Fort Street High School

2014 Assessment Task 3

Extension 1 Mathematics

Outcomes Assessed	Questions
Applies appropriate techniques to solve problems involving algebra, co-ordinate geometry and polynomials.	**1**
Applies appropriate techniques to solve problems involving trigonometry.	**2,5**
Applies appropriate techniques to solve problems in calculus.	**3,4**
Synthesises mathematical solutions to harder problems and communicates these solutions in an appropriate form.	**6**

Question	1	2	3	4	5	6	Total	%
Marks								

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board-approved calculators may be used.
- All necessary working should be shown.
- Each question should be started in a *new* booklet

Total marks – 72
Attempt Questions 1– 6
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1. (12 marks) Start a SEPARATE writing booklet. **Marks**

(a) Let *A* be the point $(8,10)$ and *B* the point $(-2,4)$. **2**

Find the coordinates of the point P which divides the interval *AB* externally in the ratio 2:5.

(b) Solve $\dfrac{4}{x+1} \leq 3$. **3**

(c) Is $x+3$ a factor of $x^3-5x+12$? Give reasons for your answer. **2**

(d) Suppose $x^3-2x^2+a \equiv (x+2)Q(x)+3$ where $Q(x)$ is a polynomial. **2**
Find the value of *a*.

(e) The polynomial $P(x)=x^3-2x^2+kx+24$ has roots α, β and γ

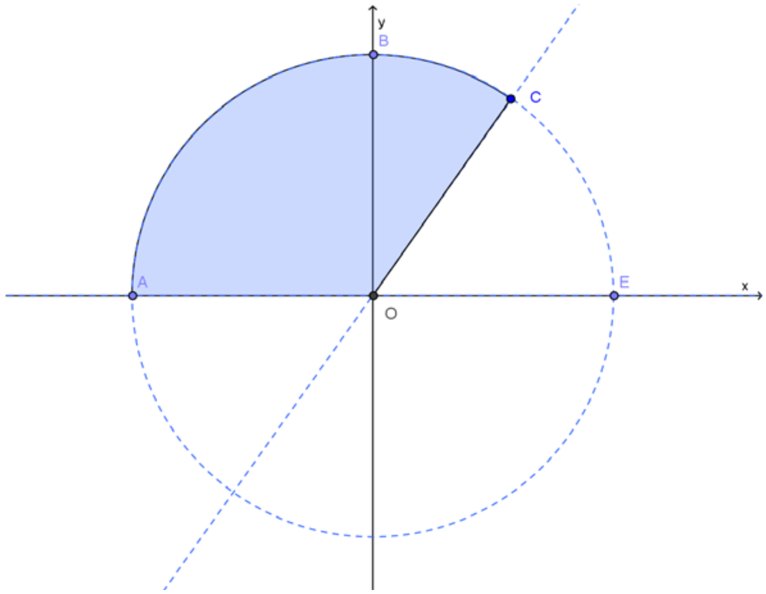
(i) Find the value of $\alpha+\beta+\gamma$ and $\alpha\beta\gamma$ **1**

(ii) It is known that two of the roots are equal in magnitude but opposite in sign. **2**

Find the third root and hence find the value of *k*.

Question 2. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The shaded region shows the intersection of $y \le \sqrt{\frac{1}{4} - x^2}$, $y \ge 0$ and $y \ge \sqrt{3}x$



i) Show that angle COE is $\frac{\pi}{3}$ radians **1**

ii) Find the minor arc length, AC. **2**

iii) Find the area of the shaded sector AOC **2**

(b) The angle between the line $y = 2x$ and the tangent to the curve $y = Ax^2 + Ax$ at $x = 1$ is $\frac{\pi}{4}$ radians. Find the values of A. **3**

(c) Sketch the curve $y = 1 - \sin\left(2x - \frac{\pi}{2}\right)$ for $0 \le x \le 2\pi$, showing all intercepts **4**

Question 3. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate with respect to x

i) $\dfrac{5}{x}$ **1**

ii) $x\sqrt{x}$ **1**

iii) $\left(2x^2+1\right)\left(3-x\right)^5$ **2**

(b) Differentiate from first principles $y = x^2 - 3x$ **3**

(c) Evaluate $\displaystyle\lim_{x\to\infty}\frac{3x^2+2x-9}{4x^2-11x+5}$ **2**

(d) At what points on the curve $y = \dfrac{x}{x-8}$ is the tangent parallel to the line $2x + y + 2 = 0$ **3**

Question 4. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ and state whether the curve is continuous or discontinuous. **3**
Give reasons for your answer.

(b) Find the values for which the function $f(x) = x^4 + 9x^3 - 15x^2 + 30x - 60$ is concave up. **3**

(c) Consider the function defined by $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 5x^2 + 12x + 1$

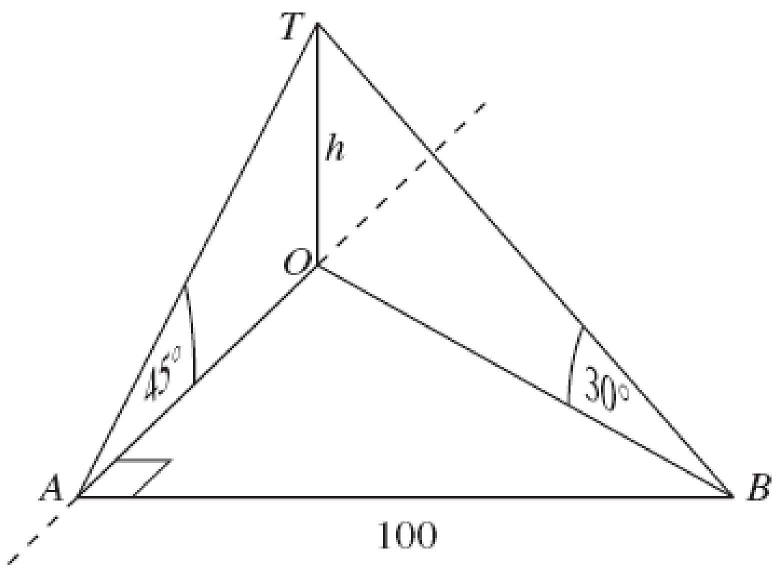
(i) Find $\frac{dy}{dx}$ **1**

(ii) Find any stationary points and determine their nature. **4**

(iii) Hence sketch the curve showing turning points. **1**

Question 5. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A surveyor stands at a point A, which is due south of a tower OT of height h m. The angle of elevation of the top of the tower from A is 45°. The surveyor then walks 100 m due east to point B, from where she measures the angle of elevation of the top of the tower to be 30°.



(i) Express the length of OB in terms of h. **1**

(ii) Show $h = 50\sqrt{2}$. **2**

(iii) Calculate the bearing of B from the base of the tower. **2**

(b) ABC is an acute angled triangle.

i) Show that $\tan(A+B) = -\tan C$. **1**

ii) Hence show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. **1**

(c) Solve

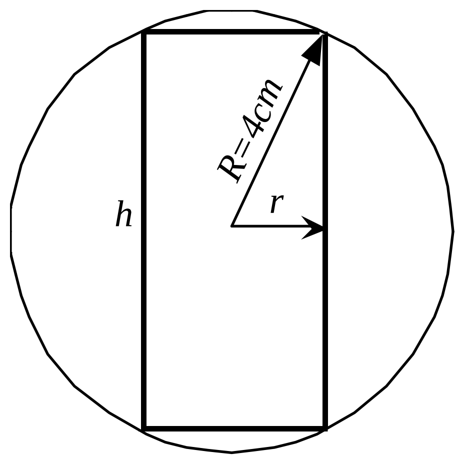
i) $\cos 2\alpha = \dfrac{1}{\sqrt{2}}$, for $0 \le \alpha \le 2\pi$ **2**

ii) $2\cos^2 x = -3\sin x$, for $0 \le x \le 2\pi$. **3**

Question 6. (12 marks) Use a SEPARATE writing booklet. **Marks**

a) A metal worker is required to cut a cylinder from a solid sphere of metal of radius $R = 4$ cm.

The diagram below shows a cross-section of the sphere and cylinder.



(i) Show that the volume V cm^3, of the cylinder is given by **1**

$$V = \frac{1}{4}\pi h\left(64 - h^2\right).$$

(ii) Find the value of h such that the volume of the cylinder is a maximum. **2**

b) Use the identity $t = \tan\frac{\theta}{2}$ to solve $3\sin\theta + 4\cos\theta = 3$. **3**

c) Express $3\cos x + 4\sin x$ in the form $R\cos\left(x - \alpha\right)$ and hence solve **3**

$$3\cos x + 4\sin x = 2, \quad \text{for } 0 \le x \le 2\pi.$$

d) Find the general solution to $\sqrt{3}\sec^2\theta - 2\tan\theta = 2\sqrt{3}$. **3**

End of examination.

Fort Street High School

2014 Assessment Task 3

Extension 1 Mathematics

Question	1	2	3	4	5	6	Total	%
Marks	12	12	12	12	12	12	60	

SOLUTIONS

Question 1.

(a) Let A be the point $(8,10)$ and B the point $(-2,4)$.

Find the coordinates of the point P which divides the interval AB externally in the ratio 2:5.

Solution

Divided externally $\Rightarrow$ the ratio is $-2:5$

$$x = \frac{mx_2 + nx_1}{m+n} = \frac{-2(-2)+5(8)}{-2+5} = \frac{44}{3}$$

$$y = \frac{my_2 + ny_1}{m+n} = \frac{-2(4)+5(10)}{-2+5} = 14$$

Marking Guideline

2 Correct answer
1 Could recall formula

Marking Notes
Common error was forgetting the negative sign for the external ratio.
Some students also applied the incorrect ratio to each coordinate.

(b) Solve $\dfrac{4}{x+1} \le 3$.

Solution

$$\frac{4(x+1)^2}{x+1} \le 3(x+1)^2$$

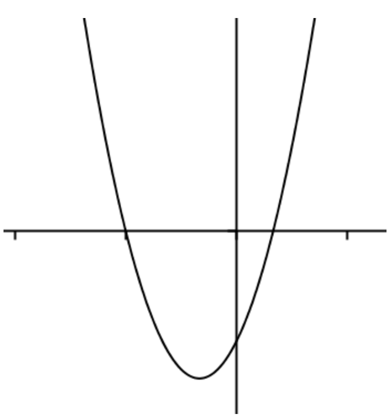
$$4(x+1) \le 3(x+1)^2$$

$$0 \le 3(x+1)^2 - 4(x+1)$$

$$0 \le (x+1)\left[3(x+1)-4\right]$$

$$0 \le (x+1)(3x-1)$$

Solving the quadratic inequality by graphing



$x < -1 \;\; or \;\; x \ge \frac{1}{3}$ Note: $x \ne -1$

Marking Guideline

3 Correct answer
2 Solve correctly the inequality but didn't recognize $x \ne -1$ or omitted justification diagram (or equivalent)
1 Could demonstrate some correct working.

Marking Notes
Main error was the omission of the justification for the inequalities.
A common error was to neglect the domain and not realize $x \ne 1$

(c) Is $x+3$ a factor of $x^3-5x+12$? Give reasons for your answer.

Solution

By the remainder theorem $x+3$ is a factor if $P(-3)=0$.

Let $P(x)=x^3-5x+12$

$$P(-3)=(-3)^3-5(-3)+12$$
$$=-27+15+12$$
$$=0$$

So $x+3$ is a factor

Marking Guideline

2 Correct answer including mention of the factor/remainder theorem or equivalent.

1 Could demonstrate some understanding but omits reasoning.

Marking Notes

Mostly well answered but some substitution of the incorrect zero.

(d) Suppose $x^3-2x^2+a \equiv (x+2)Q(x)+3$ where $Q(x)$ is a polynomial.
Find the value of a.

Solution

Let $x=-2$ $\quad (-2)^3-2(-2)^2+a \equiv ((-2)+2)Q(-2)+3$

$$-8-8+a=3$$
$$a=19$$

Marking Guideline

2 Correct answer

1 Could demonstrate some correct working.

Marking Notes

Many did not note the link to the remainder theorem.

Many tried polynomial division with little success.

(e) The polynomial $P(x)=x^3-2x^2+kx+24$ has roots α, β and γ

(i) Find the value of $\alpha+\beta+\gamma$ and $\alpha\beta\gamma$

Solution

$$\alpha+\beta+\gamma=\frac{-b}{a}$$
$$=2$$

$$\alpha\beta\gamma=-\frac{d}{a}$$
$$=-24$$

Marking Guideline

1 For both values

Marking Notes

A common mistake was $\alpha+\beta+\gamma=k$.

(ii) It is known that two of the roots are equal in magnitude but opposite in sign.

Find the third root and hence find the value of *k*.

Solution

Let the roots be $\alpha, -\alpha$ & β, then

$$\begin{aligned} \alpha - \alpha + \beta &= 2 \\ \beta &= 2 \end{aligned}$$ # The third root

$$\begin{aligned} \alpha(-\alpha)(2) &= -24 \\ \alpha^2 &= 12 \\ \alpha &= \pm 2\sqrt{3} \end{aligned}$$

Marking Guideline

2 Correct answer
1 Could find the third root.

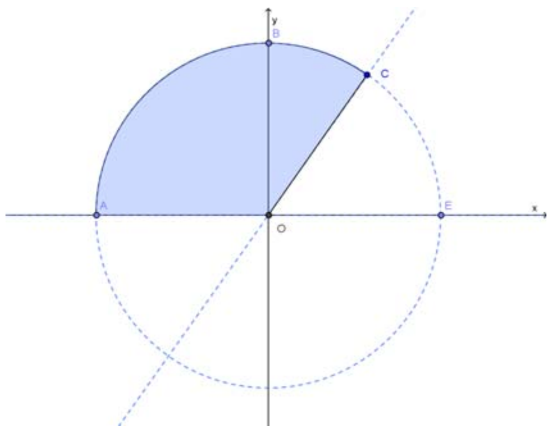
Marking Notes
Mostly well done but a common error was having $k = 12$.

Finding *k*

$$\begin{aligned} \alpha\beta + \alpha\gamma + \beta\gamma &= \frac{c}{a} \\ \alpha(-\alpha) + \alpha\beta + (-\alpha)\beta &= k \\ -\alpha^2 + \alpha\beta - \alpha\beta &= k \\ -12 &= k \\ k &= -12 \end{aligned}$$

Question 2. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The shaded region shows the intersection of $y \le \sqrt{\frac{1}{4} - x^2}$, $y \ge 0$ and $y \ge \sqrt{3}x$



i) Show that angle COE is $\frac{\pi}{3}$ radians

Solution

$\angle COE$ is formed by the line $y = \sqrt{3}x$ and the x-axis.

Noting that $m = \tan\theta$, where $\theta = \angle COE$

$$\tan\theta = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

> *Marking Guideline*
>
> **1** For correct response
>
> **Marking Notes**
> Reasonably well done

ii) Find the minor arc length, AC.

Solution

$$\angle COA = \pi - \frac{\pi}{3} \qquad \text{Supplementary angles}$$

$$= \frac{2\pi}{3}$$

Finding the radius of the circle $r^2 = \frac{1}{4}$

$$r = \frac{1}{2}$$

$$ARC\ AC = r\theta$$

$$= \frac{1}{2} \cdot \frac{2\pi}{3}$$

$$= \frac{\pi}{3}$$

> *Marking Guideline*
>
> **2** Correct answer
> **1** Could find the radius or $\angle COA$ & substitute it into the formula for arc length.
>
> **Marking Notes**

iii) Find the area of the shaded sector *AOC*

Solution

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}$$
$$= \frac{1}{2}\left(\frac{1}{2}\right)^2 \frac{2\pi}{3}$$
$$= \frac{\pi}{12}$$

Marking Guideline

2 Correct answer
1 Could find the radius or $\angle COA$ & substitute it into the formula for arc length.

Marking Notes

(b) The angle between the line $y = 2x$ and the tangent to the curve $y = Ax^2 + Ax$ at $x = 1$ is $\frac{\pi}{4}$ radians. Find the values of *A*.

Solution

$$y = Ax^2 + Ax$$
$$\frac{dy}{dx} = 2Ax + A$$

at $x = 1$ $\frac{dy}{dx} = 2A(1) + A$
$$= 3A$$

$$\Rightarrow m_1 = 3A$$

$$y = 2x \quad \Rightarrow m_2 = 2$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$
$$\tan\frac{\pi}{4} = \left|\frac{3A - 2}{1 + 3A(2)}\right|$$
$$|1 + 6A| = |3A - 2|$$

$1 + 6A = 2 - 3A$ *or* $1 + 6A = -(2 - 3A)$

$9A = 1$ $\qquad$ $1 + 6A = -2 + 3A$

$A = \frac{1}{9}$ $\qquad$ $A = -1$

Marking Guideline

3 Correct answer
2 Correctly substituting into the formula to find an angle
1 Could state m_1 & m_2

Marking Notes
Many students missed the second value of *A*.

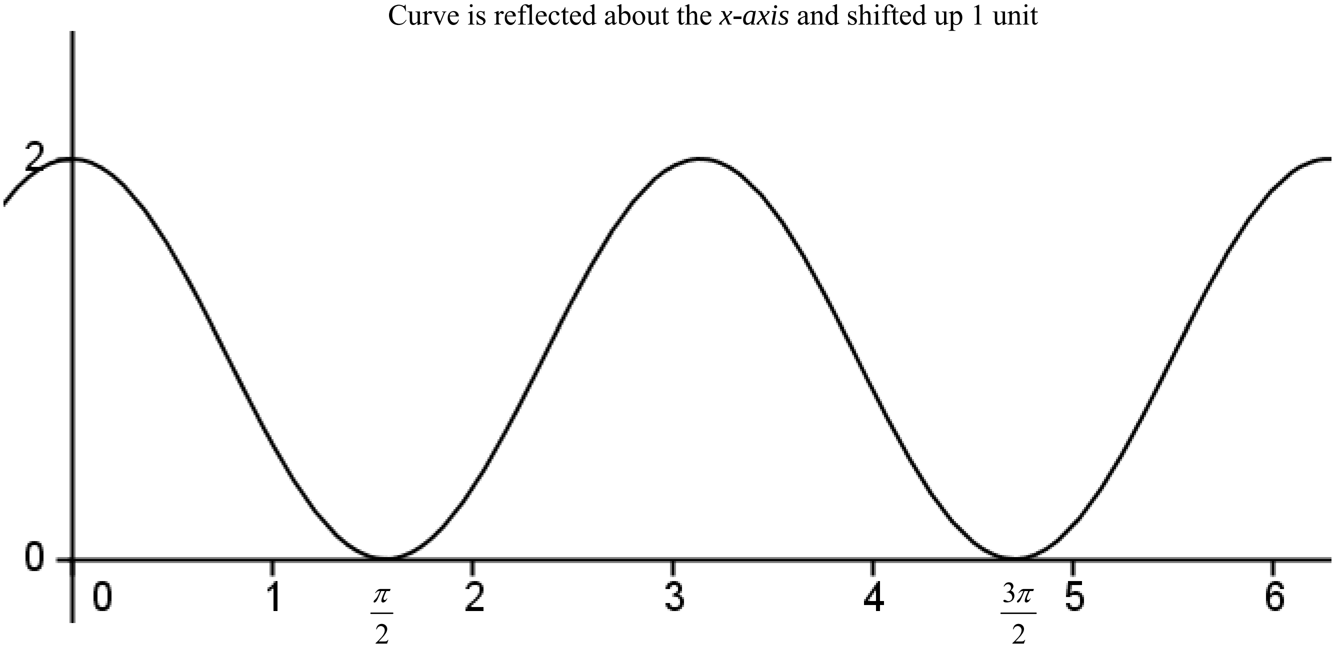
(c) Sketch the curve $y = 1 - \sin\left(2x - \frac{\pi}{2}\right)$ for $0 \le x \le 2\pi$, showing all intercepts

Solution

$$y = 1 - \sin\left(2x - \frac{\pi}{2}\right)$$
$$= 1 - \sin\left[2\left(x - \frac{\pi}{4}\right)\right]$$

Period $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ Amplitude $a = 2$ Centre of motion $y = 1$

Curve is reflected about the *x-axis* and shifted up 1 unit



Marking Guideline

4 Correct answer including the elements - shift up 1 unit, reflection, shift right by $\frac{\pi}{4}$, amplitude

3 3 correct elements
2 2 correct elements
1 1 correct element

Marking Notes

Graph was poorly answered by most students

Marks were awarded for frequency & amplitude.

Question 3. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate with respect to x

i) $\dfrac{5}{x}$

Solution

$$\frac{d}{dx}\left(\frac{5}{x}\right) = -\frac{5}{x^2}$$

> *Marking Guideline*
>
> **1** Correct response
>
> **Marking Notes**
> Well done

ii) $x\sqrt{x}$

Solution

$$\frac{d}{dx}\left(x\sqrt{x}\right) = \frac{3}{2}\sqrt{x}$$

> *Marking Guideline*
>
> **1** Correct response
>
> **Marking Notes**
> Well done

iii) $\left(2x^2+1\right)\left(3-x\right)^5$

Solution

$$\begin{aligned}\frac{d}{dx}\left[\left(2x^2+1\right)\left(3-x\right)^5\right] &= \left(3-x\right)^5 \times 4x \ + \left(2x^2+1\right)\times 5\left(3-x\right)^4\left(-1\right)\\ &= -\left(3-x\right)^4\left(14x^2-12x+5\right)\end{aligned}$$

> *Marking Guideline*
>
> **2** Correct answer
> **1** Partial solution.
>
> **Marking Notes**
> Many students neglected the - sign

(b) Differentiate from first principles $y = x^2 - 3x$

Solutions

$$\begin{aligned} y &= x^2 - 3x\\ \frac{dy}{dx} &= \lim_{h\to 0}\frac{\left(x+h\right)^2 - 3\left(x+h\right) - \left[x^2-3x\right]}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2-3h}{h}\\ &= \lim_{h\to 0} 2x+h-3\\ &= 2x+\left(0\right)-3\\ &= 2x-3\end{aligned}$$

> *Marking Guideline*
>
> **3** Correct answer
> **2** For finding $\lim_{h\to 0}\dfrac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}$
> **1** For finding $f\left(x+h\right) = \left(x+h\right)^2 - 3\left(x+h\right)$.
>
> **Marking Notes**

(c) Evaluate $\lim_{x\to\infty}\frac{3x^2+2x-9}{4x^2-11x+5}$

Solution

$$\lim_{x\to\infty}\frac{3x^2+2x-9}{4x^2-11x+5}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{9}{x^2}}{\frac{4x^2}{x^2}-\frac{11x}{x^2}+\frac{5}{x^2}}$$

$$=\lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{9}{x^2}}{4-\frac{11}{x}+\frac{5}{x^2}}$$

$$=\frac{3}{4}$$

Marking Guideline

2 Correct answer

1 Dividing throughout by x^2.

Marking Notes

Well done

(d) At what points on the curve $y=\frac{x}{x-8}$ is the tangent parallel to the line $2x+y+2=0$

Solution

$2x+y+2=0 \Rightarrow m_1=-2$

$$y=\frac{x}{x-8}$$

$$\frac{dy}{dx}=\frac{(x-8)\times1-x\times1}{(x-8)^2}$$

$$=\frac{-8}{(x-8)^2}$$

Parallel lines $\Rightarrow m_1=m_2$

$$\frac{-8}{(x-8)^2}=-2$$

$$(x-8)^2=4$$

$$x-8=\pm2$$

$$x=6,10$$

$$y=-3,5$$

Marking Guideline

3 Correct answer

2 For finding the *x* values but not *y*

1 For differentiating *y*.

Marking Notes

Some students equated the gradient of the line to the function rather than its derivative.

Many failed to find the *y* coordinates.

Question 4.

(a) Find $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ and state whether the curve is continuous or discontinuous.
Give reasons for your answer.

Solution

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \to 1} x + 1$$

$$= 2$$

The curve $\dfrac{x^2 - 1}{x - 1}$ is *discontinuous* as it is undefined for $x = 1$

Marking Guideline

3 Correct answer
2 Finding the limit but not communicating the reason for discontinuity
1 Partial solution

Marking Notes
Many students lost a mark because they did not state the *reason* for discontinuity.

(b) Find the values for which the function $f(x) = x^4 + 9x^3 - 15x^2 + 30x - 60$ is concave up.

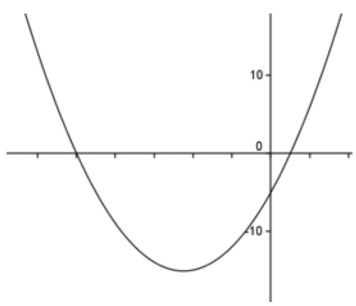
Solution

Concave up when

$$f(x) = x^4 + 9x^3 - 15x^2 + 30x - 60$$
$$f'(x) = 4x^3 + 27x^2 - 30x + 30$$
$$f''(x) = 12x^2 + 54x - 30$$

$$12x^2 + 54x - 30 > 0$$
$$2x^2 + 9x - 5 > 0$$
$$(2x-1)(x+5) > 0$$

$$x < -5 \ \ or \ \ x > \frac{1}{2}$$



Marking Guideline

3 Correct answer
2 Not correctly solving the inequality.
1 For finding the second derivative or stating $\dfrac{d^2y}{dx^2} > 0$ when concave up

Marking Notes
Poorly answered indicating a poor understanding of concavity.

(c) Consider the function defined by $f(x)=\frac{1}{2}x^4-\frac{4}{3}x^3-5x^2+12x+1$

(i) Find $\frac{dy}{dx}$

Solution

$$f(x)=\frac{1}{2}x^4-\frac{4}{3}x^3-5x^2+12x+1$$
$$f'(x)=2x^3-4x^2-10x+12$$

Marking Guideline

1 Correct response

Marking Notes
Generally well answered

(ii) Find any stationary points and determine their nature.

Solution

Stationary points occur when $f'(x)=0$

$$2x^3-4x^2-10x+12=0$$
$$x^3-2x^2-5x+6=0$$

Finding a zero: Let $P(x)=x^3-2x^2-5x+6$

$$P(1)=(1)^3-2(1)^2-5x+6=0 \quad \Rightarrow \quad x-1$$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$$P(x)=(x-1)(x^2-x-6)$$
$$=(x-1)(x-3)(x+2)$$

So the stationary points are $x=-2,1,3$

Determining the nature

$$f'(x)=2x^3-4x^2-10x+12$$
$$f''(x)=6x^2-8x-10$$

$$f''(-2)=6(-2)^2-8(-2)-10>0$$
$$f''(1)=6(1)^2-8(1)-10<0$$
$$f''(3)=6(3)^2-8(3)-10>0$$

$\left(-2,-24\frac{1}{3}\right)$ minimum turning point

$\left(1,7\frac{1}{6}\right)$ maximum turning point

$\left(3,-3\frac{1}{2}\right)$ minimum turning point

Marking Guideline

4 Correct answer

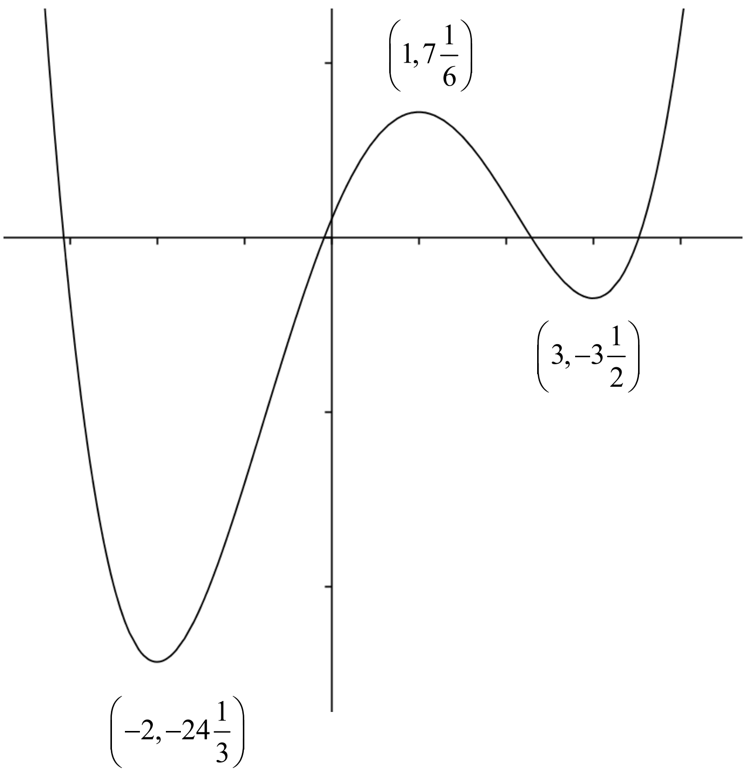
3 Correct working with one error

2 Could factorise $f(x)$

1 Limited understanding

Marking Notes
Generally well answered but could set out working more clearly.

(iii) Hence sketch the curve showing turning points.



Marking Guideline

1 Correct response

Marking Notes

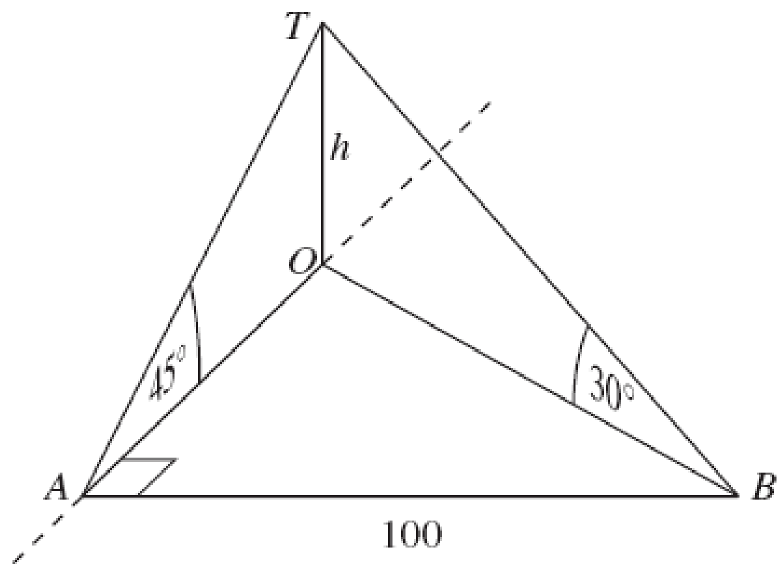
This graph was poorly drawn.

Many students did not label the coordinates of the turning points.

Some graphs were too small; graphs should be 1/3 of a page.

Question 5.

(a) A surveyor stands at a point A, which is due south of a tower OT of height h m. The angle of elevation of the top of the tower from A is 45°. The surveyor then walks 100 m due east to point B, from where she measures the angle of elevation of the top of the tower to be 30°.



(i) Express the length of OB in terms of h.

Solution

$\tan 30° = \dfrac{h}{OB}$ or $\tan 60° = \dfrac{OB}{h}$

$OB = \dfrac{h}{\tan 30}$ $\quad OB = h \tan 60°$

$OB = \sqrt{3}h$

> *Marking Guideline*
>
> **1** Correct response
>
> **Marking Notes**
> Reasonably well done.

(ii) Show $h = 50\sqrt{2}$.

Solution

In ΔAOB $\quad (AO)^2 + (AB)^2 = (OB)^2$ [Pythagoras]

Now, $AO = h$ [$\angle AOT$ is isosceles]

$AB = 100$ [Given]

$OB = h \tan 60°$

$(h)^2 + (100)^2 = (h \tan 60)^2$

$h^2 + 10\,000 = \left(h\sqrt{3}\right)^2$

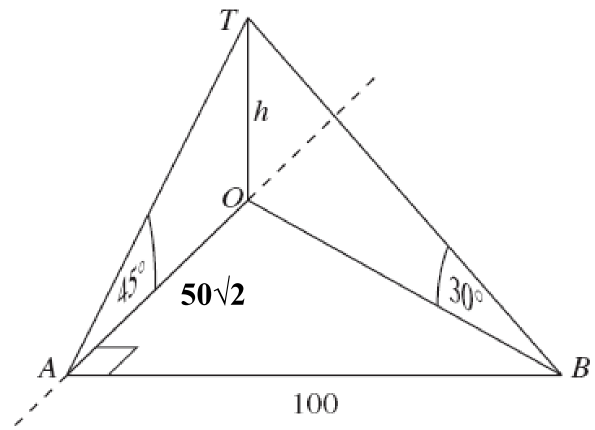
$h^2 = 5\,000$

$h = \pm\sqrt{5000}$

But since $h > 0$ $\quad h = 50\sqrt{2}$ #

> *Marking Guideline*
>
> **2** Correct answer
> **1** Correct Pythagorean statement.
>
> **Marking Notes**
> Many students did not use the correct Pythagorean result.
>
> Don't forget to state $h > 0$
>
> *Show that* questions mean you should be explicit in all steps of your working.

(iii) Calculate the bearing of B from the base of the tower.



Solution

Firstly calculate $\angle AOB$ in ΔAOB.

$AO = h = 50\sqrt{2}$ [From part (ii)]

Let $\theta = \angle AOB$

$$\tan\theta = \frac{100}{50\sqrt{2}}$$

$= 55°$ (to nearest degree)

$\therefore$ Bearing of B from the tower is $180° - 55° = 125°$

Marking Guideline

2 Correct answer

1 For finding angle AOB.

Marking Notes

Poorly answered.

Many students received a CTE (carry through error).

Also accepted compass bearing $S55^{\circ}E$

(b) ABC is an acute angled triangle.

i) Show that $\tan(A+B) = -\tan C$.

Solution

In any triangle $A+B = 180 - C$

$$\begin{aligned} LHS &= \tan(A+B) \\ &= \tan(180 - C) \\ &= -\tan C \end{aligned}$$

Note: $\tan(180^{\circ} - \theta) = -\tan\theta$

Marking Guideline

1 Correct response including communicating a relationship between A, B & C.

Marking Notes

Very poorly answered.

Students could not make the connection with an angles sum of a triangle and the identity.

ii) Hence show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Solution

$$
\begin{aligned}
LHS &= \tan A + \tan B + \tan C \\
&= \tan A + \tan B - (-\tan C) \\
&= \tan A + \tan B - \tan(A+B) \\
&= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= \frac{(1 - \tan A \tan B)(\tan A + \tan B) - \tan A - \tan B}{1 - \tan A \tan B} \\
&= \frac{\tan A + \tan B - \tan^2 A \tan B - \tan A \tan^2 B - \tan A - tamB}{1 - \tan A \tan B} \\
&= \frac{-\tan A \tan B(\tan A + \tan B)}{1 - \tan A \tan B} \\
&= \tan A \tan B \times -\frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= \tan A \times \tan B \times -\tan(A+B) \\
&= \tan A \times \tan B \times \tan C \\
&= RHS
\end{aligned}
$$

Marking Guideline

1 Correct answer

Marking Notes

Many students simply did not attempt this question.

Alternatively

$$
\begin{aligned}
\tan(A+B) &= -\tan C \\
\frac{\tan A + \tan B}{1 - \tan A \tan B} &= -\tan C \\
\tan A + \tan B &= -\tan C(1 - \tan A \tan B) \\
\tan A + \tan B &= -\tan C + \tan A \tan B \tan C \\
\tan A + \tan B + \tan C &= \tan A \tan B \tan C \\
LHS &= RHS
\end{aligned}
$$

(c) Solve i) $\cos 2\alpha = \frac{1}{\sqrt{2}}$, for $0 \le \alpha \le 2\pi$

Solution

Note: $0 \le 2\alpha \le 4\pi$

Finding the related acute angle A

$$\cos A = \frac{1}{\sqrt{2}}$$
$$A = \frac{\pi}{4}$$

$$\cos 2\alpha = \frac{1}{\sqrt{2}}$$
$$2\alpha = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$$
$$\alpha = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$$

Marking Guideline

2 Correct answer
1 For finding the related acute angle OR Giving their answers in degrees

Marking Notes
Common mistakes include
* not adjusting the domain
* answering in degrees

ii) $2\cos^2 x = -3\sin x$, for $0 \le x \le 2\pi$.

Solution

$$2\cos^2 x = -3\sin x$$
$$2\left(1-\sin^2 x\right) = -3\sin x$$
$$2 - 2\sin^2 x = -3\sin x$$
$$2\sin^2 x - 3\sin x - 2 = 0$$
$$\left(2\sin x + 1\right)\left(\sin x - 2\right) = 0$$
$$\sin x = \frac{-1}{2} \quad or \quad \sin x \ne 2$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Marking Guideline

3 Correct answer
2 For finding the related acute angle $\frac{\pi}{6}$ and $\sin x \ne 2$
1 For finding $2\sin^2 x - 3\sin x - 2 = 0$

Marking Notes
You need to state why $\sin x = 2$ is not a solution. Do not use the abbreviation N/S.

Question 6. (12 marks) Use a SEPARATE writing booklet. **Marks**

a) A metal worker is required to cut a cylinder from a solid sphere of metal of radius $R = 4$ cm.

The diagram below shows a cross-section of the sphere and cylinder.

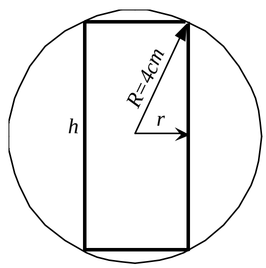
(i) Show that the volume V cm^3, of the cylinder is given by $V = \frac{1}{4}\pi h\left(64 - h^2\right)$.

Solution

$$V = \pi r^2 h$$
$$= \pi\left(16 - \frac{h^2}{4}\right)h$$
$$= \pi h\left(\frac{64}{4} - \frac{h^2}{4}\right)$$
$$= \frac{\pi h}{4}\left(64 - h^2\right)$$

Note: $r^2 = 4^2 - \left(\frac{h}{2}\right)^2$

$$= 16 - \frac{h^2}{4}$$



Marking Guideline

1 Correct response including communicating a relationship between A, B & C.

Marking Notes

Many students were unable to apply Pythagoras' theorem to find the radius

(ii) Find the value of h such that the volume of the cylinder is a maximum.

Solution

Max/min occurs when $V' = 0$

$$V = \frac{1}{4}\pi h\left(64 - h^2\right)$$
$$\frac{dV}{dh} = \left(64 - h^2\right) \times \frac{\pi}{4} + \frac{\pi h}{4} \times (-2h)$$
$$= \frac{\pi}{4}\left(64 - h^2 - 2h^2\right)$$
$$= \frac{\pi}{4}\left(64 - 3h^2\right)$$

$$\frac{\pi}{4}\left(64 - 3h^2\right) = 0$$
$$3h^2 = 64$$
$$h^2 = \frac{64}{3}$$
$$h = \pm\frac{8}{\sqrt{3}} \quad or \quad \pm\frac{8\sqrt{3}}{3}$$

Testing for a maximum

$$\frac{d^2V}{dh^2} = \frac{\pi}{4} \times (-6h) = -\frac{3\pi}{2}h$$

So when $h > 0$ $V'' < 0$ and therefore a maximum when $h = \frac{8\sqrt{3}}{3}$

Marking Guideline

2 Correct answer

1 For finding the related h but not reasoning why it is a maximum.

Marking Notes

Common errors were not correctly applying the product rule when differentiating.

Students should be aware that they must show the value is a maximum. Many students simply assumed the value for when V'=0 would be a maximum and consequently lost a mark.

b) Use the identity $t = \tan\dfrac{\theta}{2}$ to solve $3\sin\theta + 4\cos\theta = 3$.

Solution

$$\begin{aligned}
3\sin\theta + 4\cos\theta &= 3 \\
3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) &= 3 \\
6t + 4 - 4t^2 &= 3 + 3t^2 \\
7t^2 - 6t - 1 &= 0 \\
(7t+1)(t-1) &= 0
\end{aligned}$$

$$\begin{aligned}
t &= 1 \\
\tan\frac{\theta}{2} &= 1 \\
\frac{\theta}{2} &= \frac{\pi}{4} + n\pi \\
\frac{\theta}{2} &= \pi\left(\frac{1}{4} + n\right) \\
\theta &= 2\pi\left(\frac{1}{4} + n\right)
\end{aligned}$$

where n is an integer

$$\begin{aligned}
t &= -\frac{1}{7} \\
\tan\frac{\theta}{2} &= -\frac{1}{7} \\
\frac{\theta}{2} &= \tan^{-1}\left(-\frac{1}{7}\right) + n\pi \\
\frac{\theta}{2} &= n\pi - \tan^{-1}\left(\frac{1}{7}\right) \\
\theta &= 2\left(n\pi - \tan^{-1}\left(\frac{1}{7}\right)\right) \\
&\approx 2n\pi - 0\cdot 2838
\end{aligned}$$

where n is an integer

Check to see whether $\theta = \pi$ is an answer.

$3\sin\pi + 4\cos\pi \neq 3$
$\theta \neq \pi$

Marking Guideline

3	Correct answer
2	For finding one general solution.
1	For finding $(7t+1)(t-1) = 0$

Marking Notes
Poorly answered.

Errors include:
Not substituting the *t*-results into the equation correctly.

Answering in degrees (although no penalty was applied).

Not testing whether $\theta = \pi$ is a solution. (again no penalty was applied)

Not recognizing there is more than one answer – answers should have been given as general solutions.

c) Express $3\cos x + 4\sin x$ in the form $R\cos(x-\alpha)$ and hence solve

$$3\cos x + 4\sin x = 2, \quad \text{for } 0 \le x \le 2\pi .$$

Solution

$$3\cos x + 4\sin x = R\cos(x-\alpha)$$
$$= R\cos\alpha\cos x + R\sin\alpha\sin x$$

$$R\cos\alpha = 3$$
$$R\sin\alpha = 4$$

$$R^2\cos^2\alpha = 9$$
$$R^2\sin^2\alpha = 16$$
$$R^2 = 25$$
$$R = 5 \text{ since } R > 0$$

$$\cos\alpha = \frac{3}{5}$$
$$\sin\alpha = \frac{4}{5}$$
$$\tan\alpha = \frac{4}{3}$$
$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$
$$\alpha \approx 0\cdot 92729$$

Marking Guideline

3 Correct answer
2 For solution with one error.
1 For finding **R or α**

Marking Notes

Poorly answered

Many students tried to take short cuts in evaluating *R* and the auxiliary angle. Consequently many errors were made. Commonly, students incorrectly stated $\tan\alpha = \frac{3}{4}$. Better to show working and get the correct answer.

Many students lost a mark because they answered in degrees although the question's domain was stated as *radians*. Therefore the answer must be in *radians*.

Many students could not provide answers in the correct domain and commonly just provided one answer.

Now,

$$3\cos x + 4\sin x = 2,$$
$$5\cos\left(x - \tan^{-1}\left(\frac{4}{3}\right)\right) = 2$$
$$\cos\left(x - \tan^{-1}\left(\frac{4}{3}\right)\right) = \frac{2}{5}$$

$$0 \le x \le 2\pi$$
$$-\tan^{-1}\left(\frac{4}{3}\right) \le x - \tan^{-1}\left(\frac{4}{3}\right) \le 2\pi - \tan^{-1}\left(\frac{4}{3}\right)$$
$$-0\cdot 9273 \le x \le 5\cdot 3559$$

Related acute angle $A = \cos^{-1}\left(\frac{2}{5}\right) \approx 1\cdot 1593$

$$x - \tan^{-1}\left(\frac{4}{3}\right) = \cos^{-1}\left(\frac{2}{5}\right)$$
$$x = \cos^{-1}\left(\frac{2}{5}\right) + \tan^{-1}\left(\frac{4}{3}\right)$$
$$\approx 2\cdot 0866$$

OR

$$x - \tan^{-1}\left(\frac{4}{3}\right) = 2\pi - \cos^{-1}\left(\frac{2}{5}\right)$$
$$x = 2\pi - \cos^{-1}\left(\frac{2}{5}\right) + \tan^{-1}\left(\frac{4}{3}\right)$$
$$\approx 6\cdot 0512$$

d) Find the general solution to $\sqrt{3}\sec^2\theta - 2\tan\theta = 2\sqrt{3}$.

Solution

$$\sqrt{3}\sec^2\theta - 2\tan\theta = 2\sqrt{3}$$
$$\sqrt{3}\left(\tan^2\theta + 1\right) - 2\tan\theta = 2\sqrt{3}$$
$$\sqrt{3}\tan^2\theta + \sqrt{3} - 2\tan\theta - 2\sqrt{3} = 0$$
$$\sqrt{3}\tan^2\theta - 2\tan\theta - \sqrt{3} = 0$$
$$\left(\tan\theta - \sqrt{3}\right)\left(\sqrt{3}\tan\theta + 1\right) = 0$$

$$\tan\theta = \sqrt{3} \qquad \tan\theta = -\frac{1}{\sqrt{3}}$$

$$\theta = n\pi + \frac{\pi}{3} \qquad \theta = n\pi + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$
$$= n\pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= n\pi - \frac{\pi}{6}$$

where n is an integer

Marking Guideline

3 Correct answer
2 For solution with one error.
1 For simplifying the expression in terms of one ratio.

Marking Notes

Poorly answered.

Many students could not simplify the equation and converted $\sec^2\theta$ to $\frac{1}{\cos^2\theta}$ rather than taking the clue that $\sec^2\theta$ and $\tan\theta$ alluded to the Pythagorean identity.

Many students made errors in factorizing or solving using the quadratic formula.

Many students were not able to state the general solutions.

Again many students wanted to answer in degrees rather than radians. Some students incorrectly mixed units e.g. $\theta = n\pi + 60^\circ$