VIRILE AGITUR

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**Student Number**

# Knox Grammar School

## 2011

**Preliminary**
**Yearly Examination**

# Mathematics Extension 1

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen only
- Board approved calculators may be used
- All necessary working should be shown in every question

**Total Marks – 84**

- Attempt Questions 1 – 7
- Answer each question in a **separate writing booklet**
- All questions are of equal value

**Subject Teachers**
Mr Bradford
Mr Sedgeman
Ms Yamaner
Mr Harnwell
Mr Naidoo

**This paper MUST NOT be removed from the examination room**

**Number of Students in Course:**

**Number of Writing Booklets Per Student (Four Page): 7**

**Total Marks – 84**

**Attempt Questions 1 – 7**

**All questions are of equal value**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

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**Question 1 (12 marks)**

(a) Simplify $\dfrac{4}{x^2-4}+\dfrac{1}{2-x}$ **2**

(b) Solve for $x$ : $\dfrac{4x+7}{x-2}\geq -3$ **3**

(c) Find the coordinates of the point of contact of the tangent $4x - 3y + 25 = 0$ with the circle $x^2+y^2=25$

. **3**

(d) Find two possible values of $m$ if the lines $2x - y - 5 = 0$ and $y = mx + 1$ intersect at an angle of 45°. **4**

**Question 2 (12 marks) Start a new Answer Booklet**

(a) Given the function, $f(x) = \dfrac{2x}{2 - x^2}$, determine

(i) the natural domain of *f(x)* **2**

(ii) whether the function is odd, even or neither. **2**

(b) Solve for $x$ : $3^{2x+1} - 4(3^x) + 1 = 0$ **3**

(c) Given the parabola $y = 6x^2 - 24x + 26$, find the coordinates of the focus and the equation of the directrix. **3**

(d) The third term of a geometric progression is $\dfrac{1}{2}$ and the eighth term is -16.

Find the common ratio. **2**

**Question 3 (12 marks) Start a new Answer Booklet**

(a)

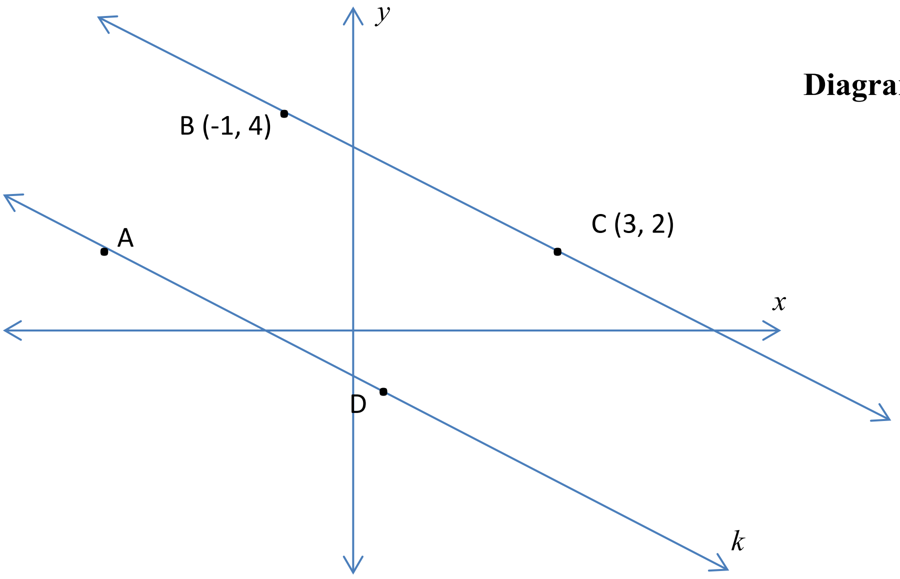



**Diagram Not to Scale**

Line $\boldsymbol{k}$ has equation $x + 2y + 1 = 0$

B is the point (-1, 4) and C is the point (3, 2)

Points A and D are chosen on line $k$ such that ABCD is a parallelogram.

(i) Verify that BC is parallel to line $k$. **1**

(ii) Find, correct to the nearest degree, the angle, $\theta$, that the line makes with the positive direction of the $x$-axis. **2**

(iii) Find the length of BC in simplest surd form. **2**

(iv) Find the shortest distance from B to line $k$ **2**

(v) Find the area of parallelogram ABCD. **2**

**Question 3 continued on page 6**

**Question 3 continued …**

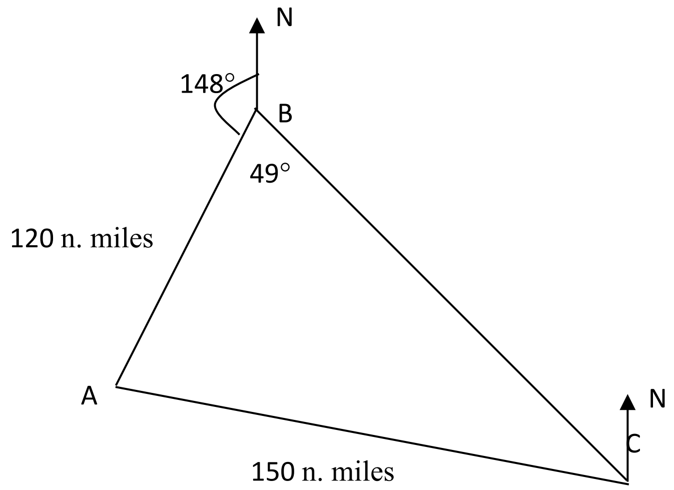
(b) Find the exact value of $\cos 210^{\circ}$ **1**

(c) Find a primitive of $y = \dfrac{1}{2x^3}$ **2**

**Question 4 (12 marks) Start a new Answer Booklet**

(a) The diagram below has been constructed from information in the question below.

Copy and complete the question, underlining all replacements for the blanks.




**3**

A ship sails from point A on a bearing of ____________ for 5 hours at a speed of __________ nautical miles per hour where it reaches point B, whose bearing from lighthouse C is __________.

Given that C is ___________ nautical miles from A, find the distance from _____ to ______.

(Note : you are **not** required to solve this problem)

(b) (i) The tangent to the curve $x^2 = 4ay$ at $P(2ap, ap^2)$ cuts the $x$-axis at M . Derive the equation of this tangent and hence find the coordinates of M. **3**

(ii) The normal to the curve $x^2 = 4ay$ at the same point $P$ cuts the $y$-axis at N. Find the coordinates of N. **2**

(iii) Hence show that the locus of the midpoint of MN is a parabola. **2**

(c) Differentiate $y = \dfrac{2}{\sqrt{x^2 - 3}}$ **2**

**Question 5 (12 marks) Start a new Answer Booklet**

(a) Let $A$ (-1, 3) and $B$ (2, -4) be two fixed points.

(i) Find the coordinates of the point M which divides AB externally in the ratio 2 : 1. **2**

(ii) (α) If point $P$ $(x, y)$ moves so that $PA$ is perpendicular to $PB$.
Show that the locus of P is a circle. **3**

(β) Determine the centre and radius of this circle. **2**

(b) Given the following information, sketch a possible graph of $y = f(x)$

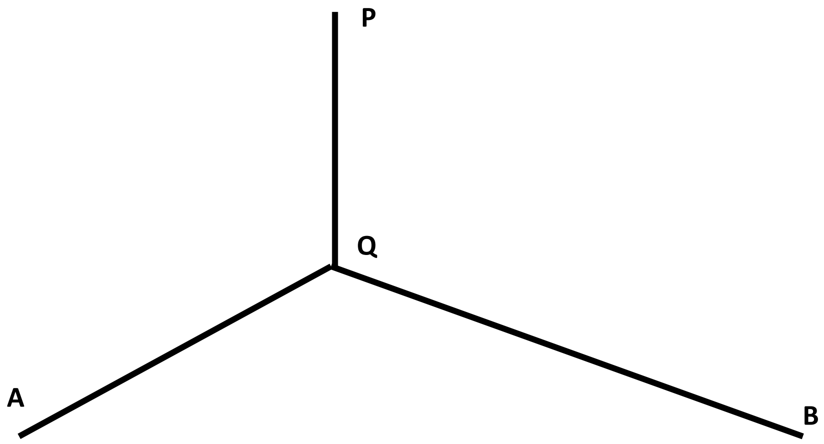
- The curve has stationary points at $x = 0$, $x = 2$ and $x = 4$
- $f(0) = f(4) = 0$
- $f''(x) < 0$ for $x < 0$
- $f''(x) > 0$ for $0 < x < 1$
- The curve is concave up for $x > 3$ **3**

(c) Given the function, $y = 2x^3 - \dfrac{x^2}{2} - 2x + 1$, determine the values of $x$ for which the function is <u>decreasing</u>. **2**

**Question 6 (12 marks) Start a new Answer Booklet**

(a) From a point A, due south of a mast PQ, the angle of elevation of PQ is 40˚.
From B, due east of A, the angle of elevation of PQ is 19˚.

PQ is 35 metres high while Q, A and B are on level ground.




(i) Copy the diagram into your booklet and add all the given information. **1**

(ii) Find, correct to one decimal place, the distance AB. **3**

(iii) Find the bearing of the mast from B, giving your answer correct to the nearest degree. **3**

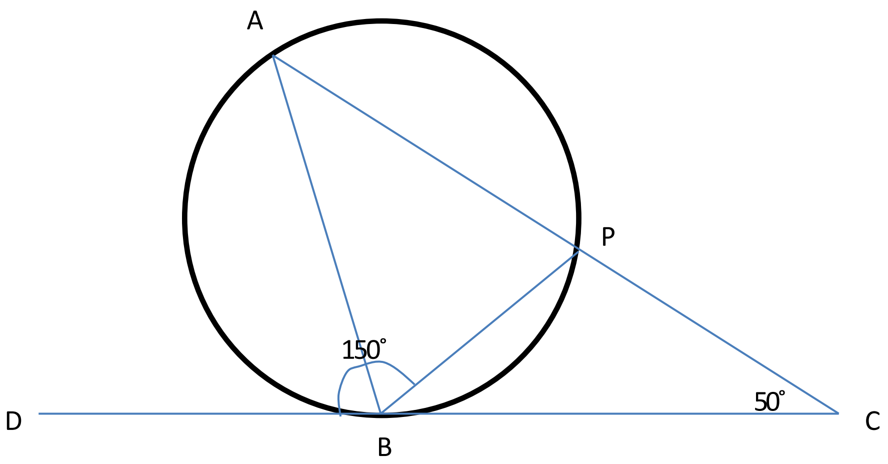
(b) Find the exact values of $k$ for which $x^2 + 2kx + k + 5 = 0$ has **real roots**. **3**

(c) Evaluate $\lim\limits_{x \to \infty} \dfrac{3x^2 - 5x^3 + 6}{5 + 4x - 7x^3}$ **2**

**Question 7** (12 marks) **Start a new Answer Booklet**

(a) Solve for θ in the domain $0^\circ \leq \theta \leq 360^\circ$ : $\cos^3\theta + 2\cos^2\theta + \cos\theta = 0$ **3**

(b)




DBC is a tangent to the circle at B. APC is a straight line. $\angle PCB = 50^\circ$ and $\angle PBD = 150^\circ$.
Find $\angle BAP$ and $\angle APB$. **4**

**Question 7 continues on page 10**

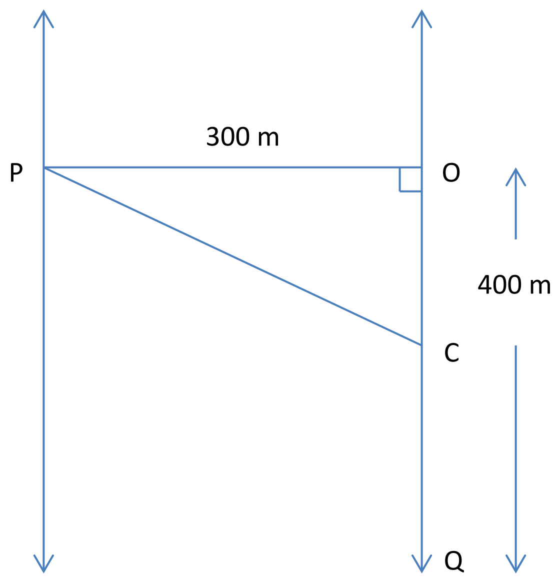
(c) An athlete can swim at 50 m/min and run at 200 m/min. She completes a course which starts from point P, situated on one bank of a 300 metre wide straight river.
The course finishes at point Q, situated on the other bank, 400 metres downstream from O, as shown in the diagram. The river is not flowing.

(i) Show that the time, $T$, taken to complete the course is given by

$$T = \frac{4\sqrt{x^2 + 90000} + 400 - x}{200} \qquad \text{where } x = \text{the distance OC}$$ **1**

(ii) If the athlete wishes to complete the course in the shortest possible time, find how far downstream she should swim (correct to the nearest metre)

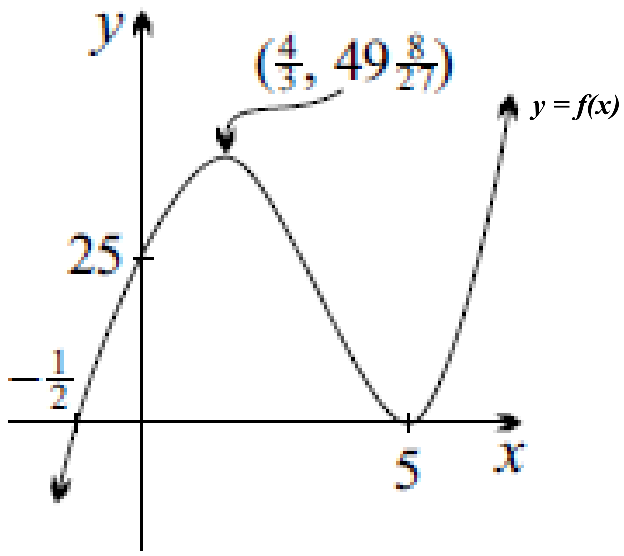
**4**




**- End of Paper -**

**Alternative Q5 (c)**

The graph of $y = f(x)$ is shown below. Copy the graph into your answer booklet and, on axes below it, draw a neat sketch of $y = f\,'(x)$



# 2011 EXTENSION 1 EXAM SOLUTIONS

## Question 1

(a) $\dfrac{4}{x^2-4} + \dfrac{1}{2-x}$

$= \dfrac{4}{(x+2)(x-2)} - \dfrac{1}{x-2}$

$= \dfrac{4-(x+2)}{(x+2)(x-2)}$

$= \dfrac{2-x}{(x+2)(x-2)}$

$= \dfrac{-1}{x+2}$

(b) $\dfrac{4x+7}{x-2} \geq -3$

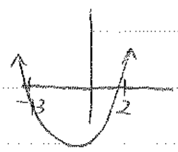
$(4x+7)(x-2) \geq -3(x-2)^2$

$4x^2 - x - 14 \geq 3(x^2-4x+4)$

$x^2+11x-26 \geq 0$

$(x+13)(x-2) \geq 0$

$x \leq -13$ or $x > 2$



(c) $4x - 3y + 25 = 0$ ① $\quad x^2+y^2 = 25$ ②

$3y = 4x+25$

$y = \dfrac{4x+25}{3}$

sub into ② $\Rightarrow$ $x^2 + \left(\dfrac{4x+25}{3}\right)^2 = 25$

$9x^2+16x^2+200x+625 = 225$

$25x^2+200x+400 = 0$

$x^2+8x+16 = 0$

$(x+4)^2 = 0$

$\therefore x = -4$

sub into ① $\Rightarrow$ $4(-4) - 3y + 25 = 0$

$3y = 9$

$y = 3$

$\therefore$ point of contact $(-4, 3)$

(d) $\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1m_2}\right|$

$2x - y - 5 = 0$

$y = 2x - 5 \quad \therefore m_1 = 2$

$y = mx + 1, \quad m_2 = m$

$\tan 45° = \left|\dfrac{2-m}{1+2m}\right|$

$1+2m = |2-m|$

$\therefore 1+2m = 2-m$ or $1+2m = m-2$

$3m = 1 \qquad m = -3$

$m = \frac{1}{3}$

Question 2

(a) $f(x) = \frac{2x}{2-x^2}$

(i) $D_f$ : all $x \in \mathbb{R}$, $x \neq \pm\sqrt{2}$

(ii) $f(-x) = \frac{2(-x)}{2-(-x)^2}$

$= \frac{-2x}{2-x^2}$

$= -f(x)$

$\therefore$ function is odd.

(b) $3^{2x+1} - 4(3^x) + 1 = 0$

$(3^x)^2 . 3 - 4(3^x) + 1 = 0$

Let $u = 3^x$

$\therefore 3u^2 - 4u + 1 = 0$

$(3u-1)(u-1) = 0$

$u = \frac{1}{3}$ or $u = 1$

$3^x = \frac{1}{3}$ $\quad$ $3^x = 1$

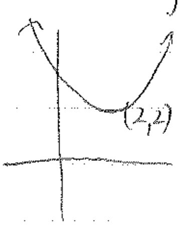
$x = -1$ $\quad$ $x = 0$.

(c) $y = 6x^2 - 24x + 26$

$y - 26 = 6(x^2 - 4x)$

$y - 26 + 24 = 6(x^2 - 4x + 4)$

$y - 2 = 6(x-2)^2$




$\therefore (x-2)^2 = \frac{1}{6}(y-2)$

vertex $(2, 2)$

$4a = \frac{1}{6}$

$\therefore a = \frac{1}{24}$

$\therefore$ focus $(2, 2\frac{1}{24})$

directrix : $y = 1\frac{23}{24}$.

(d) $T_3 = \frac{1}{2}$ $\quad$ $T_8 = -16$

$\therefore ar^2 = \frac{1}{2}$

$ar^7 = -16$

$\therefore r^5 = -32$

$r = -2$.

Question 3

(i) $m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{2-4}{3+1}$

$= -\frac{1}{2}$

$k$ : $x + 2y + 1 = 0$

$y = \frac{-x-1}{2}$

$\therefore m = -\frac{1}{2}$

$\therefore BC \parallel k$.

(ii) $m = \tan\theta$

$\therefore -\frac{1}{2} = \tan\theta$

$\theta = 180° - 27°$

$= 153°$

(iii) $d_{BC} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{(\;.\;)^2 + (4-2)^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$ units

(iv) $d_\perp = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$= \left| \frac{1(-1) + 2(4) + 1}{\sqrt{1^2+2^2}} \right|$

$= \frac{8}{\sqrt{5}}$ units.

(v) Area $= 2\sqrt{5} \times \frac{8}{\sqrt{5}}$

$= 16\ u^2$

(b) $\cos 210° = -\cos 30°$

$= -\frac{\sqrt{3}}{2}$

(c) $y = -\frac{1}{2x^3}$

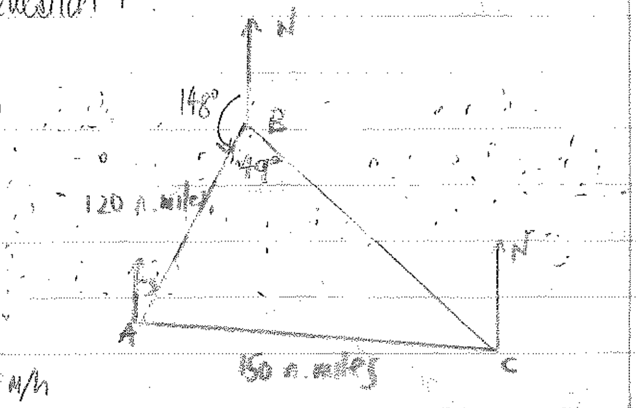
$= -\frac{1}{2}x^{-3}$

primitive $= \frac{1}{4}x^{-2}$

$= \frac{1}{4x^2}$

Question 4

(a)




speed $= \frac{120}{5}$

$= 24$ N/h

A ship sails from point A on a bearing of 032° for 5 hours at a speed of 24 nautical miles/hour where it reaches point B, whose bearing from light house C is 163°. Given that C is 150 nautical miles from A, find the distance from B to C

(b) $x^2 = 4ay$ $\quad P(2ap, ap^2)$

$y = \frac{x^2}{4a}$

gradient fn, $\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

at P, $m = \frac{2ap}{2a}$

$= p$

eqn: $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

x-int: $y = 0$

$0 = px - ap^2$

$x = ap$

$\therefore M(ap, 0)$

(ii) $m_\perp = -\frac{1}{p}$

eqn of normal: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

y-int: $x = 0$

$\therefore py = 2ap + ap^3$

$y = 2a + ap^2$

$\therefore N(0, 2a + ap^2)$

(iii) $M_{MN} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{0 + ap}{2}, \frac{0 + 2a + ap^2}{2}\right)$

$= \left(\frac{ap}{2}, \frac{a(2 + p^2)}{2}\right)$

$x = \frac{ap}{2}$

$p = \frac{2x}{a}$

$\therefore y = a\frac{\left(2 + \left(\frac{2x}{a}\right)^2\right)}{2}$

$= a\left(\frac{2a^2 + 4x^2}{2a^2}\right)$

$= \frac{2a^2 + 4x^2}{2a}$

$\therefore 4x^2 = 2ay - 2a^2$

$x^2 = \frac{2a(y - a)}{4}$

which is a parabola, vertex $(0, a)$

(c) $y = \frac{2}{\sqrt{x^2 - 3}}$

$= 2(x^2 - 3)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -1(x^2 - 3)^{-\frac{3}{2}} \cdot 2x$

$= \frac{-2x}{(\sqrt{x^2 - 3})^3}$

$= \frac{-2x}{(x^2 - 3)\sqrt{x^2 - 3}}$

## Question 5

(a) $A(-1,3)$ $B(2, \ )$

(i) $P = \left( \frac{nx_1 - my_1}{m-n}, \frac{nx_2 - my_2}{m-n} \right)$

$= \left( \frac{-1\times1 - 2\times2}{-2+1}, \frac{3\times1 - 2\times(-4)}{-2+1} \right)$

$= (5, -11)$

(b) (i) $m_{PA} \times m_{PB} = -1$

$\frac{y-3}{x+1} \times \frac{y+4}{x-2} = -1$

$y^2 + y - 12 = -(x^2 - x - 2)$

$y^2 + y - 12 = -x^2 + x + 2$

$x^2 + y^2 - x + y - 14 = 0$

$x^2 - x + (\frac{1}{2})^2 + y^2 + y + (\frac{1}{2})^2 = 14 + \frac{1}{4} + \frac{1}{4}$

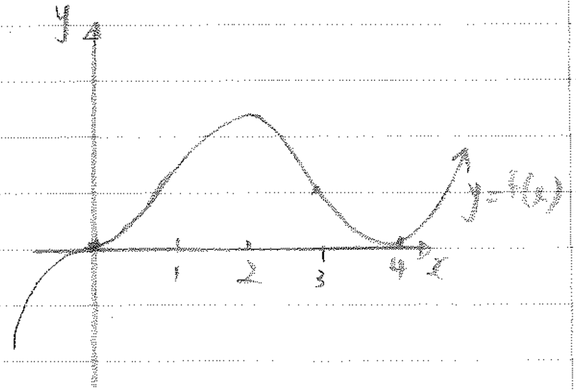
$(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 = \frac{29}{2}$

(ii) Centre $(\frac{1}{2}, -\frac{1}{2})$, $r = \sqrt{\frac{29}{2}}$

(iii) $MN = 4$ units

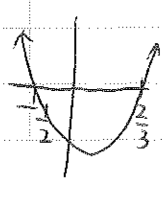
$=$ diameter of circle

(b)




(c) $y = 2x^3 - \frac{x^2}{2} - 2x + 1$

decreasing: $\frac{dy}{dx} < 0$



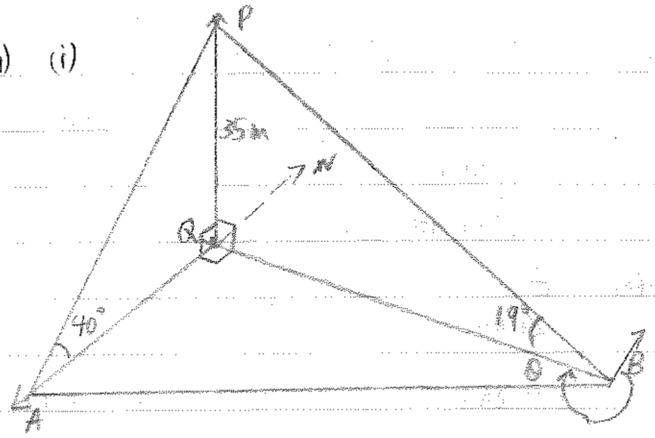

$6x^2 - x - 2 < 0$

$(3x-2)(2x+1) < 0$

$-\frac{1}{2} < x < \frac{2}{3}$

## Question 6

(a) (i)




(ii) $\tan 40° = \frac{35}{AQ}$ $\tan 19° = \frac{35}{BQ}$

$\therefore AQ = \frac{35}{\tan 40°}$ $\therefore BQ = \frac{35}{\tan 19°}$

$\therefore AB^2 = AQ^2 + BQ^2$

$= \left(\frac{35}{\tan 40°}\right)^2 + \left(\frac{35}{\tan 19°}\right)^2$

$= 12072.02\ldots$

$\therefore AB = 109.872\ldots$

$= 109.9$ m (to 1dp)

(iii) $\tan \angle QBA = \frac{35}{\tan 40°} \div \frac{35}{\tan 19°}$

$= \frac{\tan 19°}{\tan 40°}$

$= 0.410\ldots$

$\therefore \angle QBA = 22°$ (to nearest deg)

$\therefore$ bearing $= 270° + 22°$

$= 292°$

(b) $x^2 + 2kx + k + 5 = 0$

for real roots, $b^2 - 4ac \geq 0$

$(2k)^2 - 4(k+5) \geq 0$

$4k^2 - 4k - 20 \geq 0$

$k^2 - k - 5 \geq 0$

$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{1 \pm \sqrt{1^2 - 4(-5)}}{2}$

$= \frac{1 \pm \sqrt{21}}{2}$

$\therefore k \leq \frac{1 - \sqrt{21}}{2}$ or $k \geq \frac{1 + \sqrt{21}}{2}$

(c) $\lim_{x \to \infty} \frac{3x^2 - 5x^3 + 6}{5 + 4x - 7x^3}$

$= \lim_{x \to \infty} \frac{\frac{3}{x} - 5 + \frac{6}{x^3}}{\frac{5}{x^3} + \frac{4}{x^2} - 7}$

$= \frac{5}{7}$

Question 7.

(a) $0^\circ \le \theta \le 360^\circ$

$\cos^3\theta + 2\cos^2\theta + \cos\theta = 0$

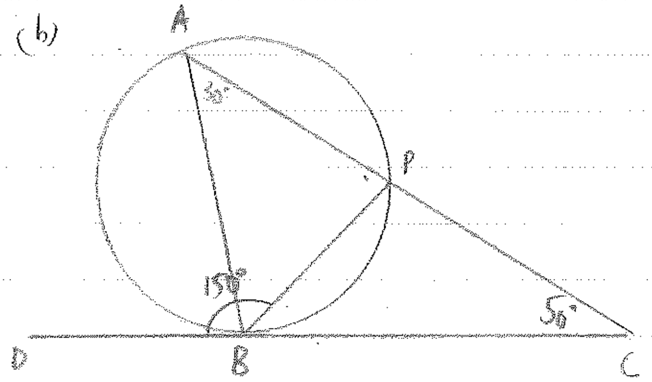
$\cos\theta(\cos^2\theta + 2\cos\theta + 1) = 0$

$\cos\theta\,(\cos\theta + 1)^2 = 0$

$\cos\theta = 0$ or $\cos\theta = -1$

$\theta = 90^\circ, 270^\circ$ $\quad \theta = 180^\circ$

(b)




$\angle PBC + 150^\circ = 180^\circ$ (straight angle)

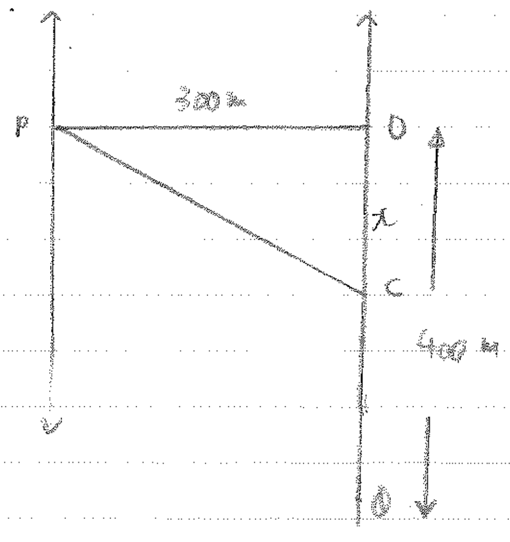
$\angle PBC = 30^\circ$

$\therefore \angle BAP = 30^\circ$ (angle in alternate segment)

Now, $\angle APB = 30^\circ + 50^\circ$ (exterior angle of $\Delta$ = sum of interior opp. angles)

$= 80^\circ$

(c)




Let $OC = x$

$\therefore PC^2 = 300^2 + x^2$

$PC = \sqrt{x^2 + 90000}$

$CQ = 400 - x$

$\dfrac{D}{S \times T}$

$\therefore$ time, $t = \dfrac{\sqrt{x^2+90000}}{50} + \dfrac{400-x}{200}$

for minimum time, $\dfrac{dt}{dx} = 0$, $\dfrac{d^2t}{dx^2} > 0$

$t = \dfrac{(x^2+90000)^{\frac{1}{2}}}{50} + \dfrac{400-x}{200}$

$\dfrac{dt}{dx} = \dfrac{\frac{1}{2}(x^2+90000)^{-\frac{1}{2}} \cdot 2x}{50} - \dfrac{1}{200}$

$0 = \dfrac{x}{50\sqrt{x^2+90000}} - \dfrac{1}{200}$

$\dfrac{1}{200} = \dfrac{x}{50\sqrt{x^2+90000}}$

$50\sqrt{x^2+90000} = 200x$

$\sqrt{x^2+90000} = 4x$

$x^2 + 90000 = 16x^2$

$15x^2 = 90000$

$x^2 = 6000$

$x = 77.459\ldots$

$= 77$ m (to nearest m)

$\therefore$ she should swim 77 m downstream from O.