x}{x}$?

(A) $\frac{1}{3}$

(B) 1

(C) 3

(D) undefined

4 The domain of the function $y = \ln(x^2 + e)$ is:

(A) $x \geq 0$

(B) $x \leq 0$

(C) all real values of x

(D) all real x, $x \neq 0$

5 The range of the function $y = \ln(x^2 + e)$ is:

(A) $y \geq 0$

(B) $y \geq 1$

(C) $y \geq e$

(D) all real y, $y \neq 0$

6 The area under the curve $y = \sin x$ between $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ and the x - axis is:

(A) -1

(B) 0

(C) 1

(D) 2

7 To find $\int_0^1 x\sqrt{x^2 + 1}\ dx$ using the substitution $u = x^2 + 1$, in terms of u the correct expression is:

(A) $\int_0^1 \sqrt{u}\ du$

(B) $\frac{1}{2}\int_0^1 \sqrt{u}\ du$

(C) $\int_1^2 \sqrt{u}\ du$

(D) $\frac{1}{2}\int_1^2 \sqrt{u}\ du$

8 The exact value of $\sin\left(\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(-\frac{4}{3}\right)\right)$ is:

(A) - 1

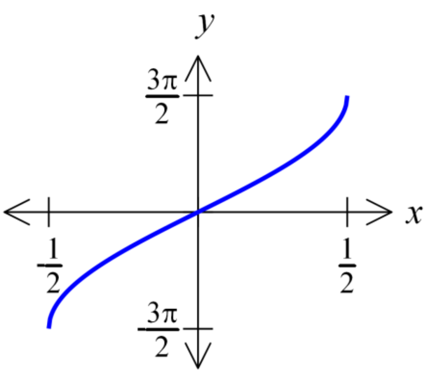
(B) $-\frac{7}{25}$

(C) $\frac{7}{25}$

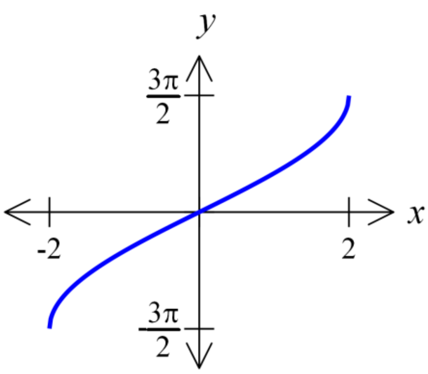
(D) 1

9 Which of the graphs below represents $y = 3\sin^{-1}\frac{x}{2}$?

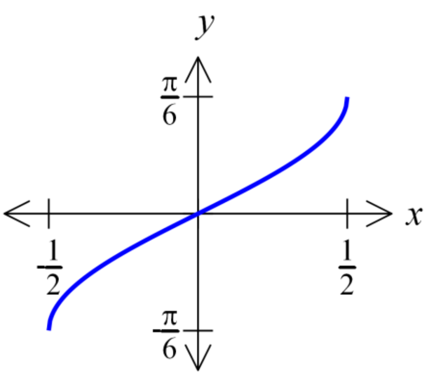
(A)



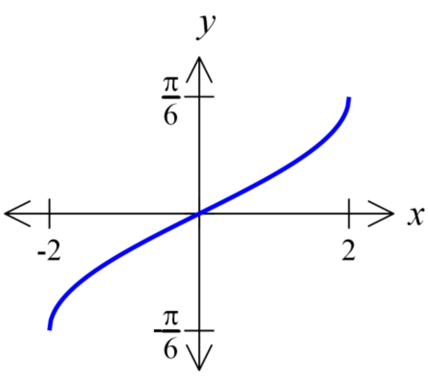
(C)



(B)



(D)



10 Below is the solution to the integral $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$.

$$\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 + \cos 2x)\, dx \qquad \textit{LINE I}$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x \right]_0^{\frac{\pi}{4}} \qquad \textit{LINE II}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} \right) - \left(0 - \frac{1}{2}\sin 0 \right) \right] \qquad \textit{LINE III}$$

$$= \frac{\pi}{8} - \frac{1}{4} \qquad \textit{LINE IV}$$

The line containing the FIRST mistake is:

(A) LINE I

(B) LINE II

(C) LINE III

(D) LINE IV

End of Section I

Section II 60 marks

- Begin each question in a new writing booklet or on a new answer sheet
- Show all necessary working
- Each question is worth 15 marks

Question 11 (15 marks)

a) Find:

i) $\dfrac{d}{dx}\sin(3x - 2)$ **1**

ii) $\displaystyle\int \sec^2 3x \, dx$ **1**

iii) $\displaystyle\int \sin^2 2x \, dx$. **2**

b) Find the value of $\displaystyle\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$ **2**

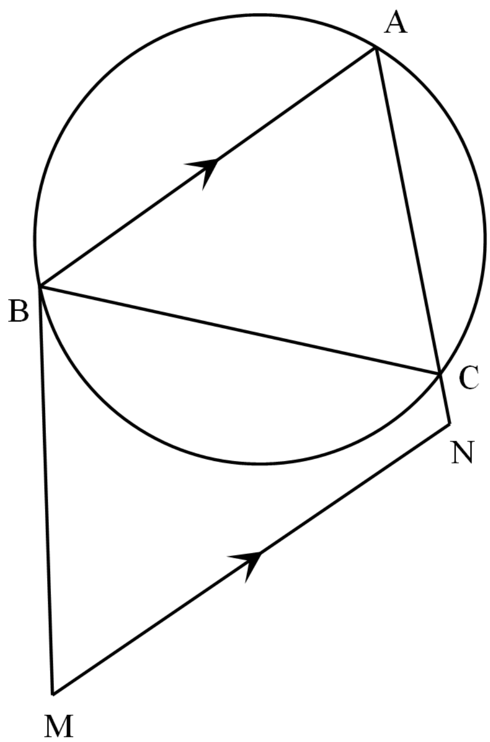
c)

i) Show that $\dfrac{1 + \cos 2x}{\sin 2x} = \cot x$. **2**

ii) Hence, find the exact value of $\cot 15^\circ$. **1**

d) Use the principle of mathematical induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers of n. **3**

e) *ABC* is a triangle inscribed in a circle. *M* is a point on the tangent to the circle at *B* and *N* is a point on *AC* produced so that *MN* is parallel to *BA*.



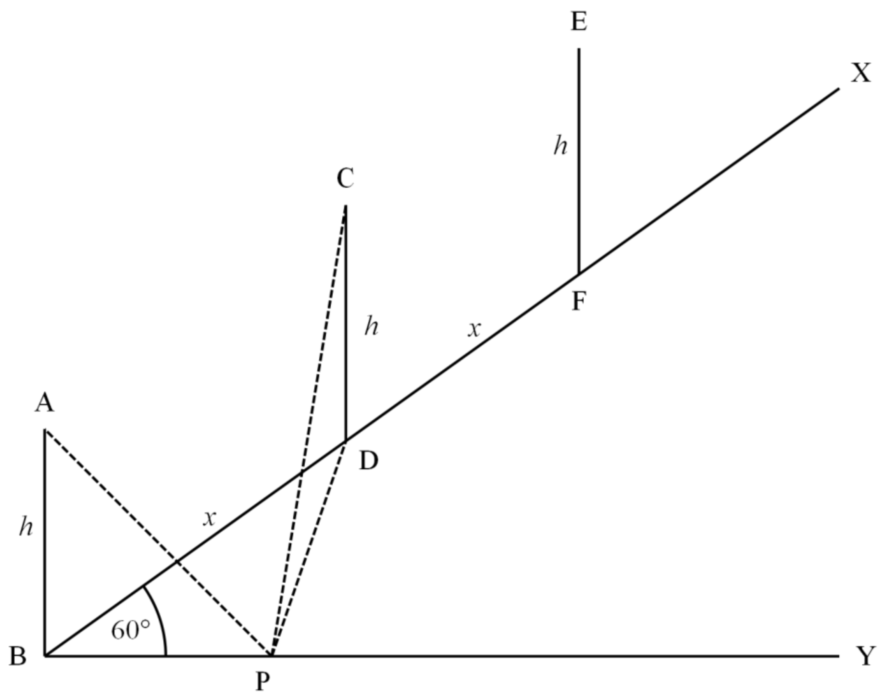
i) Copy or trace the diagram into your answer booklet.

ii) Give a reason why $\angle MBC = \angle BAC$. **1**

iii) Show that *MNCB* is a cyclic quadrilateral. **2**

Question 12 **(15 marks) Begin a NEW answer booklet or answer sheet**

a) In the diagram below, BX and BY represent two roads intersecting at an angle of 60°. On the road BX are situated three telegraph poles AB, CD and EF, all of equal height, the same distance, x metres, apart (i.e. $BD = DF = x$). P is a point on the road BY and the angles of elevation to A and C are 45° and 30° respectively.



i) Show that $DP = h\sqrt{3}$. **1**

ii) Show that $\angle BDP = 30^\circ$ and hence that ΔBPD is right angled at P. **2**

iii) Prove that $x = 2h$. **1**

iv) Show that $PF = h\sqrt{13}$ and hence show that the angle of elevation of E from P is approximately $15{\cdot}5^\circ$. **3**

b) State the domain and range of $y = \sin^{-1}(2x + 1)$ and then sketch the curve. **2**

c) The position x cm of a particle moving along an x-axis is given by $x = 3t + e^{-2t}$ where t is the time in seconds.

i) What is the position of the particle when $t = \frac{1}{2}$ second? Give your answer in exact form. **1**

ii) What is the initial velocity of the particle? **2**

iii) Show the initial acceleration of the particle is 4 cm/s^2. **2**

iv) Explain why the particle will never come to rest. **1**

Question 13 **(15 marks) Begin a NEW answer booklet or answer sheet**

a) Find the slope of the tangent to $y = \tan^{-1}\left(\sqrt{1-x}\right)$ at the point $y = \frac{\pi}{4}$. **2**

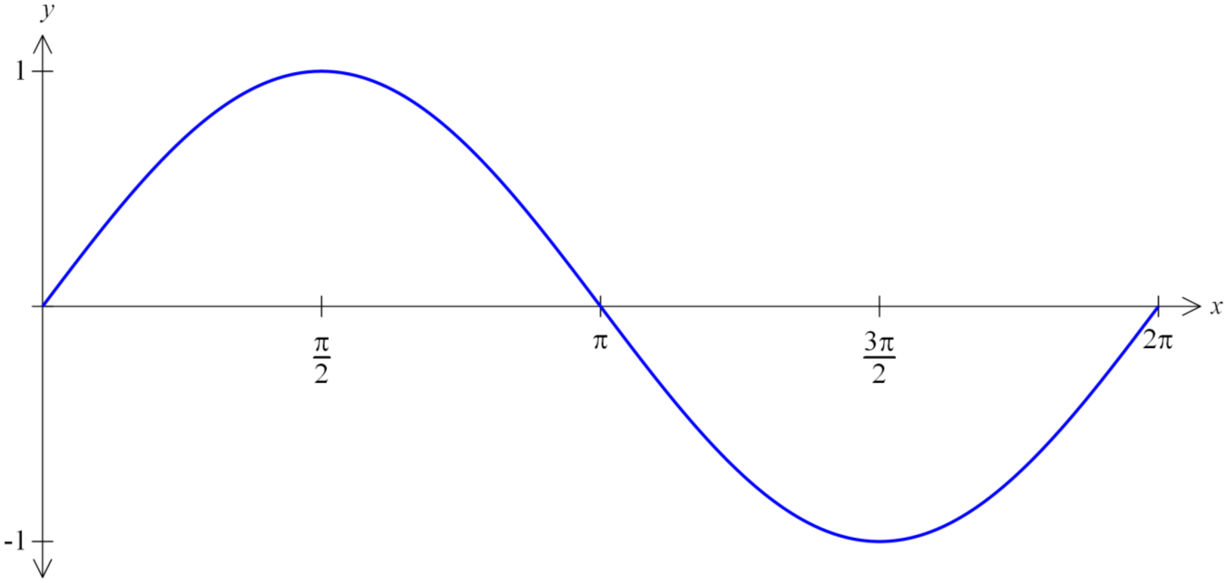
b) Solve $\frac{2}{2-x} \geq 3$. **2**

c) Evaluate $\int_0^{\frac{\pi}{6}} \frac{2\cos x}{1 + 4\sin^2 x}\, dx$ using the substitution $u = 2\sin x$. **4**

d) The region bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution about the x-axis. Find the volume of the solid formed. **3**

e) The curves $y = \ln x$ and $xy = e$ intersect at the point P $(e, 1)$. Show that at the point P $\tan\theta = \frac{2e}{e^2 - 1}$. **2**

f) The diagram shows the graph of $y = \sin x$ for $0 \leq x \leq 2\pi$. Copy or trace the diagram onto your answer sheet.



i) On the same set of axes, sketch the graph of $y = \cos 2x$ for $0 \leq x \leq 2\pi$. **1**

ii) Hence, state the number of solutions in $0 \leq x \leq 2\pi$ to the equation $\sin x = \cos 2x$. **1**

Question 14 **(15 marks)** **Begin a NEW answer booklet or answer sheet**

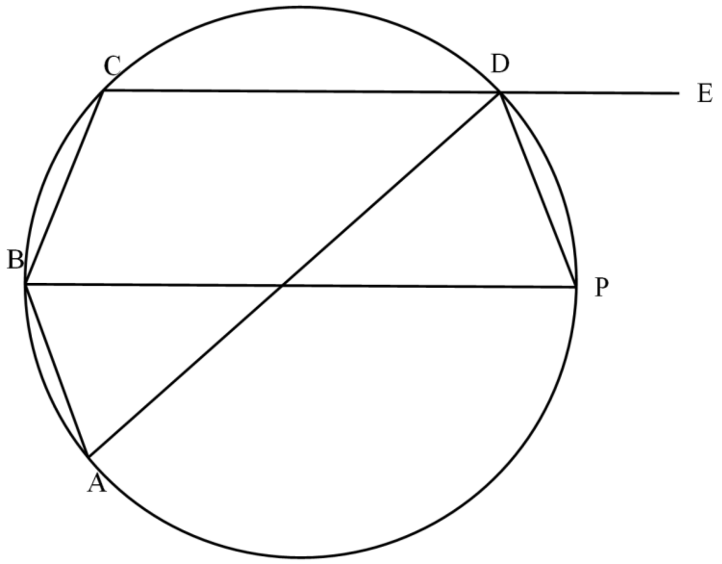
a) An isotope of carbon, C_{14} decays at a rate proportional to the mass present. The rate of change is given by $\frac{dM}{dt} = -kM$ where k is a positive constant and M is the mass present.

i) Show $M = M_0 e^{-kt}$ is a solution to this equation. **1**

ii) The half-life of this isotope is 5600 years. This means it takes 5600 years for 100 grams of C_{14} to decay to 50 grams. Find the value of k correct to 3 significant figures. **1**

iii) Archaeologists use radiocarbon dating to establish the age of discoveries. Calculate the age of an item in which only one-eighth of the original carbon remains. **1**

b) In the diagram below $ABCD$ is a cyclic quadrilateral. CD is produced to E. P is a point on the circle through A, B, C, D such that $\angle ABP = \angle PBC$.



Copy or trace the diagram into your answer booklet marking all the information given.

i) Explain why $\angle ABP = \angle ADP$. **1**

ii) Show that PD bisects $\angle ADE$. **2**

iii) If, in addition, $\angle BAP = 90^\circ$ and $\angle APD = 90^\circ$, explain where the centre of the circle is located. **1**

c) A is the point (-2, 1) and B is the point (x, y). The point P (13, -9) divides AB externally in the ratio 5:3. Find the values of x and y. **2**

d) Find the area under the curve $y = \cos^{-1} x$ between $x = 0$, $x = \frac{1}{2}$ and the x – axis. **2**

e) For the function $f(x) = \dfrac{1}{1 - x^2}$

i) Find the inverse function. **2**

ii) Using Simpsons Rule, find the area under the inverse function between $x = 1$ and $x = 3$ using five function values. **2**

End of Section II

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

TRINITY GRAMMAR SCHOOL
2012, Year 12 Mathematics Extension 1
Half Yearly Examination
SECTION I
ANSWER SHEET

Student Number: …………………………… Class Teacher: ………………………..

Be sure to write your answers for Section I on this answer sheet. After you have selected an answer, **CIRCLE** the correct answer. To change an answer, erase your previous mark completely, and then record your new answer. Mark only one answer for each question.

Q1.	A	B	C	D
Q2.	A	B	C	D
Q3.	A	B	C	D
Q4.	A	B	C	D
Q5.	A	B	C	D
Q6.	A	B	C	D
Q7.	A	B	C	D
Q8.	A	B	C	D
Q9.	A	B	C	D
Q10.	A	B	C	D

TRINITY GRAMMAR SCHOOL

Mathematics Department

2012

HALF YEARLY EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 – 14
- Write your Board of Studies Student Number **and** Class Teacher on the writing booklet(s) **or** sheet(s) submitted
- **Weighting**: 30%

Total marks – 70

Section I Pages 3 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 14

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

Section I 10 marks

- Circle the correct response on the answer sheet provided
- Each question is worth 1 mark

---

1 The exact value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is:

(A) $\frac{\pi}{3}$

(B) $-\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$

(D) $-\frac{2\pi}{3}$

2 State the domain for which the function $y = 4x - x^2$ is an increasing, one-to-one function.

(A) $x \le 2$

(B) $x < 2$

(C) $x > 0$

(D) $x < 4$

3 What is the value of $\lim\limits_{x \to 0} \frac{\sin 3x}{x}$?

(A) $\frac{1}{3}$

(B) 1

(C) 3

(D) undefined

4 The domain of the function $y = \ln\left(x^2 + e\right)$ is:

(A) $x \geq 0$

(B) $x \leq 0$

(C) all real values of x

(D) all real x, $x \neq 0$

5 The range of the function $y = \ln\left(x^2 + e\right)$ is:

(A) $y \geq 0$

(B) $y \geq 1$

(C) $y \geq e$

(D) all real y, $y \neq 0$

6 The area under the curve $y = \sin x$ between $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ and the x - axis is:

(A) -1

(B) 0

(C) 1

(D) 2

7 To find $\int_0^1 x\sqrt{x^2 + 1}\, dx$ using the substitution $u = x^2 + 1$, in terms of u the correct expression is:

(A) $\int_0^1 \sqrt{u}\, du$

(B) $\frac{1}{2}\int_0^1 \sqrt{u}\, du$

(C) $\int_1^2 \sqrt{u}\, du$

(D) $\frac{1}{2}\int_1^2 \sqrt{u}\, du$

8 The exact value of $\sin\left(\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(-\frac{4}{3}\right)\right)$ is:

(A) -1

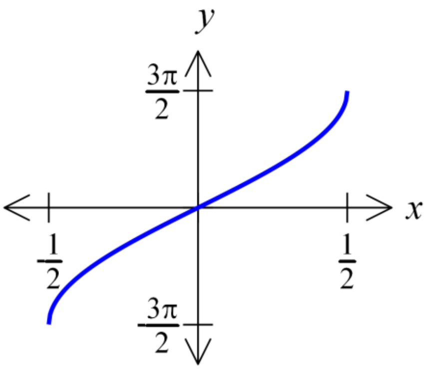
(B) $-\frac{7}{25}$

(C) $\frac{7}{25}$

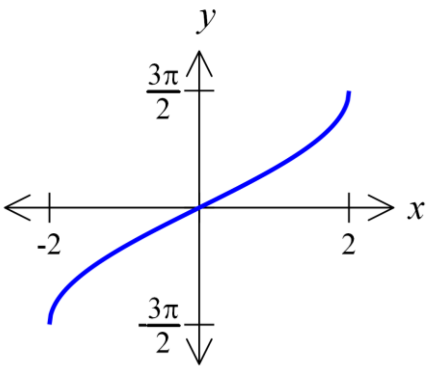
(D) 1

9 Which of the graphs below represents $y = 3\sin^{-1}\frac{x}{2}$?

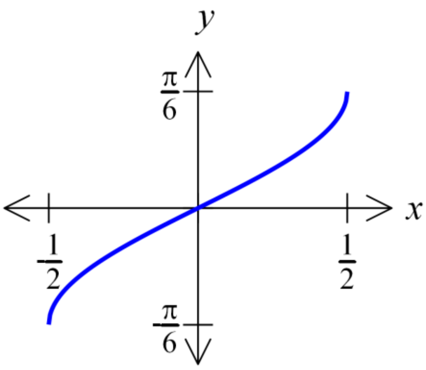
(A)



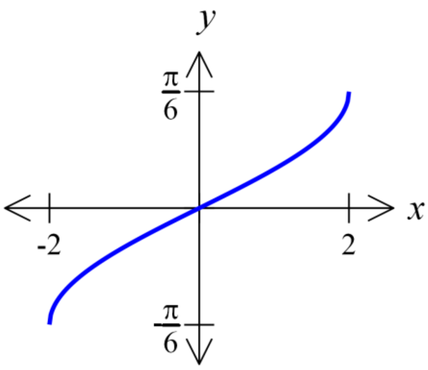
(C)



(B)



(D)



10 Below is the solution to the integral $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$.

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \cos^2 x \, dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 + \cos 2x) \, dx && \textit{LINE I} \\
&= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x \right]_0^{\frac{\pi}{4}} && \textit{LINE II} \\
&= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} \right) - \left(0 - \frac{1}{2}\sin 0 \right) \right] && \textit{LINE III} \\
&= \frac{\pi}{8} - \frac{1}{4} && \textit{LINE IV}
\end{aligned}$$

The line containing the FIRST mistake is:

(A) LINE I

(B) LINE II

(C) LINE III

(D) LINE IV

End of Section I

Section II 60 marks

- Begin each question in a new writing booklet or on a new answer sheet
- Show all necessary working
- Each question is worth 15 marks

Question 11 (15 marks)

a) Find:

i) $\dfrac{d}{dx}\sin(3x-2)$ **1**

ii) $\displaystyle\int \sec^2 3x\, dx$ **1**

iii) $\displaystyle\int \sin^2 2x\, dx$. **2**

b) Find the value of $\displaystyle\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$ **2**

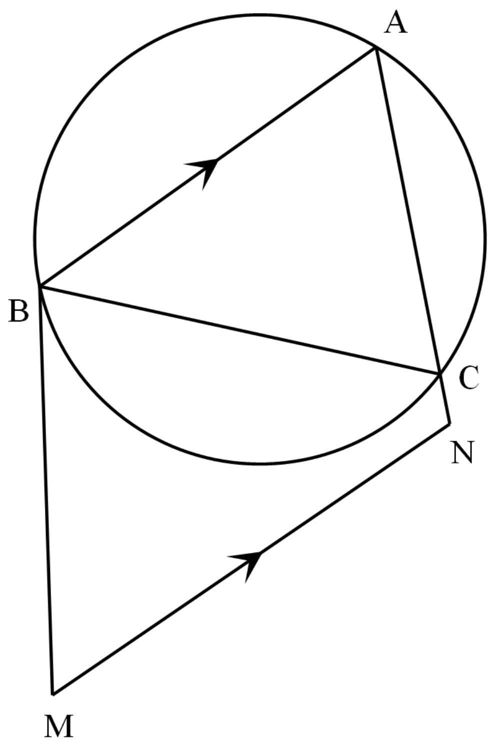
c)

i) Show that $\dfrac{1+\cos 2x}{\sin 2x} = \cot x$. **2**

ii) Hence, find the exact value of $\cot 15^\circ$. **1**

d) Use the principle of mathematical induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers of n. **3**

e) ABC is a triangle inscribed in a circle. M is a point on the tangent to the circle at B and N is a point on AC produced so that MN is parallel to BA.



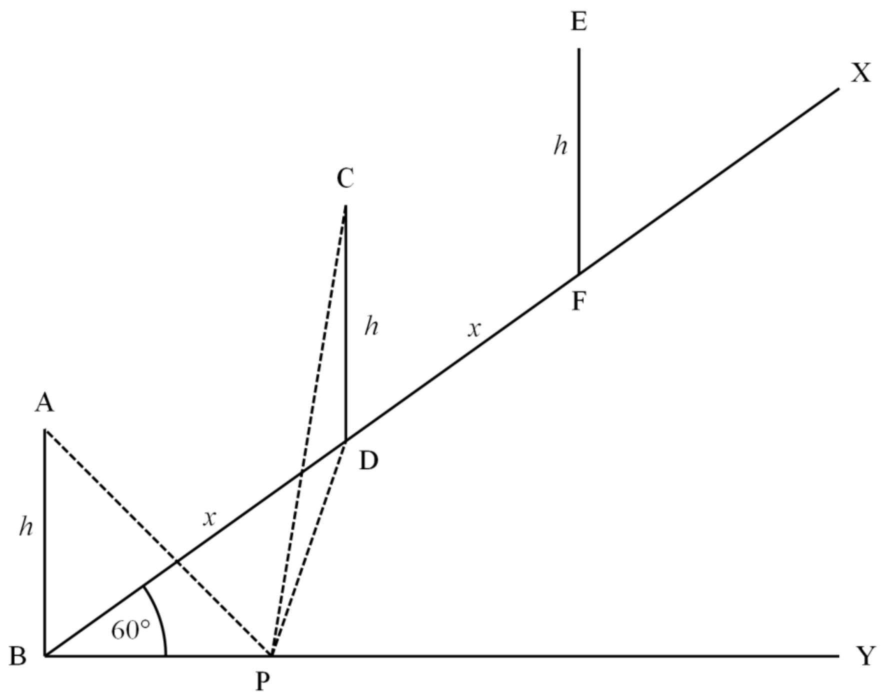
i) Copy or trace the diagram into your answer booklet.

ii) Give a reason why $\angle MBC = \angle BAC$. **1**

iii) Show that $MNCB$ is a cyclic quadrilateral. **2**

Question 12 (15 marks) Begin a NEW answer booklet or answer sheet

a) In the diagram below, BX and BY represent two roads intersecting at an angle of 60°. On the road BX are situated three telegraph poles AB, CD and EF, all of equal height, the same distance, x metres, apart (i.e. $BD = DF = x$). P is a point on the road BY and the angles of elevation to A and C are 45° and 30° respectively.



i) Show that $DP = h\sqrt{3}$. **1**

ii) Show that $\angle BDP = 30^\circ$ and hence that ΔBPD is right angled at P. **2**

iii) Prove that $x = 2h$. **1**

iv) Show that $PF = h\sqrt{13}$ and hence show that the angle of elevation of E from P is approximately $15{\cdot}5^\circ$. **3**

b) State the domain and range of $y = \sin^{-1}(2x + 1)$ and then sketch the curve. **2**

c) The position x cm of a particle moving along an x-axis is given by $x = 3t + e^{-2t}$ where t is the time in seconds.

i) What is the position of the particle when $t = \frac{1}{2}$ second? Give your answer in exact form. **1**

ii) What is the initial velocity of the particle? **2**

iii) Show the initial acceleration of the particle is 4 cm/s^2. **2**

iv) Explain why the particle will never come to rest. **1**

Question 13 (15 marks) Begin a NEW answer booklet or answer sheet

a) Find the slope of the tangent to $y = \tan^{-1}\left(\sqrt{1-x}\right)$ at the point $y = \frac{\pi}{4}$. **2**

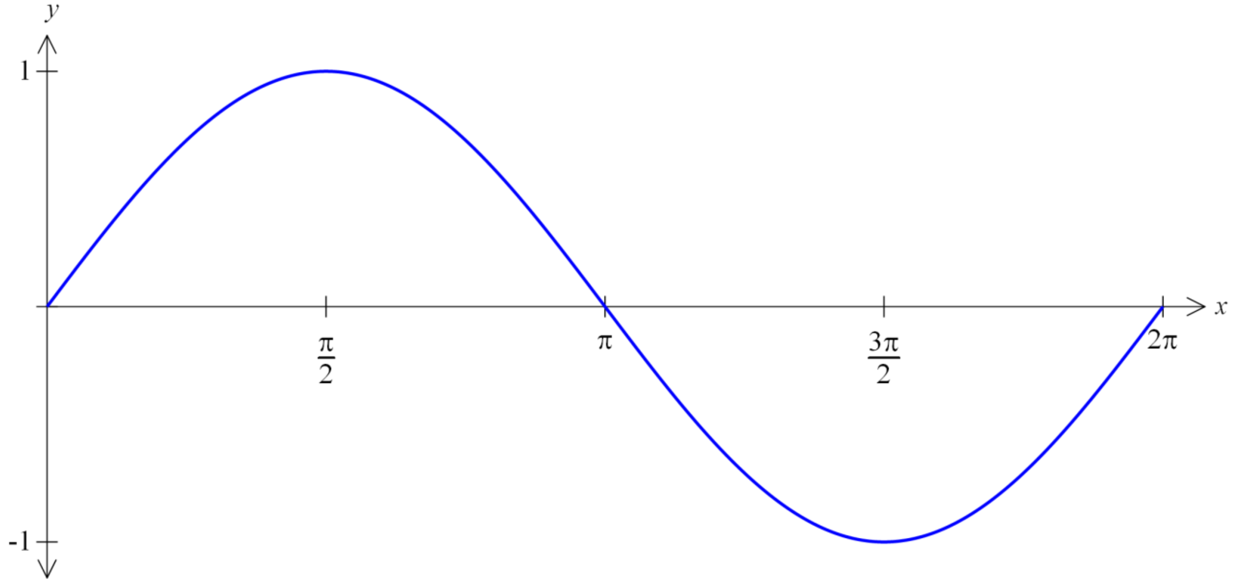
b) Solve $\frac{2}{2-x} \geq 3$. **2**

c) Evaluate $\int_0^{\frac{\pi}{6}} \frac{2\cos x}{1 + 4\sin^2 x}\,dx$ using the substitution $u = 2\sin x$. **4**

d) The region bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution about the x-axis. Find the volume of the solid formed. **3**

e) The curves $y = \ln x$ and $xy = e$ intersect at the point P (e, 1). Show that at the point P $\tan\theta = \frac{2e}{e^2 - 1}$. **2**

f) The diagram shows the graph of $y = \sin x$ for $0 \le x \le 2\pi$. Copy or trace the diagram onto your answer sheet.



i) On the same set of axes, sketch the graph of $y = \cos 2x$ for $0 \le x \le 2\pi$. **1**

ii) Hence, state the number of solutions in $0 \le x \le 2\pi$ to the equation $\sin x = \cos 2x$. **1**

Question 14 **(15 marks)** **Begin a NEW answer booklet or answer sheet**

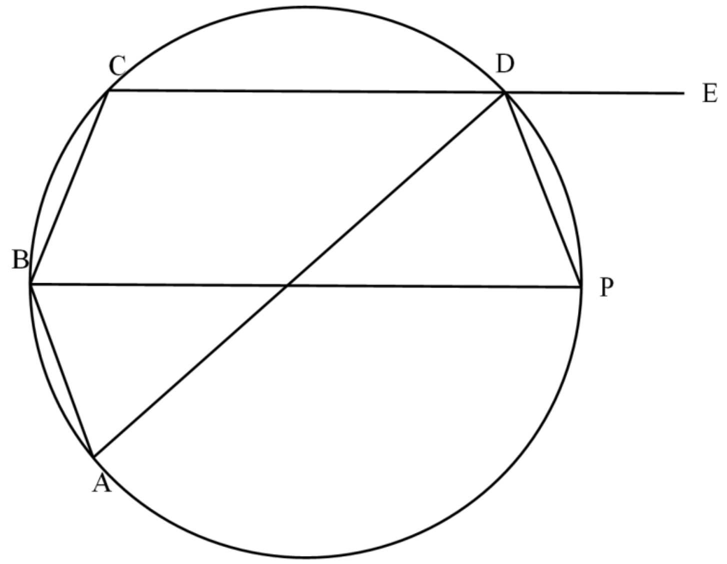
a) An isotope of carbon, C_{14} decays at a rate proportional to the mass present. The rate of change is given by $\frac{dM}{dt} = -kM$ where k is a positive constant and M is the mass present.

i) Show $M = M_0 e^{-kt}$ is a solution to this equation. **1**

ii) The half-life of this isotope is 5600 years. This means it takes 5600 years for 100 grams of C_{14} to decay to 50 grams. Find the value of k correct to 3 significant figures. **1**

iii) Archaeologists use radiocarbon dating to establish the age of discoveries. Calculate the age of an item in which only one-eighth of the original carbon remains. **1**

b) In the diagram below $ABCD$ is a cyclic quadrilateral. CD is produced to E. P is a point on the circle through A, B, C, D such that $\angle ABP = \angle PBC$.



Copy or trace the diagram into your answer booklet marking all the information given.

i) Explain why $\angle ABP = \angle ADP$. **1**

ii) Show that PD bisects $\angle ADE$. **2**

iii) If, in addition, $\angle BAP = 90^\circ$ and $\angle APD = 90^\circ$, explain where the centre of the circle is located. **1**

c) A is the point (-2, 1) and B is the point (x, y). The point P (13, -9) divides AB externally in the ratio 5:3. Find the values of x and y. **2**

d) Find the area under the curve $y = \cos^{-1} x$ between $x = 0$, $x = \frac{1}{2}$ and the x – axis. **2**

e) For the function $f(x) = \dfrac{1}{1 - x^2}$

i) Find the inverse function. **2**

ii) Using Simpsons Rule, find the area under the inverse function between $x = 1$ and $x = 3$ using five function values. **2**

End of Section II

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

TRINITY GRAMMAR SCHOOL
2012, Year 12 Mathematics Extension 1
Half Yearly Examination
SECTION I
ANSWER SHEET

Student Number: ………………………… **Class Teacher:** ………………………..

Be sure to write your answers for Section I on this answer sheet. After you have selected an answer, **CIRCLE** the correct answer. To change an answer, erase your previous mark completely, and then record your new answer. Mark only one answer for each question.

Q1.	A	B	C	D
Q2.	A	B	C	D
Q3.	A	B	C	D
Q4.	A	B	C	D
Q5.	A	B	C	D
Q6.	A	B	C	D
Q7.	A	B	C	D
Q8.	A	B	C	D
Q9.	A	B	C	D
Q10.	A	B	C	D

TRINITY GRAMMAR SCHOOL
2012, Year 12 Mathematics Extension 1
Half Yearly Examination
SECTION I
ANSWER SHEET

Student Number: ANSWERS............................ **Class Teacher:** ....

Be sure to write your answers for Section I on this answer sheet. After you have selected an answer, **CIRCLE** the correct answer. To change an answer, erase your previous mark completely, and then record your new answer. Mark only one answer for each question.

Q1.	A	B	C	D
Q2.	A	B	C	D
Q3.	A	B	C	D
Q4.	A	B	C	D
Q5.	A	B	C	D
Q6.	A	B	C	D
Q7.	A	B	C	D
Q8.	A	B	C	D
Q9.	A	B	C	D
Q10.	A	B	C	D

Question 11

a) i $\frac{d}{dx} = 3.\cos(3x-2)$ ✓

ii) $\int \sec^2 3x\, dx$

$= \frac{1}{3}\tan 3x + c$ ✓

Do not penalise for c

iii) $\int \sin^2 2x\, dx$

$= \frac{1}{2}\int 1-\cos 4x\, dx$ ✓

$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x + c_1\right]$ ✓

$= \frac{1}{2}x - \frac{1}{8}\sin 4x + c$

b) $\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$

$= \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$ ✓

$= 2.1.1 = 2.$ ✓

c) i) $LHS = \frac{1+\cos 2x}{\sin 2x}$

$= \frac{\cancel{2}\cos^2 x}{\cancel{2}\sin x \cancel{\cos x}}$ ✓ ✓

$= \frac{\cos x}{\sin x}$

$= \cot x = RHS$

ii) $\cot 15^\circ = \frac{1+\cos 2(15)}{\sin 2(15)}$

$= \frac{1+\cos 30}{\sin 30}$

$= \left(1+\frac{\sqrt{3}}{2}\right) \div \frac{1}{2} = 2+\sqrt{3}.$ ✓

d)

For $n=1$

$9^{n+2} - 4^n$

$= 9^{1+2} - 4^1$

$= 9^3 - 4$

$= 725$

which is divisible by 5. ✓

Assume that when $n=k$.

$9^{k+2} - 4^k$ is divisible by 5.

i.e. $9^{k+2} - 4^k = 5M$ ✓

where M is an integer.

or $9^{k+2} = 5M + 4^k$.

When $n = k+1$.

$9^{k+1+2} - 4^{k+1}$

$= 9.9^{k+2} - 4.4^k$

$= 9(5M + 4^k) - 4.4^k$

$= 9.5M + 9.4^k - 4.4^k$

$= 5.9M + 5.4^k$

$= 5(9M + 4^k)$ ✓

Since $9M + 4^k$ is an integer $5(9M+4^k)$ is divisible by 5.

$\therefore$ By the process of M I. It has been shown that $9^{n+2} - 4^n$ is divisible by 5 for all $n \geq 1$

Question 11 cont.

e) ii) the angle between a tangent + a chord is equal to the angle subtended by the chord in the alternate segment.

iii) $M\hat{N}C = 180 - B\hat{A}C$ (co-int <'s, AB//NM)

$\therefore M\hat{N}C = 180 - M\hat{B}C$ (from above, $B\hat{A}C = M\hat{B}C$)

$\therefore$ MBCN is cyclic quad, opposite angles supplementary.

Question 12.

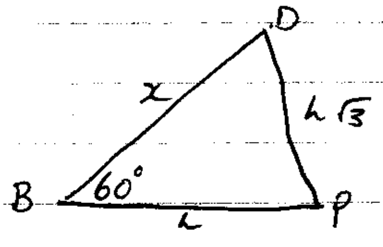
a) i) In $\triangle CDP$, $C\hat{P}D = 30°$

$\therefore \quad \tan 30 = \frac{h}{DP}$

$\frac{1}{\sqrt{3}} = \frac{h}{DP}$

$DP = \sqrt{3}h.$

ii)



BP in right angle isos $\triangle$.

$\frac{\sin B\hat{D}P}{h} = \frac{\sin 60}{h\sqrt{3}}$

$\sin B\hat{D}P = \cancel{h} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{\cancel{h}\sqrt{3}}$

$\sin B\hat{D}P = \frac{1}{2}$

$\therefore B\hat{D}P = 30°$

$\therefore B\hat{P}D = 90°$ ($\angle$ sum $\triangle$)

iii) In $\triangle BPD$

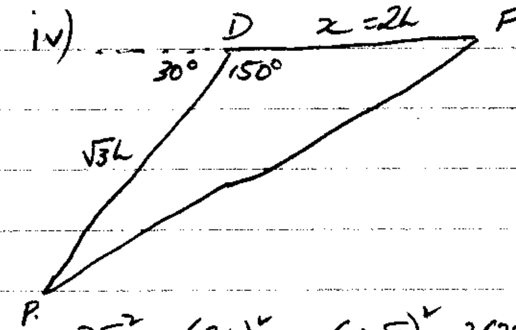
$x^2 = (h\sqrt{3})^2 + h^2$

$= 3h^2 + h^2$

$x^2 = 4h^2$

$\therefore \quad x = 2h.$

iv)

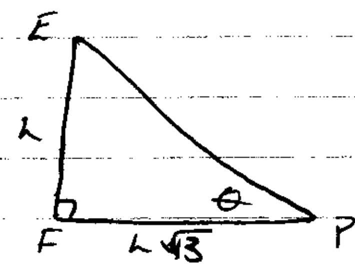


$PF^2 = (2h)^2 + (h\sqrt{3})^2 - 2(2h)(h\sqrt{3})\cos 150$

$= 7h^2 - \cancel{4}^{2}\sqrt{3}h^2\left(-\frac{\sqrt{3}}{\cancel{2}}\right)$

$= 13h^2$

$PF = h\sqrt{13}$



$\tan\theta = \frac{h}{h\sqrt{13}}$

$\tan\theta = 0.277$

$\theta = 15.5°$

b) $y = \sin^{-1}(2x+1)$.

Domain:

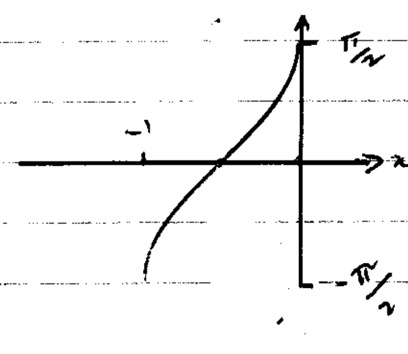
$-1 \leq 2x+1 \leq 1$

$-2 \leq 2x \leq 0$

$-1 \leq x \leq 0$

Range

$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



Q12

c) $x = 3t + e^{-2t}$

$\dot{x} = 3 - 2e^{-2t}$

$\ddot{x} = 4e^{-2t}$

i) at $t = \frac{1}{2}$

$x = 3\left(\frac{1}{2}\right) + e^{-2\left(\frac{1}{2}\right)}$

$= \frac{3}{2} + e^{-1}$ or $\frac{3}{2} + \frac{1}{e}$.

ii) initial velocity at $t = 0$

$\dot{x} = 3 - 2e^{-2(0)}$

$= 3 - 2$

$= 1$ cm/s.

iii) initial accel. at $t = 0$

$\ddot{x} = 4e^{-2(0)}$

$= 4$ cm/s^2

iv) the acceleration $4e^{-2t}$ is always greater than zero, so the particle is always speeding up.

Question 13

a) $y = \tan^{-1}\left(\sqrt{1-x}\right)$

$\therefore \sqrt{1-x} = \tan y$

$1 - x = \tan^2 y$

$x = 1 - \tan^2 y$

$\dfrac{dx}{dy} = -2\sec^2 y \tan y$ ✓

at $y = \dfrac{\pi}{4}$

$\dfrac{dx}{dy} = -2\sec^2\left(\dfrac{\pi}{4}\right)\tan\left(\dfrac{\pi}{4}\right)$

$= -2\left(\dfrac{\sqrt{2}}{1}\right)^2 \cdot 1$

$= -4$

$\therefore m = \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{-1}{4}$ ✓

b) $\dfrac{2}{2-x} \geq 3$

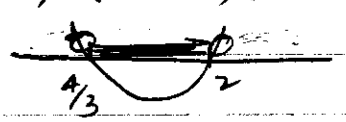
$2(2-x) \geq 3(2-x)^2$

$4 - 2x \geq 12 - 12x + 3x^2$

$0 \geq x^2 - 10x + 8$

$0 \geq 3x^2 - 6x - 4x + 8$

$0 \geq (3x-4)(x-2)$ ✓



$\therefore \dfrac{4}{3} \leq x < 2$ ✓

c) $\displaystyle\int_0^{\pi/6} \frac{2\cos x}{1 + 4\sin^2 x}\, dx$

$u = 2\sin x$

$du = 2\cos x\, dx$ ✓

at $x = 0 \quad u = 0$

$x = \pi/6 \quad u = 1$ ✓

$\therefore = \displaystyle\int_0^1 \frac{du}{1+u^2}$

$= \left[\tan^{-1} u\right]_0^1$ ✓

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \dfrac{\pi}{4} - 0 = \dfrac{\pi}{4}$ ✓

d) $V = \pi \displaystyle\int_{\pi/12}^{\pi/4} (\sin x)^2\, dx$

$= \dfrac{\pi}{2}\displaystyle\int_{\pi/12}^{\pi/4} 1 - \cos 2x\, dx$ ✓

$= \dfrac{\pi}{2}\left[x - \dfrac{1}{2}\sin 2x\right]_{\pi/12}^{\pi/4}$ ✓

$= \dfrac{\pi}{2}\left[\left(\dfrac{\pi}{4} - \dfrac{1}{2}\sin 2\cdot\dfrac{\pi}{4}\right) - \left(\dfrac{\pi}{12} - \dfrac{1}{2}\sin 2\cdot\dfrac{\pi}{12}\right)\right]$

$= \dfrac{\pi}{2}\left(\dfrac{\pi}{4} - \dfrac{1}{2} - \dfrac{\pi}{12} + \dfrac{1}{4}\right)$

$= \dfrac{\pi^2}{12} - \dfrac{\pi}{8}$ units^3. ✓

Q13

e) $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}$$

at $x = e$.

$$m_1 = \frac{1}{e}.$$

For $y = \frac{e}{x} = e x^{-1}$

$$\frac{dy}{dx} = -e x^{-2}$$

at $x = e$.

$$m_2 = \frac{-e}{e^2} = \frac{-1}{e}$$

For angle between lines:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

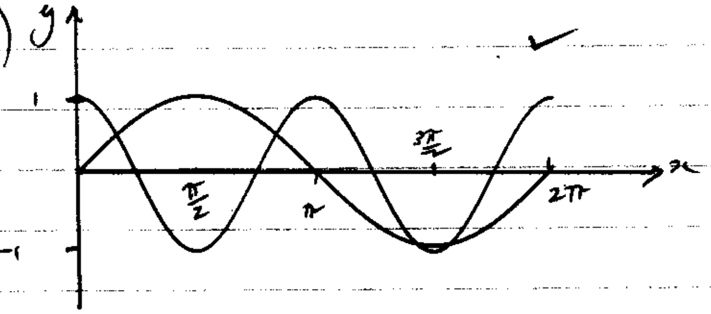
$$= \left(\frac{1}{e} - \left(\frac{-1}{e}\right)\right) \div \left(1 + \frac{1}{e} \cdot \frac{-1}{e}\right)$$

$$= \frac{2}{e} \div \left(1 - \frac{1}{e^2}\right)$$

$$= \frac{2}{\cancel{e}} \times \frac{e^{\cancel{2}}}{e^2 - 1}$$

$$\therefore \tan\theta = \frac{2e}{e^2 - 1}$$

f) i)



ii) 3 solutions

Question 14.

a) $\frac{dM}{dt} = -kM$

i) $M = M_0 e^{-kt}$

$\frac{dM}{dt} = -k M_0 e^{-kt}$

$= -kM.$ ✓

ii) $50 = 100\, e^{-k(5600)}$

$\frac{1}{2} = e^{-k5600}$

$-5600k = \ln 0.5$

$k = \frac{\ln 0.5}{-5600}$

$= 0.000124.$ ✓

iii) $\frac{M}{M_0} = e^{-kt}$

$\frac{1}{8} = e^{-kt}$

$-kt = \ln 0.125$

$t = \frac{\ln 0.125}{-k}$

$= 16\,800$ years. ✓

b) i) angles at the circumference standing on the same arc are equal. ✓

ii) $P\hat{D}E = C\hat{B}P$ (ext. ∠ of cyclic quad equal to int. opp. ∠).

$\therefore P\hat{D}E = A\hat{D}P$

$\therefore PD$ bisects $A\hat{D}E$

✓✓

iii Since $B\hat{A}P = 90°$, BP is a diametre (∠ in a semi circle).
Since $A\hat{P}D = 90°$, AD is a diametre (∠ in a semi circle).

$\therefore$ the centre is where BP & AD intersect. ✓

c)

$(-2,1) : (x,y)$

$5 : -3$

$13 = \frac{5x - 3(-2)}{5 + (-3)}$

$13 = \frac{5x + 6}{2}$

$26 = 5x + 6$

$5x = 20$

$B\ x = 4$ ✓

For y:

$-9 = \frac{5y - 3(1)}{2}$

$-18 = 5y - 3$

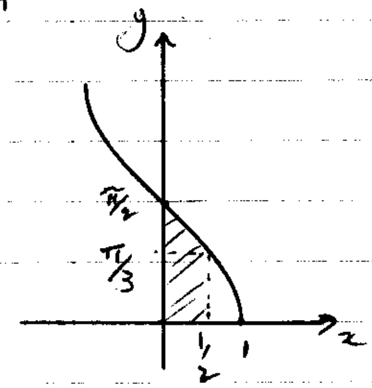
$5y = -15$

$B\ y = -3.$ ✓

$\therefore B(4, -3)$

Q 14

d)



$$A = \int_0^{1/2} \cos^{-1} x \, dx$$

$$= \frac{1}{2} \times \frac{\pi}{3} + \int_{\pi/3}^{\pi/2} \cos y \, dy \quad ✓$$

$$= \frac{\pi}{6} + \left[\sin y\right]_{\pi/3}^{\pi/2}$$

$$= \frac{\pi}{6} + \left(\sin\frac{\pi}{2} - \sin\pi/3\right)$$

$$= \frac{\pi}{6} + \left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{6} + \frac{2-\sqrt{3}}{2} \quad ✓$$

ii

x	1	1.5	2	2.5	3
y	0	0.577	$\frac{1}{\sqrt{2}}$	0.774	0.816

$$A = \frac{0.5}{3}\left(0 + 0.816 + 4(0.577 + 0.774) + 2\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= 1.27 \text{ units}^2 \quad (2dp).$$

e) i

$f(x)$: $y = \frac{1}{1-x^2}$

$f^{-1}(x)$: $x = \frac{1}{1-y^2}$

$$1 - y^2 = \frac{1}{x}$$

$$y^2 = 1 - \frac{1}{x}$$

$$y = \sqrt{\frac{x-1}{x}}$$