Trinity Grammar School

Mathematics Department

2017

HALF-YEARLY EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 1

NESA Student Number
(Year 12 only)

Class Teacher:

……………………………………………………………………

Name:
(Year 10 or 11 only)

……………………………………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination:

Thursday, 4 May 2017

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- NESA approved calculators for this course may be used
- A Reference Sheet for this course is supplied
- In Questions 11-14, show all relevant mathematical reasoning and/or calculations
- Write your NESA Student Number (Year 12 HSC) or Name (Year 11) and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- **HSC Assessment Weighting:** 30%

Total marks – 70

Section I

10 marks

- Attempt all Questions
- Allow about **15** minutes for this section

Section II

60 marks

- Attempt all Questions
- Allow about **1** hour **45** minutes for this section

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Section I 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Shade the correct response on the multiple-choice answer sheet for Questions 1 – 10
- Each question is worth 1 mark

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1 What is the solution to $\frac{x+1}{x-5} \leq 0$?

(A) $x \leq -1, x > 5$

(B) $x \leq -1, x \geq 5$

(C) $-1 \leq x \leq 5$

(D) $-1 \leq x < 5$

2 Which of the following does NOT have an inverse function?

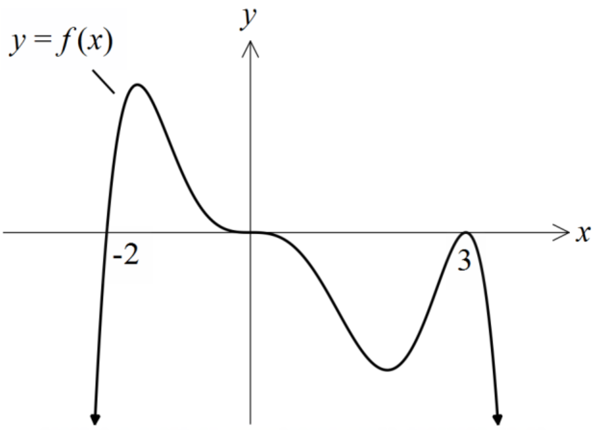
(A) $y = \sqrt{x}$

(B) $y = x^2$

(C) $x = \sqrt{y}$

(D) $x = y^2$

3 A polynomial $y = f(x)$ is sketched on the axis below.



Which of the following could be the equation of the polynomial?

(A) $P(x) = -x^3(x+2)(x-3)^2$

(B) $P(x) = x(x+2)(x-3)$

(C) $P(x) = x^3(x+2)(x-3)^2$

(D) $P(x) = -x^3(x-2)(x+3)^2$

4 What is the size of the acute angle between the line $y = \frac{x}{2} - 1$ and the line $3x - y - 7 = 0$?

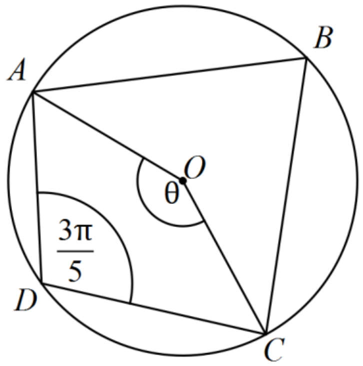
(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

5 $ABCD$ is a cyclic quadrilateral, with all vertices lying on a circle with centre O. The size of $\angle ADC$ is $\dfrac{3\pi}{5}$ radians.



NOT TO SCALE

What is the size of the angle marked θ?

(A) $\dfrac{\pi}{5}$

(B) $\dfrac{2\pi}{5}$

(C) $\dfrac{3\pi}{5}$

(D) $\dfrac{4\pi}{5}$

6 Which expression is equal to $\int \sin^2 3x \, dx$?

(A) $\dfrac{1}{2}\left(x - \dfrac{1}{6}\sin 6x\right) + c$

(B) $\dfrac{1}{2}\left(x + \dfrac{1}{6}\sin 6x\right) + c$

(C) $\dfrac{\sin^3 3x}{9} + c$

(D) $\dfrac{-\cos^3 3x}{9} + c$

7 Evaluate $\lim_{x \to 0} \dfrac{\sin \frac{x}{4}}{2x}$.

(A) 0

(B) $\dfrac{1}{8}$

(C) $\dfrac{1}{2}$

(D) 8

8 It is known that $\tan A = 4$ and $\tan(A+B) = \dfrac{1}{2}$.

What is the value of $\tan B$?

(A) $-3\dfrac{1}{2}$

(B) $-\dfrac{7}{6}$

(C) -1.292

(D) -0.862

9 What is the derivative of 5^{2x+3}?

(A) $2\left(5^{2x+3}\right)$

(B) 10^{2x+3}

(C) $(2\ln 5).5^{2x+3}$

(D) $(\ln 5).5^{2x+3}$

10 The point $P(9, 8)$ divides the interval from $A(4,-2)$ to B externally in the ratio 5 : 3. What are the coordinates of B?

(A) $\left(\frac{6}{5},\frac{22}{5}\right)$

(B) $\left(\frac{84}{5},\frac{58}{5}\right)$

(C) $(12, 14)$

(D) $(6, 2)$

End of Section I
Section II commences on the next page

Section II 60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.
- In Questions 11 – 14, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Solve simultaneously: **3**

$$\ln x^2 + \ln y = 15$$

$$\ln x + \ln y^2 = 10$$

(b) Show that $\dfrac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A} = 2 - \sin 2A$ if $\sin A + \cos A \neq 0$. **3**

(c) The parabolas $y = x^2$ and $y = (x-2)^2$ intersect at P (1, 1). Find the acute angle between the tangents at P, correct to the nearest minute. **3**

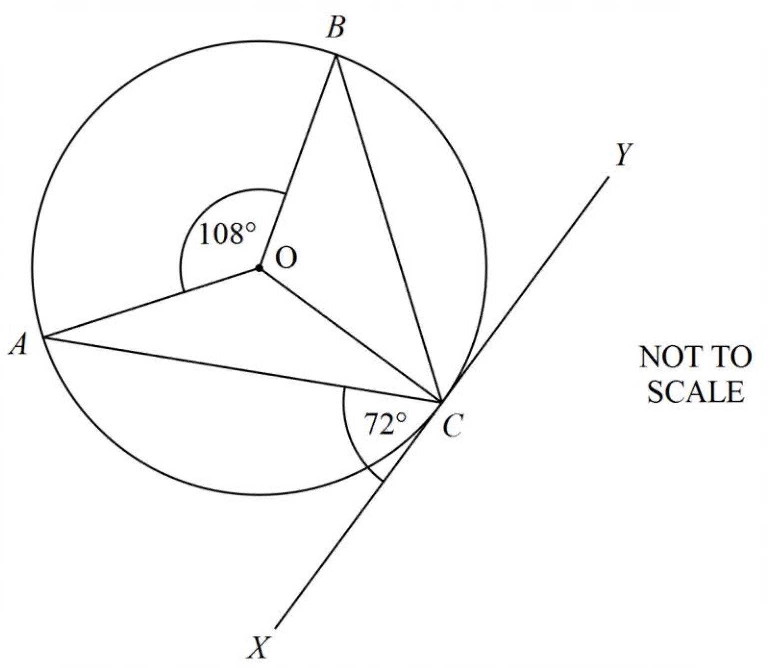
(d) Consider the function $f(x) = \sqrt{e^{x+5}}$.

(i) Find the range of $f(x)$. **1**

(ii) Find the inverse function, $f^{-1}(x)$, **and** state its domain. **2**

Question 11 continues on the next page

(e) The points A, B and C lie on a circle with centre O. The line XY is tangent to the circle at C. Also, $\angle AOB = 108°$ and $\angle ACX = 72°$.



(i) Explain why $\angle BCA = 54°$. **1**

(ii) Hence, or otherwise, find the size of $\angle BCO$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) The polynomial $P(x) = x^3 - ax + b$ has a zero at $x = 2$. When $P(x)$ is divided by $(x + 3)$, it has a remainder of -10. Determine the values of a and b. **3**

(b) Consider the even function $f(x) = \dfrac{2x^2 + 9}{x^2 + 4}$.

(i) State the equation of any asymptotes of the curve. **1**

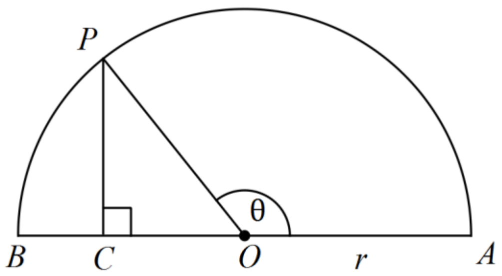
(ii) Find the coordinates of the stationary point on $y = f(x)$. **1**

(iii) Hence, sketch $y = f(x)$, showing the features found above. **2**

(c) Using the substitution $u = x^2 - 1$, or otherwise, find $\displaystyle\int_{-2}^{1} x\sqrt{x^2 - 1}\, dx$. **4**

Question 12 continues on the next page

(d) The point P lies on the circumference of a semicircle of radius r and diameter AB, as shown below. The point C lies on AB and PC is perpendicular to AB. The arc AP subtends an angle θ at the centre, O, and the length of the arc AP is twice the length of the interval PC.



(i) Show that $2\sin\theta = \theta$. **2**

(ii) Taking $\theta = 1 \cdot 8$ as an approximation for the solution to the equation $2\sin\theta = \theta$ between $\frac{\pi}{2}$ and π, use one application of Newton's method to give a better approximation, giving your final answer correct to 3 significant figures. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

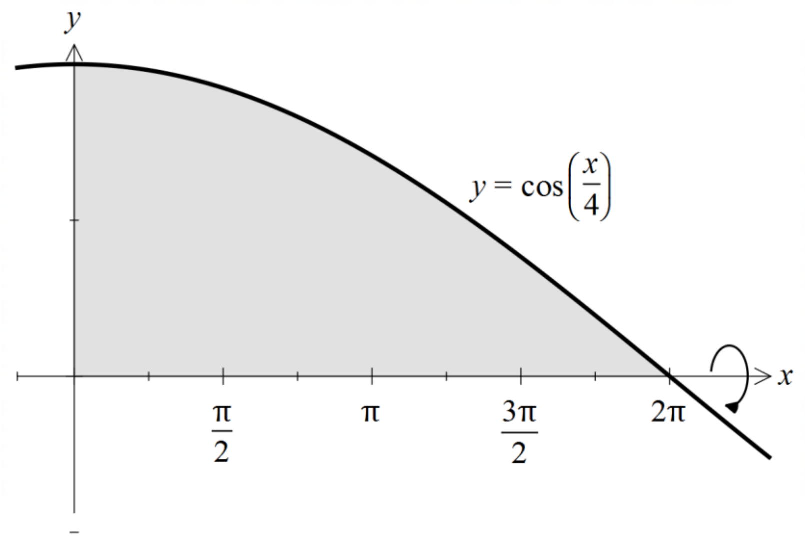
(a) Solve $3\sec^2 2x = 4$ for $0 \le x \le \pi$. **3**

(b)

(i) Express $3\cos x + \sqrt{3}\sin x$ in the form $R\cos(x - \alpha)$, where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. **2**

(ii) Hence, or otherwise, find the general solution of the equation $3\cos x + \sqrt{3}\sin x = \sqrt{6}$. **2**

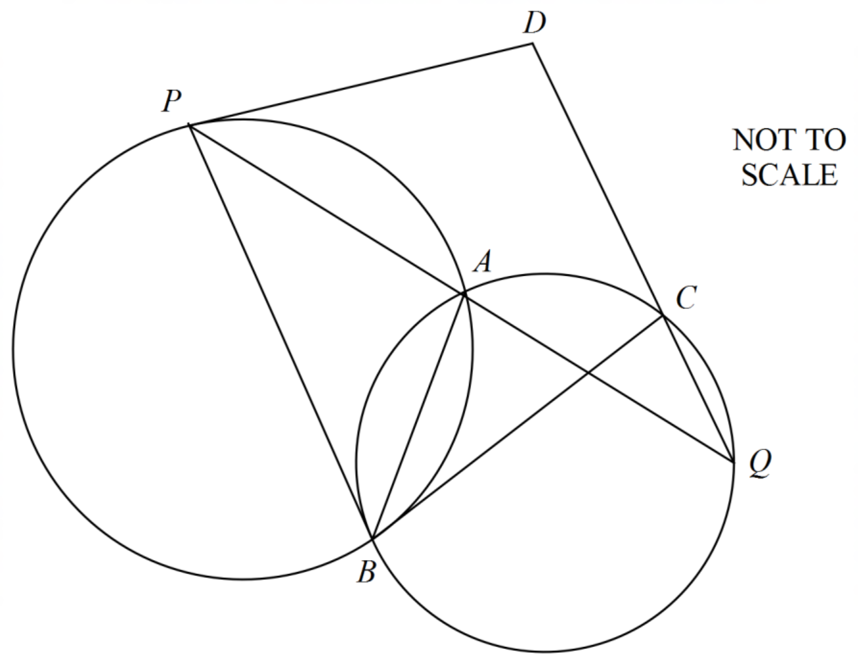
(c) The region bounded by the curve $y = \cos\frac{x}{4}$, the x-axis, and the y-axis is rotated about the x-axis to form a solid. **4**



Find the exact volume of the solid of revolution?

Question 13 continues on the next page

(d) Two circles intersect at A and B. P is a point on the first circle and Q is a point on the second circle such that PAQ is a straight line. C is a point on the second circle. The line QC produced and the tangent to the first circle at P meet at D, as shown on the diagram below.



(i) Give a reason why $\angle DPA = \angle PBA$. **1**

(ii) Show that $BCDP$ is a cyclic quadrilateral. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. The normal to the parabola at P meets the y-axis at Q.

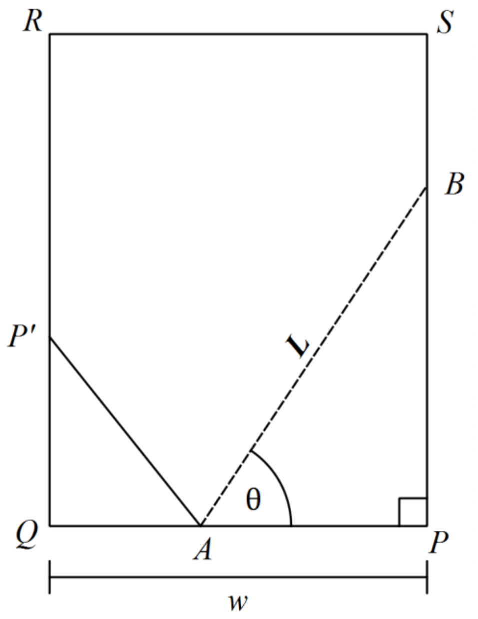
(i) Find the coordinates of Q. **1**

(ii) Determine the coordinates of R, which divides PQ externally in the ratio 2 : 1. **2**

(iii) Prove that the Cartesian equation of the locus of R is $x^2 = 4a(y - 4a)$, **and** state the coordinates of the focus of this parabola. **3**

Question 14 continues on the next page

(b) The rectangular piece of paper $PQRS$, shown below, is folded along a line AB, where A and B lie on edges PQ and PS respectively. This line is positioned so that, after folding, P coincides with a point P' which lies on the edge QR. This fold line AB makes an acute angle θ with the edge PQ. The length of AB is L and that of PQ is w.



(i) Show that $\angle P'AQ = \pi - 2\theta$. **1**

(ii) Prove that $L = \dfrac{w}{\cos\theta(1 - \cos 2\theta)}$. **3**

(iii) More than one fold line exists such that P coincides with a point on QR after folding. **4**

Show that the fold line of minimum length occurs when $\cos\theta = \dfrac{1}{\sqrt{3}}$.

(iv) Let CD be the fold line of minimum length, where C lies on PQ and D lies on PS. Show that the length of CP is given by $\dfrac{3w}{4}$. **1**

End of Paper

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Trinity Grammar School
Mathematics Department
2017
Year 12 Mathematics Extension 1
HALF-YEARLY EXAMINATION

HSC Assessment Task 3

SOLUTIONS

1	A	B	C	●
2	A	●	C	D
3	●	B	C	D
4	A	●	C	D
5	A	B	C	●
6	●	B	C	D
7	A	●	C	D
8	A	●	C	D
9	A	B	●	D
10	A	B	C	●

Multiple Choice Worked Solutions

1. $\dfrac{x+1}{x-5} \le 0 \qquad x \ne 5$

$$(x-5)^{\cancel{2}} \times \frac{(x+1)}{\cancel{(x-5)}} \le 0 \times (x-5)^{\cancel{2}}$$

$$(x-5)(x-2) \le 0$$

2. $x = y^2$ is not a function – does not pass the vertical line test.

3. Single root at x = -2
Double root at x = 3.
Triple root at x = 0

4. $m_1 = \dfrac{1}{2} \qquad m_2 = 3$

$$\tan\theta = \frac{(3) - \left(\frac{1}{2}\right)}{1 + (3)\left(\frac{1}{2}\right)}$$

$$= 1$$

$\therefore \quad \theta = \dfrac{\pi}{4}$ (acute only)

5. obtuse $\angle AOC = \dfrac{6\pi}{5}$

(angle at the centre is twice the angle at the circumference when standing on the same arc.)

$\therefore \theta = 2\pi - \dfrac{6\pi}{5}$ (angles at a point)

$$= \frac{4\pi}{5}$$

6. $\sin^2 3x = \dfrac{1}{2}(1 - \cos 6x)$

$$\therefore \int \sin^2 3x \, dx$$

$$= \frac{1}{2}\int 1 - \cos 6x \, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + c$$

7. $\displaystyle\lim_{x \to 0} \frac{\sin\frac{x}{4}}{2x}$

$$= \frac{1}{2}\cdot\frac{1}{4}\lim_{x \to 0}\frac{\sin\frac{x}{4}}{\frac{x}{4}}$$

$$= \frac{1}{8}$$

8. $\tan(A+B) = \dfrac{1}{2}$

$$\frac{1}{2} = \frac{4 + \tan B}{1 - 4.\tan B}$$

$$1 - 4\tan B = 8 + 2\tan B$$

$$-7 = 6\tan B$$

$$\tan B = -\frac{7}{6}$$

9.

$$y = 5^{2x+3}$$

$$\therefore \quad 2x + 3 = \log_5 y$$

$$2x + 3 = \frac{\log_e y}{\log_e 5}$$

$$\log_e 5(2x+3) = \log_e y$$

$$\therefore \quad y = e^{\log_e 5(2x+3)}$$

$$\frac{dy}{dx} = (2\log_e 5).e^{\log_e 5(2x+3)}$$

$$= (2\log_e 5).\, y$$

$$= (2\log_e 5).\, 5^{2x+3}$$

10. $9 = \dfrac{5x - 3.4}{5-3}$

$$18 = 5x - 12$$

$$30 = 5x$$

$$x = 6$$

$$8 = \frac{5y - 3.-2}{5-3}$$

$$16 = 5y + 6$$

$$10 = 5y$$

$$y = 2$$

$\therefore B(6, 2)$

Question 11 (15 marks)

Question 11 (a)

Criteria	Marks
Provides correct solution for x and y	**3**
Provides one correct value for x OR y and makes progress towards a second value. OR states correctly the value of ln$x$ AND ln$y$.	**2**
Attempts to use a valid simultaneous equation method	**1**

Sample answer:

$$\ln x^2 + \ln y = 15 \quad (1)$$
$$\ln x + \ln y^2 = 10 \quad (2)$$

rearrange (2)

$$\ln x = 10 - \ln y^2$$
$$= 10 - 2\ln y$$

subst into (1)

$$2(10 - 2\ln y) + \ln y = 15$$
$$20 - 4\ln y + \ln y = 15$$
$$-3\ln y = -5$$
$$\ln y = \frac{5}{3}$$
$$\therefore \quad y = e^{\frac{5}{3}}$$

$$\ln x = 10 - 2.\frac{5}{3}$$
$$= \frac{20}{3}$$
$$\therefore \quad x = e^{\frac{20}{3}}$$

Question 11 (b)

Criteria	Marks
Provides correct solution	**3**
Provides two correct substitutions	**2**
Correctly factorises sum of two cubes.	**1**

Sample answer:

$$\text{LHS} = \frac{2\sin^3 A + 2\cos^3 A}{\sin A + \cos A}$$
$$= \frac{2\cancel{(\sin A + \cos A)}(\sin^2 A - \sin A \cos A + \cos^2 A)}{\cancel{\sin A + \cos A}}$$
$$= 2(\sin^2 A + \cos^2 A - \sin A \cos A)$$
$$= 2(1 - \sin A \cos A)$$
$$= 2 - 2\sin A \cos A$$
$$= 2 - \sin 2A$$
$$= RHS$$

Question 11 (c)

Criteria	Marks
Provides correct solution	**3**
Finds two gradients OR one gradient and correctly substitutes into the formula.	**2**
Correctly finds the gradient of one tangent.	**1**

Sample answer:

$$y = x^2 \qquad\qquad y = (x-2)^2$$

$$\frac{dy}{dx} = 2x \qquad\qquad \frac{dy}{dx} = 2(x-2)$$

$$\text{at } x = 1 \qquad\qquad \text{at } x = 1$$

$$m = 2 \qquad\qquad m = -2$$

$$\tan\theta = \left| \frac{2-(-2)}{1+2.(-2)} \right|$$

$$= \left| \frac{4}{-3} \right|$$

$$\therefore \quad \theta = 53°8'$$

Question 11 (d) (i)

Criteria	Marks
Provides correct solution	**1**

Sample answer:

$y > 0$

Question 11 (d) (ii)

Criteria	Marks
Provides correct solution with *y* as the subject.	**2**
States the domain OR finds the inverse function.	**1**

Sample answer:

$$y = \sqrt{e^{x+5}}$$

$$f^{-1} : x = \sqrt{e^{y+5}}$$

$$x^2 = e^{y+5}$$

$$y + 5 = \ln x^2$$

$$y = \ln x^2 - 5$$

OR

$$= 2\ln x - 5$$

Domain: $x > 0$

Question 11 (e) (i)

Criteria	Marks
Provides correct solution.	**1**

Sample answer:

The angle at the circumference is half the angle at the centre when subtended by the same arc.

Question 11 (e) (ii)

Criteria	Marks
Provides correct solution	**2**
Provides one correct reason in progress towards solution	**1**

Sample answer:

$\angle ACO = 90° - 72°$ (radius is perpendicular to the tangent)
$= 18°$

$\angle BCO = 54° - 18°$ (adjacent angles)
$= 36°$

End of Question 11

Question 12

Question 12 (a)

Criteria	Marks
Provides correct solution	**3**
Makes progress in solving a set of simultaneous equations.	**2**
Uses the Remainder Theorem OR substitutes $x = 2$ to create an equation.	**1**

Sample answer:

$$P(2) = 0 = (2)^3 - a(2) + b$$
$$2a - b = 8 \quad (1)$$

$$P(-3) = -10 = (-3)^3 - a(-3) + b$$
$$3a + b = 17 \quad (2)$$

add (1) and (2)
$$5a = 25$$
$$a = 5$$

substitute into (2)
$$3 \times 5 + b = 17$$
$$b = 2$$

$$\therefore \quad P(x) = x^3 - 5x + 2$$

Question 12 (b) (i)

Criteria	Marks
Provides correct solution	**1**

Sample answer:

$$\lim_{x \to \infty} \frac{2x^2 + 9}{x^2 + 4}$$
$$= \lim_{x \to \infty} \frac{2 + \frac{9}{x^2}}{1 + \frac{4}{x^2}}$$
$$= 2$$

Asymptote at $y = 2$.

Question 12 (b) (ii)

Criteria	Marks
Provides correct solution	**1**

Sample answer:

$$f'(x) = \frac{4x(x^2+4) - 2x(2x^2+9)}{(x^2+4)^2}$$
$$= \frac{4x^3 + 16x - 4x^3 - 18x}{(x^2+4)^2}$$
$$= \frac{-2x}{(x^2+4)^2}$$

For st. pt.

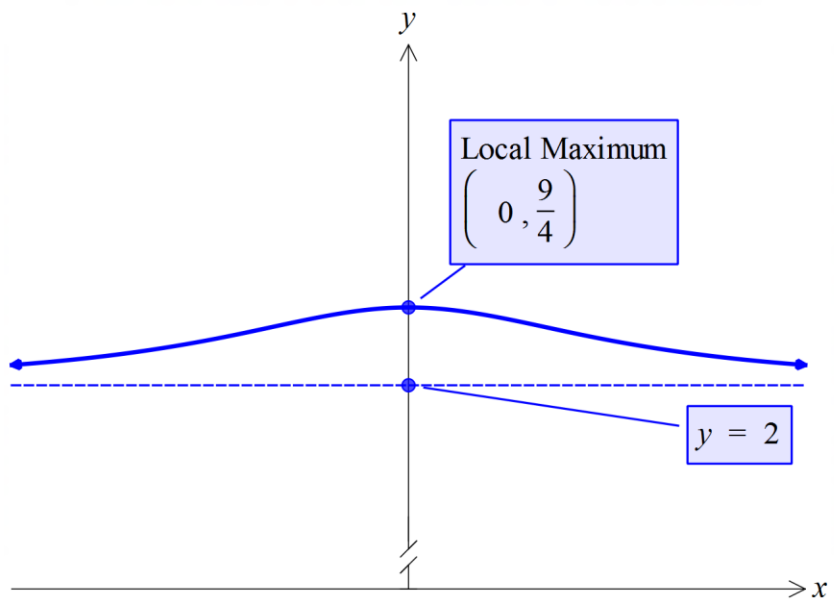
$$f'(x) = 0 = -\frac{2x}{(x^2+4)^2}$$
$$\therefore \quad x = 0$$
$$y = \frac{9}{4}$$

Stationary point at $(0, 2\frac{1}{4})$

Question 12 (b) (iii)

Criteria	Marks
Provides correct solution with correct shape, stationary point marked and the asymptote labelled as an equation.	**2**
Correct shape OR asymptote correctly labelled	**1**

Sample answer:



Question 12 (c)

Criteria	**Marks**
Provides correct solution	**4**
Correct integration of *u*	**3**
Correct substitution of *u* and bounds	**2**
Correct differential of *u* OR change in bounds	**1**

Sample answer:

$$\int_{-2}^{1} x\sqrt{x^2-1}\,dx$$

$$u = x^2 - 1$$
$$du = 2x\,dx$$
$$\frac{1}{2}du = x\,dx$$

$$x = 1 \quad u = 0$$
$$x = -2 \quad u = 3$$

$$= \int_{3}^{0} \sqrt{u}\ \frac{1}{2}du$$

$$= \int_{3}^{0} \frac{u^{\frac{1}{2}}}{2}du$$

$$= \left[\frac{\cancel{2}u^{\frac{3}{2}}}{\cancel{2}.3}\right]_3^0$$

$$= \left(\frac{0}{3}\right) - \left(\frac{3^{\frac{3}{2}}}{3}\right)$$

$$= 0 - \sqrt{3}$$
$$= -\sqrt{3}$$

Question 12 (d) (i)

Criteria	Marks
Provides correct proof	**2**
Finds a statement for *PC* OR *AP* in terms of θ	**1**

Sample answer:

Arc $AP = r\theta = 2 \times PC$

$\angle POC = \pi - \theta$ (supplementary angles)

in ΔPOC

$$\sin(\pi - \theta) = \frac{\text{PC}}{r}$$

$$\therefore \quad \sin\theta = \frac{\text{PC}}{r}$$

$$\text{PC} = r\sin\theta$$

$$\therefore \quad r\theta = 2(r\sin\theta)$$

$$r\theta = 2r\sin\theta$$

$$\theta = 2\sin\theta \quad \text{as required}$$

Question 12 (d) (ii)

Criteria	Marks
Provides correct solution	**2**
Correctly differentiates $f(\theta)$	**1**

Sample answer:

$$f(\theta) = 2\sin\theta - \theta$$
$$f(1\cdot 8) = 2\sin(1\cdot 8) - 1\cdot 8$$
$$= 0\cdot 1476\ldots$$

$$f'(\theta) = 2\cos\theta - 1$$
$$f'(1\cdot 8) = 2\cos(1\cdot 8) - 1$$
$$= -1\cdot 4544\ldots$$

$$\therefore \quad \theta_2 = 1\cdot 8 - \frac{2\sin(1\cdot 8) - 1\cdot 8}{2\cos(1\cdot 8) - 1}$$
$$= 1\cdot 9015\ldots$$
$$= 1\cdot 90 \quad \text{to 3 s.f.}$$

End of Question 12

Question 13

Question 13 (a)

Criteria	Marks
Provides correct solution	**3**
Correctly finds the values of $2x$	**2**
Correctly finds the values of $\sec 2x$	**1**

Sample answer:

$$3\sec^2 2x = 4 \qquad\qquad 0 \le x \le \pi$$
$$0 \le 2x \le 2\pi$$

$$\sec^2 2x = \frac{4}{3}$$

$$\sec 2x = \pm\frac{2}{\sqrt{3}}$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

Question 13 (b) (i)

Criteria	Marks
Provides correct solution	**2**
Correctly finds the value of R OR α	**1**

Sample answer:

$$\begin{aligned}&R\cos(x-\alpha)\\ =\ &R\cos\alpha\cos x + R\sin\alpha\sin x\\ \therefore\ &R\cos\alpha = 3 \qquad R\sin\alpha = \sqrt{3}\end{aligned}$$

$$\begin{aligned}R^2 &= 3^2 + (\sqrt{3})^2\\ &= 12\\ R &= \sqrt{12}\\ &= 2\sqrt{3} \qquad R > 0\end{aligned}$$

$$\begin{aligned}\tan\alpha &= \frac{\sqrt{3}}{3}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

$$\therefore \quad \alpha = \frac{\pi}{6} \qquad \alpha \text{ is acute}$$

$$\therefore 3\cos x + \sqrt{3}\sin x = 2\sqrt{3}\cos\left(x - \frac{\pi}{6}\right)$$

Question 13 (b) (ii)

Criteria	Marks
Provides correct solution for x	**2**
Correctly finds $\cos\left(x-\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$	**1**

Sample answer:

$$\begin{aligned}3\cos x + \sqrt{3}\sin x &= \sqrt{6}\\ 2\sqrt{3}\cos\left(x-\frac{\pi}{6}\right) &= \sqrt{6}\\ \cos\left(x-\frac{\pi}{6}\right) &= \frac{\sqrt{6}}{2\sqrt{3}}\\ \cos\left(x-\frac{\pi}{6}\right) &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$x - \frac{\pi}{6} = 2n\pi \pm \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$$

$$x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$$ (mark given at this p

$$x = 2n\pi + \frac{5\pi}{12} \qquad \text{or} \qquad x = 2n\pi - \frac{\pi}{12}$$

Question 13 (c)

Criteria	Marks
Provides correct solution	**4**
Correctly integrates the function.	**3**
Correctly finds an integrand independent of $\cos^2 \dfrac{x}{4}$	**2**
Correctly finds an integral for the volume with the correct bounds.	**1**

Sample answer:

$$V = \pi \int_0^{2\pi} \cos^2 \frac{x}{4}\, dx \qquad\qquad \cos^2 \frac{x}{4} = \frac{1}{2} + \frac{1}{2}\cos\frac{x}{2}$$

$$= \pi \int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos\frac{x}{2}\, dx$$

$$= \pi\left[\frac{x}{2} + \cancel{2}.\frac{1}{\cancel{2}}\sin\frac{x}{2}\right]_0^{2\pi}$$

$$= \pi\left[\left(\frac{2\pi}{2} + \sin\frac{2\pi}{2}\right) - \left(\frac{0}{2} + \sin\frac{0}{2}\right)\right]$$

$$= \pi[(\pi + 0) - 0]$$

$$= \pi^2 \text{ } units^3$$

Question 13 (d) (i)

Criteria	Marks
Provides correct reason	**1**

Sample answer:

The angle between a tangent and a chord is equal to the angle in the alternate segment.

Question 13 (d) (ii)

Criteria	Marks
Provides correct proof	**3**
Provides two correct reasons progressing towards a proof	**2**
Provides a correct reason	**1**

Sample answer:

Let $D\hat{P}A = P\hat{B}A = x$

$A\hat{B}C = A\hat{Q}C = y$ (Angles subtended by the same arc are equal at the circumference)

$180 = D\hat{P}A + D\hat{Q}P + P\hat{D}Q$ (Angle sum of ΔPDQ)

$180 = x + y + P\hat{D}Q$

$\therefore P\hat{D}Q = 180 - (x + y)$

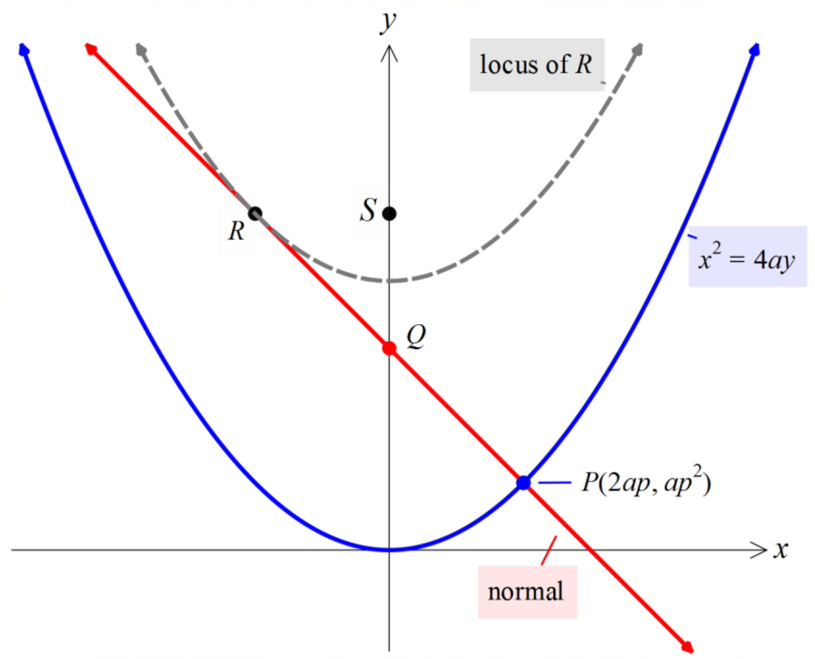
$P\hat{B}C = P\hat{B}A + C\hat{B}A$ (Adjacent angles)

$= x + y$

$\therefore P\hat{D}Q + P\hat{B}C = 180$

$\therefore$ $BCDP$ is cyclic. (Opposite angles are supplementary)

Question 14 (15 marks) Use a SEPARATE writing booklet



Question 14 (a) (i)

Criteria	Marks
Provides a correct coordinates	**1**

Sample answer:

Equation of normal: $x + py = ap^3 + 2ap$

y intercept at $x = 0$

$\therefore py = ap^3 + 2ap$

$y = ap^2 + 2a$

$\therefore Q(0, ap^2 + 2a)$

Question 14 (a) (ii)

Criteria	Marks
Provides the correct coordinates	**2**
Makes an attempt to substitute into the formula (ratio or midpoint)	**1**

Sample answer:

$$R_x = \frac{2(0) - 1(2ap)}{2 - 1} = -2ap \qquad R_y = \frac{2(ap^2 + 2a) - 1(ap^2)}{2 - 1} = ap^2 + 4a$$

$\therefore R(-2ap, ap^2 + 4a)$

Question 14 (a) (iii)

Criteria	Marks
Provides correct solution	**3**
Provides makes progress towards locus AND provides the correct focus	**2**
Provides a correct focus	**1**

Sample answer:

$$x = -2ap$$
$$p = -\frac{x}{2a}$$

$$y = a\left(-\frac{x}{2a}\right)^2 + 4a$$
$$y = \frac{x^2}{4a} + 4a$$
$$4ay = x^2 + 16a^2$$
$$x^2 = 4ay - 16a^2$$
$$x^2 = 4a(y - 4a)$$

Vertex $(0, 4a)$
Focal length = a

$S(0, 5a)$

Question 14 (b) (i)

Criteria	Marks
Provides correct solution	**1**

Sample answer:

$P'\hat{A}B = \theta$ (Fold AB bisects $P'\hat{A}P$)

$\pi = Q\hat{A}P' + \theta + \theta$ (Supplementary angles)

$Q\hat{A}P' = \pi - 2\theta$

Question 14 (b) (ii)

Criteria	**Marks**
Provides correct proof	**3**
Finds a statement for L and makes progress towards AP.	**2**
Finds a statement for :L $L = \dfrac{AP}{\cos\theta}$	**1**

Sample answer:

$AP = AP'$ due to fold
$QA = w - AP$

$$\cos\theta = \frac{AP}{L}$$

$$\therefore \quad L = \frac{AP}{\cos\theta}$$

In $\Delta AQP'$

$$\cos(\pi - 2\theta) = \frac{QA}{AP'}$$

$$\cos(\pi - 2\theta) = \frac{w - AP}{AP}$$

$$-\cos 2\theta = \frac{w - AP}{AP}$$

$-AP\cos 2\theta = w - AP$
$AP - AP\cos 2\theta = w$
$AP(1 - \cos 2\theta) = w$

$$AP = \frac{w}{1 - \cos 2\theta}$$

$$\therefore L = \frac{1}{\cos\theta} \cdot \frac{w}{1 - \cos 2\theta}$$

$$= \frac{w}{\cos\theta(1 - \cos 2\theta)} \qquad \text{as required.}$$

Question 14 (b) (iii)

Criteria	**Marks**
Provides correct solution. MUST include showing that the point is a minimum, and state why $\sin\theta = 0$ is discarded.	**4**
Correctly finds the required values, but does not use the gradient test (or similar) to demonstrate a minimum.	**3**
Changes form of L and attempts to differentiate OR correctly differentiates L	**2**
Attempts to differentiate L.	**1**

Sample answer:

$$L = \frac{w}{\cos\theta(1 - \cos 2\theta)}$$

$$= \frac{w}{\cos\theta(2\sin^2\theta)}$$

$$= \frac{w}{2\cos\theta(1 - \cos^2\theta)}$$

$$= \frac{w}{2} \cdot \frac{1}{\cos\theta - \cos^3\theta}$$

$$= \frac{w}{2} \cdot (\cos\theta - \cos^3\theta)^{-1}$$

$$\frac{dL}{d\theta} = \frac{w}{2} \,.\, -1(\cos\theta - \cos^3\theta)^{-2}(-\sin\theta + 3\sin\theta\cos^2\theta)$$

$$= \frac{-w(-\sin\theta + 3\sin\theta\cos^2\theta)}{2(\cos\theta - \cos^3\theta)^2}$$

$$= \frac{w\sin\theta(1 - 3\cos^2\theta)}{2(\cos\theta - \cos^3\theta)^2}$$

For minimum $\dfrac{dL}{d\theta} = 0$

$$0 = \frac{w\sin\theta(1 - 3\cos^2\theta)}{2(\cos\theta - \cos^3\theta)^2}$$

$\therefore \sin\theta = 0$

$\theta = 0, \pi, 2\pi, \ldots$

no solutions as θ is acute.

$$1 - 3\cos^2\theta = 0$$

$$\cos^2\theta = \frac{1}{3}$$

$$\cos\theta = \pm\frac{1}{\sqrt{3}}$$

θ is acute

$$\therefore \quad \cos\theta = +\frac{1}{\sqrt{3}}$$

$$\theta = 0{\cdot}955 \text{ radians}$$

To test for minimum using gradient test:

θ	0.9	0.955	1
$\dfrac{dL}{d\theta}$	$-0.43w$	0	$0.35w$

$\therefore$ a minimum value when $\cos\theta = \dfrac{1}{\sqrt{3}}$

Question 14 (b) (iv)

Criteria	Marks
Provides a correct solution	**1**

Sample answer:

Minimum fold length:

$$\cos\theta = \frac{1}{\sqrt{3}}$$

$$\begin{aligned}
L &= \frac{w}{\cos\theta(1 - \cos 2\theta)} \\
&= \frac{w}{\cos\theta(2\sin^2\theta)} \\
&= \frac{w}{2\cos\theta(1 - \cos^2\theta)} \\
&= \frac{w}{2 \times \frac{1}{\sqrt{3}} \times \left(1 - \left(\frac{1}{\sqrt{3}}\right)^2\right)} \\
&= \frac{w}{\frac{2}{\sqrt{3}} \times \frac{2}{3}} \\
&= \frac{3\sqrt{3}\,w}{4}
\end{aligned}$$

$$\cos\theta = \frac{CP}{L}$$

$$\begin{aligned}
CP &= L\cos\theta \\
&= \frac{3\cancel{\sqrt{3}}\,w}{4} \cdot \frac{1}{\cancel{\sqrt{3}}} \\
&= \frac{3w}{4} \text{ as required.}
\end{aligned}$$