Name: ....

Teacher: ....

Fort Street High School

2019

Preliminary HSC Assessment Task 3

Mathematics Extension 1

Syllabus Outcomes	Assessment Area
ME11-1	Uses algebraic and graphical concepts in the modelling and solving of problems involving functions
ME11-2	Manipulates algebraic expressions and graphical functions to solve problems
ME11-3	Applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems
ME11-4	Applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change
ME11-7	Communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Time allowed: 90 minutes (+ 5 minutes reading time)

General Instructions:

- Marks may be deducted for careless or poorly arranged work
- NESA-approved calculators may be used
- A Reference Sheet is provided with this paper
- Use a separate booklet for each question

Question	Mark
1	/13
2	/13
3	/12
4	/11
5	/11
Total	/60
	%

Question 1 (13 marks) Use a SEPARATE writing booklet.

---

a) Find the exact value of $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$. **1**

b) i) Show that $\dfrac{1+\cos 2A}{\sin 2A} = \cot A$ **2**

ii) Hence find the exact value of $\cot 15°$ **1**

c) The polynomial $P(x) = x^3 + px^2 + qx + r$ has roots $0, 2, -2$.

i) Find the values of p, q and r. **2**

ii) Sketch the graph of $y = P(x)$, showing any intercepts with the coordinate axes. **2**

iii) Hence, or otherwise, solve $\dfrac{x^2-4}{x} \geq 0$ **2**

d) i) Find the Cartesian equation of the curve defined by the parametric equations: **2**

$$x = 2\sin\theta$$
$$y = 3 + \cos 2\theta$$

ii) State the type of graph the Cartesian equation represents. **1**

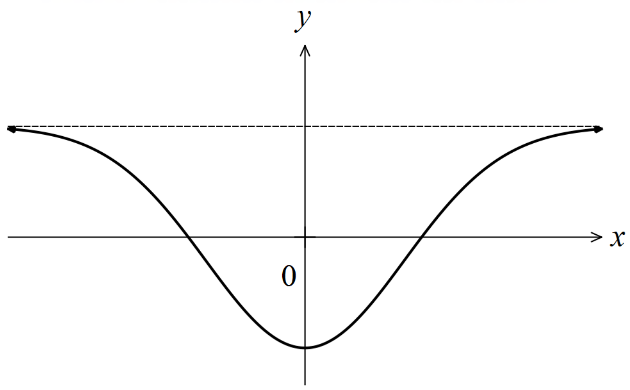
Question 2 (13 marks) Use a SEPARATE writing booklet.

---

a) If $x=3t^2+7$ and $y=4t^3-5t^2-8$ show that $\dfrac{dy}{dx}=\dfrac{6t-5}{3}$ **2**

b) A polynomial has a remainder of 1 when divided by $(x+1)$, and a remainder of 5 when divided by $(x-3)$. Find the remainder when the same polynomial is divided by $(x+1)(x-3)$. **2**

c) The diagram below shows the graph of $y=f(x)$ where $f(x)=1-2e^{-x^2}$.



i) Find the equation of the horizontal asymptote, showing relevant working out. **2**

ii) Copy the graph of $y=f(x)$ into your writing book. On the same set of axes, sketch the graph of $y=\sqrt{f(x)}$. Showing clearly the asymptote and any points of intersection between $y=f(x)$ and $y=\sqrt{f(x)}$. **2**

d) i) Show that $\sin(A-2B)\cos(A+2B)=\dfrac{1}{2}(\sin 2A-\sin 4B)$. **2**

ii) Hence, calculate the exact value of $\sin\left(22\frac{1}{2}^\circ\right)\cos\left(7\frac{1}{2}^\circ\right)$ **3**

Question 3 (12 marks) Use a SEPARATE writing booklet.

a) The zeros of $x^3 - 5x + 3 = 0$ are α, β and γ. Without evaluating α, β and γ, find a cubic equation with integer coefficients whose zeros are 2α, 2β and 2γ. **2**

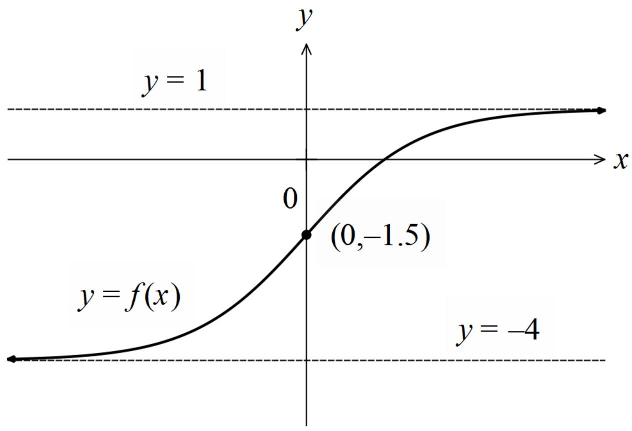
b) The water level in a reservoir rises and falls during a four-hour period of heavy rainfall. The height, h cm, of water above its normal level, t hours after it starts to rain, can be modelled by the equation

$$h = 3t^3 - 30t^2 + 51t \qquad 0 \le t \le 4$$

i) Find the rate of change of the height of water, in cm per hour, 3 hours after it starts to rain. **2**

ii) Find the values of t for which the height of the water is decreasing. **3**

c) The graph of the function $f(x) = \dfrac{e^x - 4}{e^x + 1}$ is shown below.



i) Explain why $f(x)$ has an inverse function for all values of x. **1**

ii) On the same set of axes, sketch the graphs of $y = x$, $y = f(x)$ and $y = f^{-1}(x)$. **2**

iii) Show that the equation of the inverse function is $f^{-1}(x) = \ln\left(\dfrac{x+4}{1-x}\right)$ **2**

Question 4 (11 marks) Use a SEPARATE writing booklet.

---

a) An island was found to be contaminated by fallout from a nuclear test explosion. One of the most dangerous components of nuclear fallout is strontium-90. The rate at which the amount, A, of strontium-90 decays t years after an explosion is given by:

$$\frac{dA}{dt} = -kA$$

i) Show that $A = A_0 e^{-kt}$, where A_0 is the amount of strontium-90 present immediately after an explosion, satisfies the given equation. **1**

ii) It was found that 28 years after the explosion there was half the original amount of strontium-90 present on the island. Find the value of k correct to 4 decimal places. **2**

iii) After how many years will the original amount of strontium-90 fall below 30% on the island? **2**

b) Using $t = \tan\frac{\theta}{2}$, solve the equation $\sin\theta + \cos\theta = 1$ for $0 \le \theta \le 2\pi$. **3**

c) A function $f(x)$ is defined such that $f(x) = \begin{cases} \sin^{-1}(x) & \text{for } 0 < x \le 1 \\ \cos^{-1}(x) & \text{for } -1 \le x \le 0 \end{cases}$

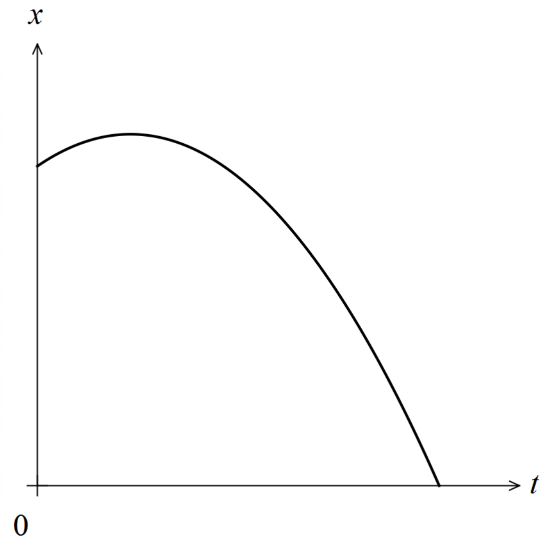
i) By considering appropriate values, determine if $f(x)$ is continuous in the domain $-1 \le x \le 1$. **1**

ii) Sketch the graph of $y = f(x)$ showing clearly any intercepts and co-ordinates of the endpoints. **2**

Question 5 (11 marks) Use a SEPARATE writing booklet.

---

a) After work, Trevor decided to go for a drive to Newport. He started at Lane Cove, drove to Newport then turned around and drove home. The graph below represents the distance, x km, he is from home after t hours and is given by $x = -3t^2 + 6t + 30$.



i) How long does it take Trevor to reach Newport? **2**

ii) What is the total distance Trevor travelled from the time he left Lane Cove until he arrived home? **2**

b) A pan of warm water of temperature $46°C$, was put into a refrigerator. Ten minutes later the water temperature of the pan was $39°C$, and ten minutes after that the water temperature was $33°C$. The temperature, T, of the water is given by:

$$T = B + Ae^{-kt}$$

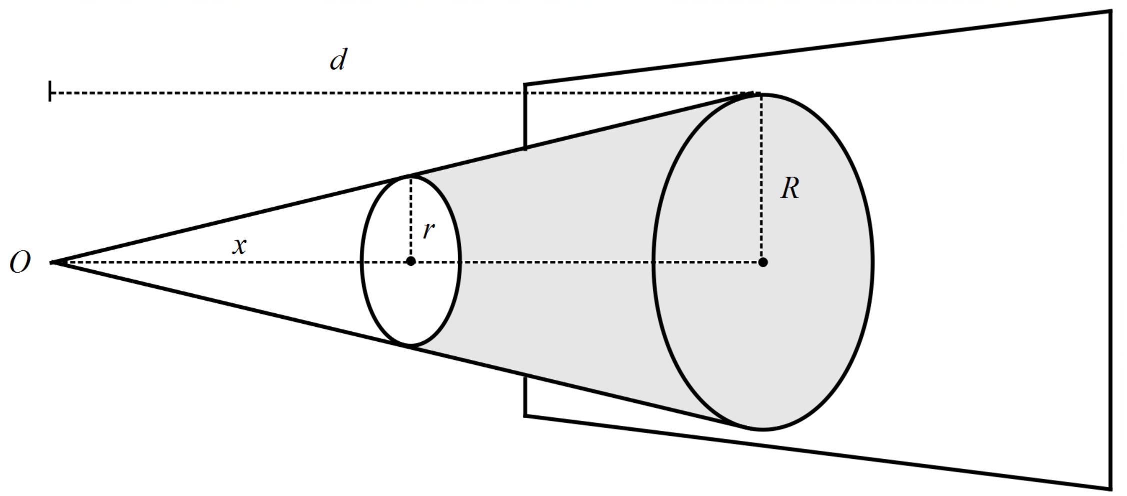
where B is the temperature of the refrigerator and A is a constant.
What was the temperature of the refrigerator? **3**

Question 5 Continued Next Page

(Question 5 continued)

c) A light source is placed d cm away from a screen. A circular coin of radius r cm is placed x cm from the light source, such that its horizontal axis of symmetry passes through O.

The coin is moving horizontally towards the source of light at a speed of 4 cm per second and casts a circular shadow of radius R cm on the screen as shown.



i) Show that the area of the shadow is $A = \dfrac{\pi d^2 r^2}{x^2}$ **2**

ii) Find the rate, in terms of d and r, at which the area of the shadow increases with respect to time when $x = \dfrac{d}{4}$. **2**

End of Examination

Name: ....

Teacher: ....

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SOLUTIONS

Question	Mark
1	/13
2	/13
3	/12
4	/11
5	/11
Total	/60
	%

Question 1 (13 marks) Use a SEPARATE writing booklet.

---

a) Find the exact value of $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$. **1**

Solution

$$\sin^{-1}\left(\sin\frac{7\pi}{6}\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$$

since range of $\sin^{-1}$ $-\dfrac{\pi}{2}\le y\le\dfrac{\pi}{2}$

Marking guideline:

1 Correct response

Marker's comments:
Mark deducted if students answered in degrees. Many students didn't consider the range of $\sin^{-1}$ when answering the question

b) i) Show that $\dfrac{1+\cos 2A}{\sin 2A}=\cot A$ **2**

Solution

$$\begin{aligned}\text{LHS} &= \frac{1+\cos 2A}{\sin 2A}\\ &= \frac{1+2\cos^2 A-1}{2\sin A\cos A}\\ &= \frac{2\cos^2 A}{2\sin A\cos A}\\ &= \frac{\cos A}{\sin A}\\ &= \cot A\\ &= \text{RHS} \ \text{ as req'd}\end{aligned}$$

Marking guideline:

2 Correct proof

1 Correctly expands $\cos 2A$ or $\sin 2A$ or equivalent merit

Marker's comments:
Generally answered well.

ii) Hence find the exact value of $\cot 15°$ **1**

Solution

$$\begin{aligned}\cot 15 &= \frac{1+\cos 2(15)}{\sin 2(15)}\\ &= \frac{1+\cos 30^o}{\sin 30^o}\\ &= \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}\\ &= \frac{2+\sqrt{3}}{2}\times 2\\ &= 2+\sqrt{3}\end{aligned}$$

Marking guideline:

1 Correct response

Marker's comments:
Generally answered well

c) The polynomial $P(x) = x^3 + px^2 + qx + r$ has roots $0, 2, -2$.

i) Find the values of p, q and r. **2**

Solution

$$P(0) = 0$$

$$0^3 + p(0)^2 + q(0) + r = 0$$

$$\therefore r = 0$$

$P(2) = 0$

$$(2)^3 + p(2)^2 + q(2) + 0 = 0$$

$$8 + 4p + 2q = 0 \quad \text{........(1)}$$

$P(-2) = 0$

$$(-2)^3 + p(-2)^2 + q(-2) + 0 = 0$$

$$-8 + 4p - 2q = 0 \quad \text{........(2)}$$

eqn $(1) + (2)$

$$0 + 8p + 0 = 0$$

$$\therefore p = 0$$

Sub p into (2):

$$-8 + 4(0) - 2q = 0$$

$$\therefore q = -4$$

Marking guideline:

2 Correct response

1 Calculates 2 out of 3 correct values for p, q *or* r

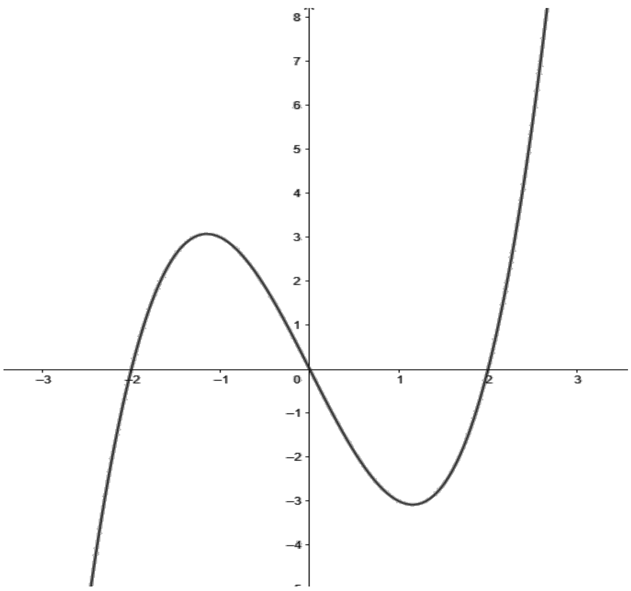
Marker's comments:

Generally answered well

ii) Sketch the graph of $y = P(x)$, showing any intercepts with the coordinate axes. **2**

Solution

$$\begin{aligned} P(x) &= x^3 - 4x \\ &= x\left(x^2 - 4\right) \\ &= x\left(x-2\right)\left(x+2\right) \end{aligned}$$



Marking guideline:

2 Correct shape and intercepts

1 Correct shape or intercepts

Marker's comments:
Generally answered well

iii) Hence, or otherwise, solve $\dfrac{x^2 - 4}{x} \geq 0$ **2**

Solution

$\dfrac{x^2 - 4}{x} \geq 0 \qquad x \neq 0$

$x^2 \times \dfrac{\left(x^2 - 4\right)}{x} \geq 0$

$x\left(x-2\right)\left(x+2\right) \geq 0$

From part (ii) $x(x-2)(x+2) \geq 0$ when $-2 \leq x < 0$ and $x \geq 2$

Marking guideline:

2 Correct response

1 Successfully deals with the denominator or equivalent merit

Marker's comments:
Most students didn't recognise $x \neq 0$ and included it in the values of x.

d) i) Find the Cartesian equation of the curve defined by the parametric equations: **2**

$$x = 2\sin\theta$$
$$y = 3 + \cos 2\theta$$

Solution

$x = 2\sin\theta$ $\qquad$ $y = 3 + \cos 2\theta$

$\qquad\qquad\qquad\quad$ $y = 3 + 1 - 2\sin^2\theta$

$x^2 = 4\sin^2\theta \ldots(1)$ $\qquad$ $y = 4 - 2\sin^2\theta \ldots(2)$

From (1) $\sin^2\theta = \dfrac{x^2}{4}$ sub into (2)

$$y = 4 - 2\left(\frac{x^2}{4}\right)$$

$$y = 4 - \frac{x^2}{2}$$

$$\therefore 2y = 8 - x^2$$

Marking guideline:

2 Correct shape and intercepts

1 Finds correct expression for $\sin^2\theta$ or equivalent merit

Marker's comments:
Not answered very well. It was apparent that some students did not understand double angles and the corresponding notation – these students took $\cos 2\theta$ to mean $2\cos\theta$ and treated it algebraically.

ii) State the type of graph the Cartesian equation represents. **1**

Solution

Parabola

Marking guideline:

1 Correct response

Marker's comments:
Many students described the graph as a circle.

Question 2 (13 marks) Use a SEPARATE writing booklet.

---

a) If $x = 3t^2 + 7$ and $y = 4t^3 - 5t^2 - 8$ show that $\dfrac{dy}{dx} = \dfrac{6t-5}{3}$ **2**

Solution

$x = 3t^2 + 7 \qquad\qquad y = 4t^3 - 5t^2 - 8$

$\dfrac{dx}{dt} = 6t \qquad \dfrac{dy}{dt} = 12t^2 - 10t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \left(12t^2 - 10t\right) \times \frac{1}{6t}$$
$$= \frac{6t-5}{3} \quad \text{as req'd}$$

Marking guideline:

2 Correct response

1 Correct derivatives for x and t

Marker's comments:

Overall, well done. As this is a 'SHOW' question, marks were deducted if all of the working opposite was not given.

b) A polynomial has a remainder of 1 when divided by $(x+1)$, and a remainder of 5 when divided by $(x-3)$. Find the remainder when the same polynomial is divided by $(x+1)(x-3)$. **2**

Solution

Let $P(x) = (x+1)(x-3)Q(x) + R(x) \qquad$ where $R(x) = (Ax + B)$

$P(-1) = 1 \qquad\qquad P(3) = 5$

$-A + B = 1 \text{ ....(1)} \qquad 3A + B = 5 \text{ .....(2)}$

$(2) - (1)$

$4A = 4$

$A = 1$ and $B = 2$

$\therefore R(x) = x + 2$

Marking guideline:

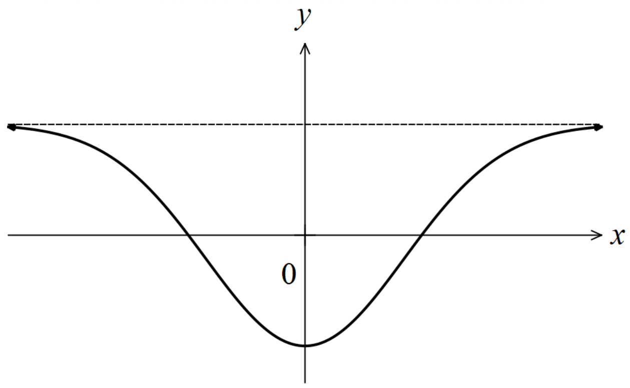
2 Correct response

1 Recognises remainder as a linear expression

Marker's comments:

Extremely poorly done!! Most students could not answer this type of question. Students need to practise using the 'division transformation' to help solve various problems such as this one. Also, please become aware of important features such as if D(x) is quadratic, then R(x) is of the form ax+b.

c) The diagram below shows the graph of $y = f(x)$ where $f(x) = 1 - 2e^{-x^2}$.



i) Find the equation of the horizontal asymptote, showing relevant working out. **2**

Solution

Let $y = f(x)$, hence

$$y = 1 - \frac{2}{e^{x^2}}$$

As $x \to +\infty$, $-\dfrac{2}{e^{x^2}} \to 0^-$

$$1 - \frac{2}{e^{x^2}} \to 1^-$$

As $x \to -\infty$, $-\dfrac{2}{e^{x^2}} \to 0^-$

$$1 - \frac{2}{e^{x^2}} \to 1^-$$

$\therefore$ Horizontal asymptote is $y = 1$

Marking guideline:

2 Correct response

1 Considers limits as x approaches $\pm$ infinity

Marker's comments:

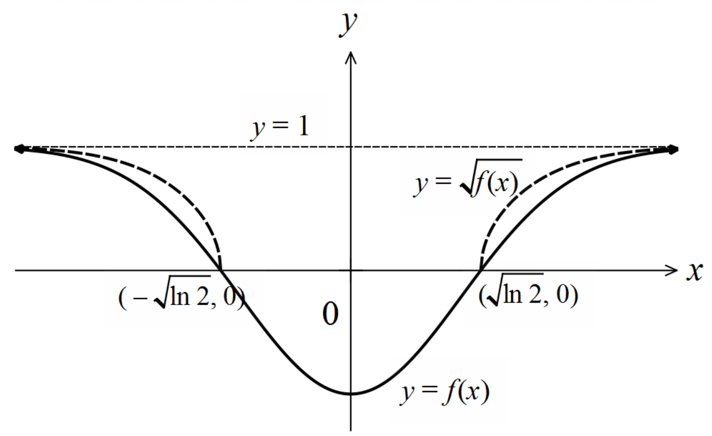
Poorly done. Students needed to give the relevant working i.e. explain as

$x \to \infty$, $\dfrac{2}{e^{x^2}} \to 0$ as $e^{x^2} \to \infty \therefore$ $f(x) \to 1$.

The equation of the asymptote is not '1'. It is y=1.

ii) Copy the graph of $y = f(x)$ into your writing book. On the same set of axes, sketch the graph of $y = \sqrt{f(x)}$. Showing clearly the asymptote and any points of intersection between $y = f(x)$ and $y = \sqrt{f(x)}$. **2**

Solution



Marking guideline:

2 Correct shape and position of $\sqrt{(f(x)}$ relative to $f(x)$

1 Correct shape or position

Marker's comments:

A lot of students had the required graph in the wrong position i.e. below f(x) which resulted in 1 mark lost.

d) i) Show that $\sin(A-2B)\cos(A+2B) = \frac{1}{2}(\sin 2A - \sin 4B)$. **2**

Solution

$$\begin{aligned}\text{LHS} &= \sin(A-2B)\cos(A+2B)\\ &= \frac{1}{2}\left[\sin(A-2B+A+2B)+\sin(A-2B-A-2B)\right]\\ &= \frac{1}{2}(\sin 2A+\sin(-4B))\\ &= \frac{1}{2}(\sin 2A-\sin 4B)\\ &= \text{RHS} \quad \text{as req'd}\end{aligned}$$

Marking guideline:

2 Correct proof

1 Attempts to simplify the expression

Marker's comments:

Most students expanded the LHS and made errors. Method opposite is much easier.
As this is a 'show' question, the line $\frac{1}{2}(\sin 2A+\sin(-4B))$ had to be given .

1 mark was deducted if not given.

ii) Hence, calculate the exact value of $\sin\left(22\frac{1}{2}^\circ\right)\cos\left(7\frac{1}{2}^\circ\right)$ **3**

Solution

$$\sin(A-2B)\cos(A+2B)=\sin\left(22\frac{1}{2}^o\right)\cos\left(7\frac{1}{2}^o\right)$$

$$A-2B=22\frac{1}{2} \quad ....(1)$$

$$A+2B=7\frac{1}{2} \quad ....(2)$$

$(1)+(2)$

$$2A=30$$

$$A=15, \quad B=-\frac{15}{4}$$

Hence,

$$\sin\left(22\frac{1}{2}^o\right)\cos\left(7\frac{1}{2}^o\right)=\frac{1}{2}\left(\sin 2\left(15^o\right)-\sin 4\left(-\frac{15}{4}\right)\right)$$

$$=\frac{1}{2}\left(\sin 30^o-\sin\left(-15^o\right)\right)$$

$$=\frac{1}{2}\left(\sin 30^o+\sin 15^o\right)$$

$$=\frac{1}{2}\left(\sin 30^o+\sin\left(45^o-30^o\right)\right)$$

$$=\frac{1}{2}\left(\sin 30^o+\sin 45^o\cos 30^o-\sin 30^o\cos 45^o\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{1}{\sqrt{2}}\right)$$

$$=\frac{\sqrt{2}+\sqrt{3}-1}{4\sqrt{2}}$$

Marking guideline:

3 Correct response

2 Observes $\sin 15^o=\sin(45^o-30^0)$ or equivalent merit

1 Correct values for A and B

Marker's comments:

Poorly done. This is a 'HENCE' question which means that the identity proven in part (i) had to be used. You cannot say A= $22\frac{1}{2}^\circ$, B= $7\frac{1}{2}^\circ$ because then you are not using result from (i).

Question 3 (12 marks) Use a SEPARATE writing booklet.

---

a) The zeros of $x^3 - 5x + 3 = 0$ are α, β and γ. Without evaluating α, β and γ, find a cubic equation with integer coefficients whose zeros are 2α, 2β and 2γ. **2**

Solution

$$\alpha+\beta+\gamma = \frac{-b}{a}$$
$$= 0$$
$$\therefore 2\alpha+2\beta+2\gamma = 2(\alpha+\beta+\gamma)$$
$$= 0$$
$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$$
$$= -5$$
$$\therefore 2\alpha 2\beta + 2\alpha 2\gamma + 2\beta 2\gamma = 4(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$= -20$$
$$\alpha\beta\gamma = -\frac{d}{a}$$
$$= -3$$
$$2\alpha 2\beta 2\gamma = 8\alpha\beta\gamma$$
$$= -24$$

$\therefore$ New cubic equation with integer coefficients is $x^3 - 20x + 24 = 0$

Marking guideline:

2 Correct response

1 Attempts to find relationships for new polynomial or equivalent merit

Marker's comments:

Poorly done. The most common error was simply doubling the coefficients to attain $x^3 - 10x + 6 = 0$

b) The water level in a reservoir rises and falls during a four-hour period of heavy rainfall. The height, h cm, of water above its normal level, t hours after it starts to rain, can be modelled by the equation

$$h = 3t^3 - 30t^2 + 51t \qquad 0 \le t \le 4$$

i) Find the rate of change of the height of water, in cm per hour, 3 hours after it starts to rain. **2**

Solution

$$\frac{dh}{dt} = 9t^2 - 60t + 51$$

when $t = 3$

$$\frac{dh}{dt} = 9(3)^2 - 60(3) + 51$$
$$= 81 - 180 + 51$$
$$= -48$$

$\therefore$ The height of water decreases at the rate of 48cm/hr

Marking guideline:

2 Correct response

1 Correct derivative

Marker's comments:

Generally well done. Some students were lazy and did not answer with units.

ii) Find the values of t for which the height of the water is decreasing. **3**

Solution

$$\frac{dh}{dt} < 0, \text{ that is}$$

$$9t^2 - 60t + 51 < 0$$

$$3t^2 - 20t + 17 < 0$$

$$(3t - 17)(t - 1) < 0$$

$$1 < t < \frac{17}{3}$$

But $0 \le t \le 4$, so $1 < t \le 4$

Marking guideline:

3 Correct response, up to $1 < t < \frac{17}{3}$

2 Correct factorisation or equivalent merit

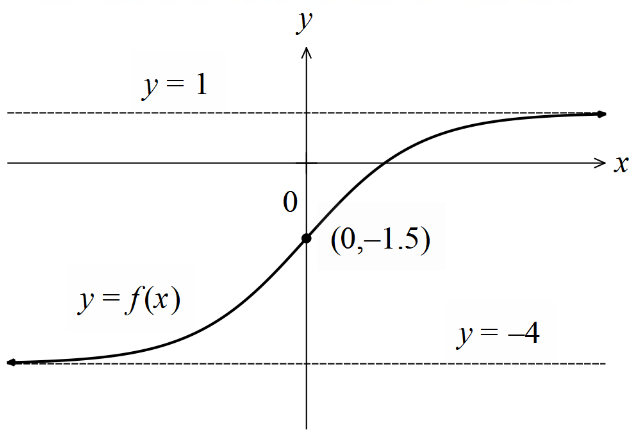
1 Recognition that $\frac{dh}{dt} < 0$ for decreasing *h*

Marker's comments:

Very poorly done. The most correct answer is $1 < t \le 4$ but students who were able to correctly work towards $1 < t < \frac{17}{3}$ were awarded 3 marks. Some students tested points for $\frac{dh}{dt}$ rather than solving the inequality.

Thorough revision of rates of change is required.

c) The graph of the function $f(x) = \dfrac{e^x - 4}{e^x + 1}$ is shown below.



i) Explain why $f(x)$ has an inverse function for all values of x. **1**

Solution

$f(x)$ is a one-to-one function

OR $f(x)$ is monotonic increasing

Marking guideline:

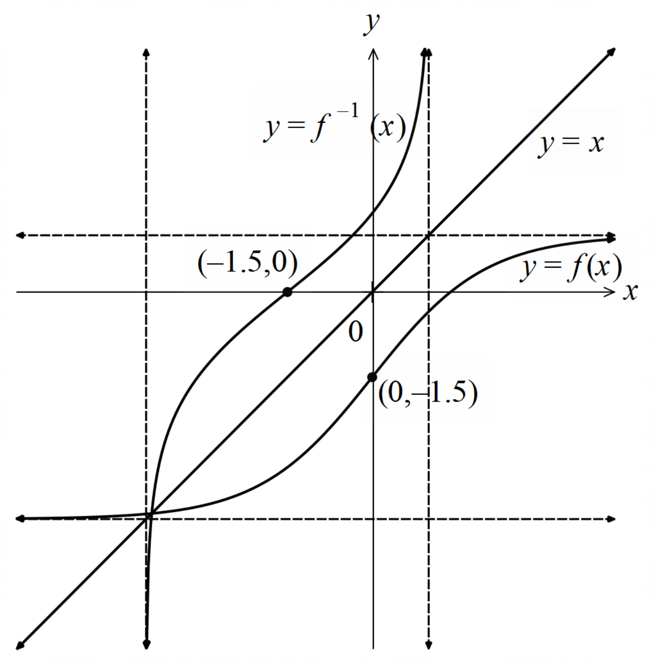
1 Correct explanation or equivalent merit

Marker's comments:

Poorly done. Many students used 'vertical line test' and/or 'horizontal line test' which are not acceptable reasons. There is some confusion about domains, reciprocal functions and inverse functions.

ii) On the same set of axes, sketch the graphs of $y = x$, $y = f(x)$ and $y = f^{-1}(x)$. **2**

Solution



Marking guideline:

2 Correct shape and asymptotes for $f^{-1}(x)$, curves meet at $y = x$

1 Correct shape and asymptotes for $f^{-1}(x)$

Marker's comments:

Poorly done. Many students' graphs did not meet at $y = x$. Some students were very slack, not labelling asymptotes, axes, intercepts.

iii) Show that the equation of the inverse function is $f^{-1}(x) = \ln\left(\dfrac{x+4}{1-x}\right)$ **2**

Solution

$$f(x): \quad y = \frac{e^x - 4}{e^x + 1}$$

$$f^{-1}(x): \quad x = \frac{e^y - 4}{e^y + 1}$$

$$e^y x + x = e^y - 4$$

$$e^y(x-1) = -x - 4$$

$$e^y = \frac{-(x+4)}{x-1}$$

$$= \frac{x+4}{1-x}$$

$$\therefore y = \ln\left(\frac{x+4}{1-x}\right)$$

Marking guideline:

2 Correct response showing clear steps

1 Correct progress

Marker's comments:

Well done by most students. Some students tried to 'log both sides' and fudge their way through the process. Students must remember to clearly communicate all steps, particularly in a *show* or *prove* question.

Question 4 (11 marks) Use a SEPARATE writing booklet.

---

a) An island was found to be contaminated by fallout from a nuclear test explosion. One of the most dangerous components of nuclear fallout is strontium-90. The rate at which the amount, A, of strontium-90 decays t years after an explosion is given by:

$$\frac{dA}{dt} = -kA$$

i) Show that $A = A_0 e^{-kt}$, where A_0 is the amount of strontium-90 present immediately after an explosion, satisfies the given equation. **1**

Solution

$$A = A_0 e^{-kt}$$

$$\frac{dA}{dt} = -k \times A_0 e^{-kt}, \quad \text{but } A_0 e^{-kt} = A$$

$$\frac{dA}{dt} = -kA$$

$\therefore A = A_0 e^{-kt}$ the given equation

Marking guideline:

1 Correct response

Marker's comments:

Mostly well done

ii) It was found that 28 years after the explosion there was half the original amount of strontium-90 present on the island. Find the value of k correct to 4 decimal places. **2**

Solution

$$\frac{1}{2}A_0 = A_0 e^{-28k}$$

$$\frac{1}{2} = e^{-28k}$$

$$-28k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-28}$$

$$\therefore k = 0.0248$$

Marking guideline:

2 Correct response

1 Attempts to use logs to find k

Marker's comments:

Some students did not set up the correct equation $\frac{1}{2}A_0 = A_0 e^{-28k}$ and many did not correctly solve for k

iii) After how many years will the original amount of strontium-90 fall below 30% on the island? **2**

Solution

$$0.3A_0 = A_0e^{-28k}$$
$$0.3 = e^{-kt}$$
$$-kt = \ln(0.3))$$
$$t = \frac{\ln(0.3)}{-k} = \frac{\ln(0.3)}{-0.0248}$$
$$= 48.547....$$

$\therefore$ Original amount falls below 30% after 49 years

Marking guideline:

2 Correct response

1 Finds t as a fraction/ decimal or rounds to 48 years

Marker's comments:

Mostly well done

b) Using $t = \tan\frac{\theta}{2}$, solve the equation $\sin\theta + \cos\theta = 1$ for $0 \le \theta \le 2\pi$. **3**

Solution

$$\sin\theta + \cos\theta = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$$

hence,

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$$
$$2t + 1 - t^2 = 1 + t^2$$
$$2t^2 - 2t = 0$$
$$2t(t-1) = 0$$
$$t = 0 \quad or \quad t = 1$$

$$\tan\frac{\theta}{2} = 0$$
$$\frac{\theta}{2} = 0,\ \pi$$
$$\theta = 0,\ 2\pi$$

$$\tan\frac{\theta}{2} = 1 \Rightarrow \frac{\theta}{2} = \tan^{-1}(1)$$
$$\frac{\theta}{2} = \frac{\pi}{4}, \frac{5\pi}{4}$$
$$\theta = \frac{\pi}{2}, \frac{5\pi}{2} \quad \text{but } \frac{5\pi}{2} \text{ is outside the domain}$$

$$\therefore \theta = 0, \frac{\pi}{2}, 2\pi$$

Marking guideline:

3 Correct response

2 Attempts to solve for θ

1 Finds $t = 0$ and $t = 1$

Marker's comments:

Students need to remember to express their answer in radian form to match the given domain

c) A function $f(x)$ is defined such that $f(x)=\begin{cases}\sin^{-1}(x) & \text{for } 0<x\le 1\\ \\ \cos^{-1}(x) & \text{for } -1\le x\le 0\end{cases}$

i) By considering appropriate values, determine if $f(x)$ is continuous in the domain $-1\le x\le 1$. **1**

Solution

$\sin^{-1}(0)=0$

$\cos^{-1}(0)=1$

$\therefore$ since $\sin^{-1}(0)\neq\cos^{-1}(0)$

then $f(x)$ is not continuous

Marking guideline:

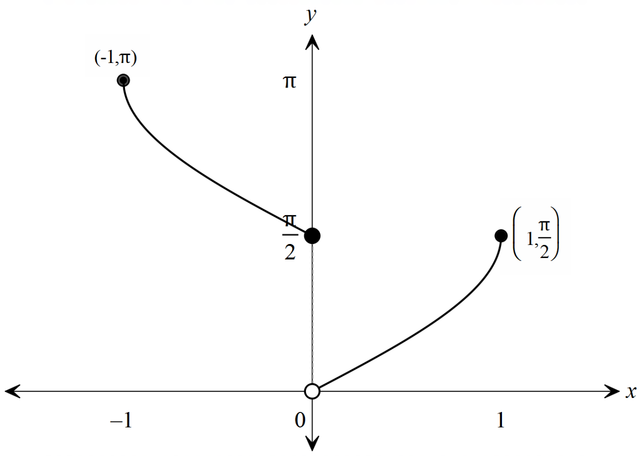
1 Correct conclusion

Marker's comments:

Mostly well done

ii) Sketch the graph of $y=f(x)$ showing clearly any intercepts and co-ordinates of the endpoints. **2**

Solution



Marking guideline:

2 Correct shape, intercepts and co-ordinates of endpoints

1 Correct intercepts or co-ordinates of endpoints

Marker's comments:

Students needed to show the coloured and uncoloured points at $x=0$

Question 5 (11 marks) Use a SEPARATE writing booklet.

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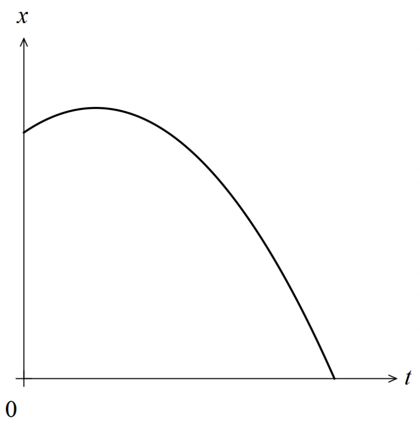
a) After work, Trevor decided to go for a drive to Newport. He started at Lane Cove, drove to Newport then turned around and drove home. The graph below represents the distance, x km, he is from home after t hours and is given by $x = -3t^2 + 6t + 30$.



i) How long does it take Trevor to reach Newport? **2**

Solution

$x = -3t^2 + 6t + 30$

$\frac{dx}{dt} = -6t + 6$

$= 0$ when reaches Newport

Hence,

$-6(t-1) = 0$

$t = 1$

$\therefore$ Trevor reaches Newport after 1 hour

Marking guideline:

2 Correct response

1 Sets $\frac{dx}{dt} = 0$ to indicate arrival at Newport or equivalent merit

Marker's comments:

- Well done

ii) What is the total distance Trevor travelled from the time he left Lane Cove until he arrived home? **2**

Solution

At $t = 0$, Trevor is 30km from home

At $t = 1$ when Trevor is at Newport

$x = -3(1)^2 + 6(1) + 30$

$= 33$ km, i.e. Trevor is 33km from home

Distance travelled = 36km

$\therefore$ Trevor travelled a total distance of 36km

Marking guideline:

2 Correct response

1 Correct calculation of distance from Lane Cove and Newport or from Newport to home

Marker's comments:

- Many students lost mark for not considering the distance from Lane Cove to home.

b) A pan of warm water of temperature $46°C$, was put into a refrigerator. Ten minutes later the water temperature of the pan was $39°C$, and ten minutes after that the water temperature was $33°C$. The temperature, T, of the water is given by:

$$T = B + Ae^{-kt}$$

where B is the temperature of the refrigerator and A is a constant.

What was the temperature of the refrigerator? **3**

Solution

$T = B + Ae^{-kt}$

At $t = 0,\ T = 46$

$46 = B + Ae^{0}$

$B + A = 46 \quad .....(1)$

At $t = 10,\ T = 39$

$B + Ae^{-10k} = 39 \quad .....(2)$

Sub (1) into (2)

$46 - A + Ae^{-10k} = 39$

$$e^{-10k} = \frac{A-7}{A} \quad .....(3)$$

At $t = 20,\ T = 33$

$B + Ae^{-20k} = 33 \quad .....(4)$

Sub (1) into (4)

$46 - A + Ae^{-20k} = 33$

$$e^{-20k} = \frac{A-13}{A}$$

$$\left(e^{-10k}\right)^2 = \frac{A-13}{A} \quad .....(5)$$

Sub (3) into (5)

$$\left(\frac{A-7}{A}\right)^2 = \frac{A-13}{A}$$

$A(A-13) = (A-7)^2$

$A^2 - 13A = A^2 - 14A - 49$

$A = 49$

$B = -3$

$\therefore$ Temperature of the refrigerator is $-3^{o}C$

Marking guideline:

3 Correct response

2 Find the correct value of k or A

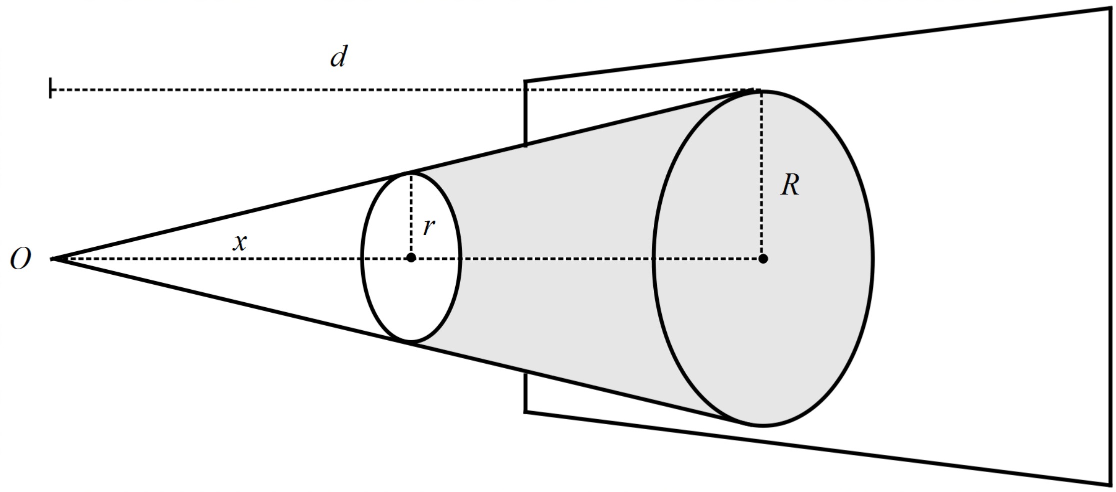
1 Sets up simultaneous equations

Marker's comments:

- Not well done.
- Most of the students couldn't solve the equations simultaneously to find the correct value of B.

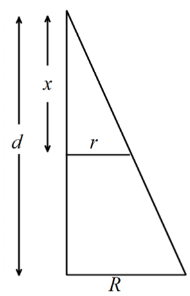
c) A light source is placed d cm away from a screen. A circular coin of radius r cm is placed x cm from the light source, such that its horizontal axis of symmetry passes through O.

The coin is moving horizontally towards the source of light at a speed of 4 cm per second and casts a circular shadow of radius R cm on the screen as shown.



i) Show that the area of the shadow is $A = \dfrac{\pi d^2 r^2}{x^2}$ **2**

Solution



Using similar triangles:

$$\frac{x}{d} = \frac{r}{R}$$

$$R = \frac{rd}{x}$$

$$A = \pi R^2$$

$$= \pi\left(\frac{rd}{x}\right)^2$$

$$\therefore A = \frac{\pi r^2 d^2}{x^2} \text{ as req'd}$$

Marking guideline:

2 Correct response

1 Uses similarity to find an expression for R or equivalent merit

Marker's comments:

- Well done

ii) Find the rate, in terms of d and r, at which the area of the shadow increases with respect to time when $x = \frac{d}{4}$. **2**

Solution

$$A = \pi d^2 r^2 x^{-2}$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$\frac{dA}{dt} = \left(\pi d^2 r^2 \times -2x^{-3}\right) \times -4$$

$$= \frac{8\pi d^2 r^2}{x^3}$$

When $x = \frac{d}{4}$

$$\frac{dA}{dt} = 8\pi d^2 r \times \frac{64}{d^3}$$

$$= \frac{512\pi r^2}{d}$$

$\therefore$ Area of the shadow increases at the rate of $\frac{512\pi r^2}{d} cm^2 / s$

Marking guideline:

2 Correct response

1 Establishes correct expression for $\frac{dA}{dt}$

Marker's comments:

- Most of the students couldn't use chain rule properly to find $\frac{dA}{dt}$ correctly

End of Examination