VIRILE AGITUR

Name: ____________________

KNOX GRAMMAR SCHOOL

Teacher: ____________________

2019

Year 11 Yearly Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- calculators approved by NESA may be used
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Teachers:
Mr Cheah
Mr Meli
Mr Vuletich
Mr Willcocks

Setter: Ms Yun

Total marks – 70

Section I: Pages 3 – 6

10 marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II: Pages 7 – 12

60 marks
- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

This paper MUST NOT be removed from the examination room

Write your full name and your teacher's name on the front cover of each writing booklet

Number of Students in Course: 106

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Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 Given $f(x)=\frac{1}{4}x^3+2$, what is the value of $f^{-1}(4)$?

A. 2

B. 5

C. 8

D. 18

2 Given that $\cos\alpha=\frac{1}{3}$, find the value of $\cos 2\alpha$.

A. $\frac{2}{3}$

B. $\frac{7}{9}$

C. $-\frac{7}{9}$

D. $\frac{1}{9}$

3 A car hire company has 10 sports cars and 16 four-wheel drive cars.
The company is to make a leaflet for a special offer.

This leaflet is to contain 4 sports cars and 6 four-wheel drive cars.

In how many different ways could the cars in the leaflet be chosen?

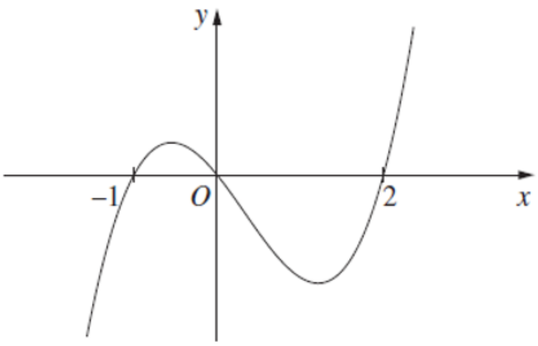
A. ${}^{10}P_4 \times {}^{16}P_6$

B. ${}^{10}C_4 \times {}^{16}C_6$

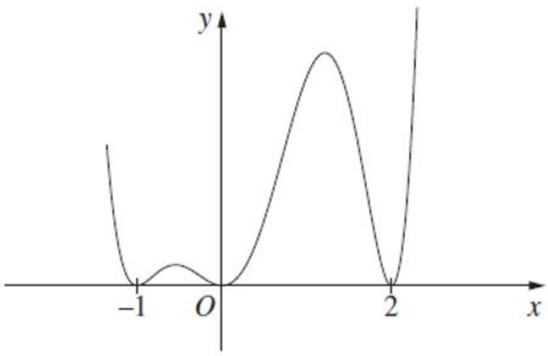
C. ${}^{10}P_4 + {}^{16}P_6$

D. ${}^{10}C_4 + {}^{16}C_6$

4 The graph of the function $y = f(x)$ is shown.



A second graph is obtained from the function $y = f(x)$.



Which equation best represents the second graph?

A. $y = \sqrt{f(x)}$

B. $y^2 = f(x)$

C. $y = [f(x)]^2$

D. $y = f(x^2)$

5 Which group of three numbers could be the roots of the polynomial equation $x^3 + ax^2 - 41x + 42 = 0$?

A. 2, 3, 7

B. 1, −6, 7

C. −1, −2, 21

D. −1, −3, −14

6 A curve has parametric equations $x = t - 3$ and $y = t^2 + 2$.

What is the Cartesian equation of this curve?

A. $y = x^2 - x - 1$

B. $y = x^2 + x - 1$

C. $y = x^2 - 6x + 11$

D. $y = x^2 + 6x + 11$

7 In how many ways can 5 people from a group of 14 people be chosen and then arranged in a circle?

A. $\dfrac{13!}{8!}$

B. $\dfrac{13!}{8!5}$

C. $\dfrac{14!}{9!}$

D. $\dfrac{14!}{9!5}$

8 Given that $f(x)$ is a non-zero odd function.

Which function below is also odd?

A. $f(x^2)\cos x$

B. $f(f(x))$

C. $f(x^3)\sin x$

D. $f(x^2) - f(x)$

9 Which of the following is the solution of $\dfrac{3}{|x+2|} \le x$?

A. $-3 \le x < -2,\ x \ge 1$

B. $x \le -3,\ x \ge 1$

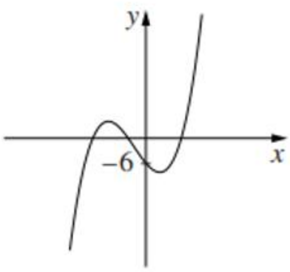
C. $-3 \le x \le -2$

D. $x \ge 1$

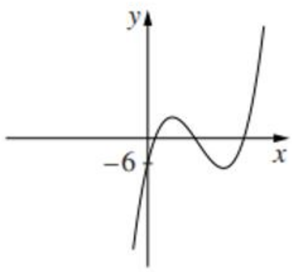
10 Consider the polynomial $p(x) = ax^3 + bx^2 + cx - 6$ with a and b positive.

Which graph could represent $p(x)$?

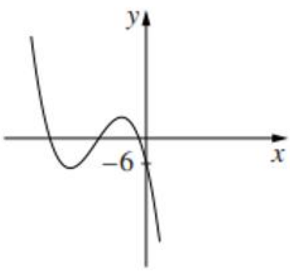
(A)



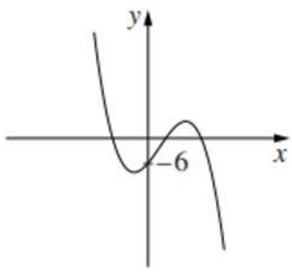
(B)



(C)



(D)



End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet.
In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) State the domain and range of the function $f(x) = 2\sin^{-1}\dfrac{x}{2}$. **2**

(ii) Sketch the graph of $f(x) = 2\sin^{-1}\dfrac{x}{2}$. **1**

(b) The cubic equation $x^3 - 4x^2 + 6 = 0$ has roots α, β and γ.

(i) What is the value of $\alpha\beta\gamma(\alpha+\beta+\gamma)$? **2**

(ii) What is the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$? **1**

(c) Prove that $\dfrac{1-\cos\theta}{1+\cos\theta} = \tan^2\dfrac{1}{2}\theta$, using the $t = \tan\dfrac{1}{2}\theta$ results. **2**

(d) The polynomial $2x^3 + ax^2 + bx + 6$ has a factor $x-1$ and leaves a remainder of -12 when divided by $x+2$. Find the values of a and b. **3**

Question 11 continues

(e) A salad, which is initially at a temperature of $25\,^\circ\mathrm{C}$, is placed in a refrigerator that has a constant temperature of $3\,^\circ\mathrm{C}$. The cooling rate of the salad is proportional to the difference between the temperature of the refrigerator and the temperature, T of the salad. That is, T satisfies the equation $\dfrac{dT}{dt} = -k(T-3)$ where t is the number of minutes after the salad is placed in the refrigerator.

(i) Show that $T = 3 + Ae^{-kt}$ satisfies this equation. **1**

(ii) The temperature of the salad is $11\,^\circ\mathrm{C}$ after 10 minutes. Find the temperature of the salad after 15 minutes. **3**

End of Question 11

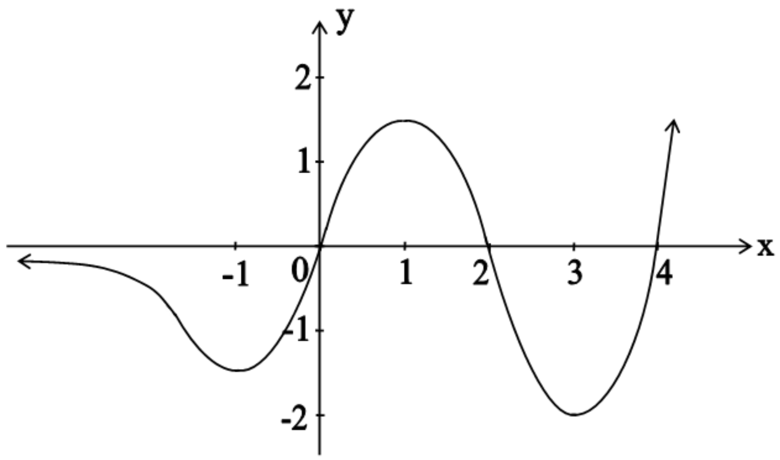
Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A chest has 3 drawers, each containing a coin. One coin is silver on both sides, one is gold on both sides and the third is silver on one side and gold on the other. **2**

A drawer is chosen at random and one face of the coin is shown to be silver.

What is the probability that the other side is also silver?

(b) The diagram shows the graph of $y = f(x)$.



Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = f(|x|)$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(c) The letters of STREET are arranged randomly in a row. Find the probability that:

(i) the two letters E are together. **2**

(ii) the Es and Ts are together in one group. **2**

(d) If $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 4b^3 = 0$. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the probability function below, evaluate *a* and *b* given that $E(X) = 2$. **2**

x	1	2	3	4
$P(X = x)$	a	b	0.2	0.1

(b) Consider the digits 7, 6, 5, 4, 3, 2, 1 and 0. Find how many four-digit numbers are possible if the digits are in:

(i) descending order, **1**

(ii) ascending order. **1**

(c) (i) Prove that $\tan 4\theta - \tan\theta = \tan 3\theta(1 + \tan 4\theta \tan\theta)$. **1**

(ii) Using part (i), or otherwise, prove that **3**

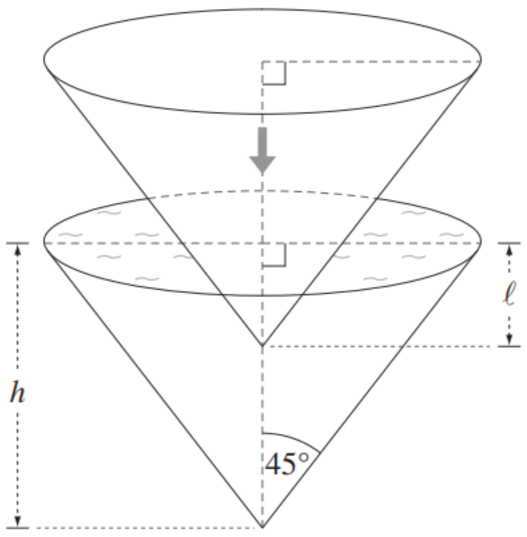
$$\tan 4\theta - \tan\theta = \frac{\sin 3\theta}{\cos 4\theta \cos\theta}.$$

(iii) Hence solve for $0 \le \theta \le \pi$ **2**

$$\tan 4\theta = \tan\theta.$$

Question 13 continues

(d) The diagram shows two identical circular cones with a common vertical axis. Each cone has height h cm and semi-vertical angle $45°$.



The lower cone is completely filled with water. The upper cone is lowered vertically into the water as shown in the diagram. The rate at which it is lowered is given by

$$\frac{dl}{dt} = 10,$$

where l cm is the distance the upper cone has descended into the water after t seconds.

As the upper cone is lowered, water spills from the lower cone. The volume of water remaining in the lower cone at time t is

$V = \frac{\pi}{3}(h^3 - l^3)$ cm^3. [Do NOT prove this]

(i) Find the rate at which V is changing with respect to time when $l = 2$. **3**

(ii) Find the rate at which V is changing with respect to time when the lower cone has lost $\frac{1}{8}$ of its water. Give your answer in terms of h. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Write $\dfrac{1}{(n+1)!}-\dfrac{1}{(n-1)!}$ as a single fraction. **1**

(b) What is the domain of the function $g(x)=\cos^{-1}(\ln x)$? **2**

(c) (i) Show that for a random variable X and constants a and b **2**

$$E(a+bX)=a+b\times E(X).$$

(ii) For the random variable X, it is known that $\mathrm{E}(X)=6$ and $\mathrm{Var}(X)=2$. **1**
Using part (i) calculate $\mathrm{E}(3X-10)$

(iii) Find $\mathrm{Var}(3X-10)$. **2**

(d) Let $f(x)=e^{x}-\dfrac{1}{e^{x}}$ for all real values of x.

(i) Show that $f(x)$ has an inverse function for all values of x. **1**

(ii) Sketch the graph of $y=f(x)$. **2**

(iii) On a separate diagram, sketch the graph of the inverse function. **1**

(iv) Find an expression for $y=f^{-1}(x)$ in terms of x. **3**

End of Examination

2019 Year 11 Mathematics Extension 1 Yearly SOLUTIONS

Suggested Solution (s)	Comments

SECTION I

Q1) To find $f^{-1}(4)$, we let $y=4$ in $f(x)$.

ie) $4=\frac{1}{4}x^3+2$ so $x^3=8$

$\therefore x=2$ (A)

Method II: By interchanging x & y, we get $x=\frac{1}{4}y^3+2$

ie) $x-2=\frac{1}{4}y^3$

$y^3=4x-8$

$\therefore y=\sqrt[3]{4x-8}$

Now for $x=4$, $y=\sqrt[3]{16-8}$
$=2$

$\therefore f^{-1}(4)=2$

Q2) $\cos 2\alpha = 2\cos^2\alpha - 1$

then $\cos 2\alpha = 2\left(\frac{1}{3}\right)^2-1$
$=-\frac{7}{9}$ (C)

Q3) There are ${}^{10}C_4$ ways to choose the 4 sports cars for the leaflet. Each of these combinations could be paired with any one of the ${}^{16}C_6$ combinations of 6 four-wheel drive cars. $\therefore {}^{10}C_4 \times {}^{16}C_6$ (B)

Q4) All single roots became double roots & the curve is non-negative. (C)

Q5) Let α, β and γ be the roots

$\alpha\beta\gamma=-42$ $\therefore$ either B or D

$\alpha\beta+\beta\gamma+\gamma\alpha=-41$ (B)

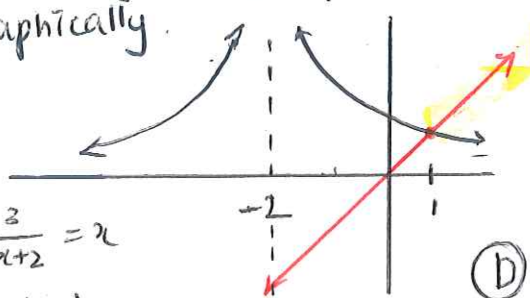
Q6) $t=x+3$

$\therefore y=(x+3)^2+2$
$=x^2+6x+11$ (D)

Q7) $\frac{14!}{5!\,9!}\times 4! = \frac{14!}{9!\,5}$ (D)

Q8) Find $f(-x)$ for each option (B)

Q9) Solving $y=\frac{3}{|x+2|}$ & $y=x$ graphically.



$\frac{3}{x+2}=x$

$x=1$ (D)

Q10) $a>0$, $\alpha\beta\gamma=-\frac{-6}{a}=\frac{6}{a}>0$

$\therefore$ either 0 negative roots or 2 negative roots. (A)

$\alpha+\beta+\gamma=-\frac{b}{a}<0$ since a & b are both positive

$\therefore$ there is at least one negative root.

Suggested Solution (s)	Comments

SECTION II

Question 11

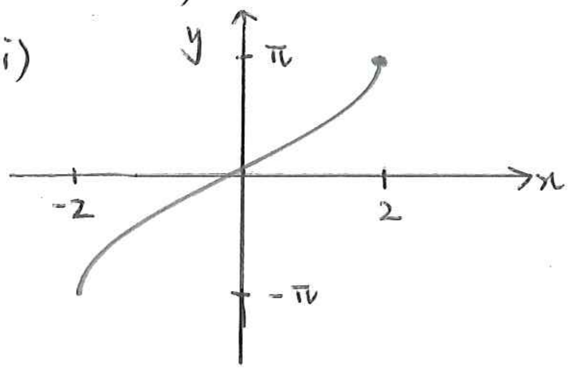
(a) (i) Domain $-1 \le \frac{x}{2} \le 1$

ie) $-2 \le x \le 2$

Range $-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

ie) $-\pi \le y \le \pi$

(ii)



(b) $x^3 - 4x^2 + 6 = 0$

(i) $\alpha + \beta + \gamma = 4$

$\alpha\beta\gamma = -6$

Hence $\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= -6 \times 4$

$= -24$

(ii) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

$= \frac{\gamma}{\alpha\beta\gamma} + \frac{\beta}{\alpha\beta\gamma} + \frac{\alpha}{\alpha\beta\gamma}$

$= \frac{\gamma+\beta+\gamma}{\alpha\beta\gamma} = \frac{4}{-6} = -\frac{2}{3}$

(c) LHS $= (1-\cos\theta) \div (1+\cos\theta)$

$= \left(1 - \frac{1-t^2}{1+t^2}\right) \div \left(1 + \frac{1-t^2}{1+t^2}\right)$

$= \left(\frac{1+t^2-1+t^2}{1+t^2}\right) \div \left(\frac{1+t^2+1-t^2}{1+t^2}\right)$

$= \frac{\cancel{2}t^2}{1+t^2} \times \frac{\cancel{1+t^2}}{\cancel{2}}$

$= t^2$ which is $\tan^2\frac{1}{2}\theta$

(d) $P(x) = 2x^3 + ax^2 + bx + 6$

$P(1) = 0$

ie) $2(1)^3 + a(1)^2 + b(1) + 6 = 0$

$a + b = -8$ — ①

Now, $P(-2) = 12$

ie) $2(-2)^3 + a(-2)^2 + b(-2) + 6 = -12$

$4a - 2b = -2$

$\therefore 2a - b = -1$ — ②

Solve ① & ② simultaneously

$3a = -9$

$a = -3$

$\therefore b = -5$

Suggested Solution (s)	Comments

Q 11 cont.

(e) (i) $T = 3 + Ae^{-kt}$

$T - 3 = Ae^{-kt}$ — ⊛

$\frac{dT}{dt} = -kAe^{-kt}$

but $Ae^{-kt} = T - 3$ ⊛

$= -k(T-3)$

$\therefore T = 3 + Ae^{-kt}$ satisfied

(ii) when $t = 0$ $T = 25$

ie) $25 = 3 + Ae^{-k(0)}$

$A = 22$

when $t = 10$ $T = 11$

ie) $11 = 3 + 22e^{-10k}$

$e^{-10k} = \frac{4}{11}$

$\therefore k = -\frac{1}{10}\ln\left(\frac{4}{11}\right)$ $(\doteqdot 0.10116\ldots)$

when $t = 15$

$T = 3 + 22e^{-15k}$

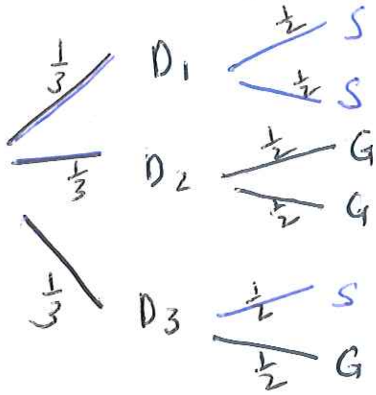
where $k = -\frac{1}{10}\ln\left(\frac{4}{11}\right)$

$= 7.8241$

$\therefore$ Temperature is approx $8°C$

Question 12

(a) Let S be the event that a face of the coin is silver & let G be the gold.



$P(SS|S) = \frac{P(SS)}{P(S)} = \frac{1/3}{1/2}$

$= \frac{2}{3}$

(b) see seperate page

(c) $\boxed{\begin{matrix} S \\ T\,T \\ R \\ E\,E \end{matrix}}$ Total arrangement $= \frac{6!}{2!2!} = 180$

(i) 2 Es together:

Consider them as 1 group & with the rest, they can be arranged in $\frac{5!}{2!} = 60$

$\therefore$ Probability $= \frac{60}{180} = \frac{1}{3}$

(ii) Es & Ts are together in one group:

4 letters as one group can be arranged in $\frac{4!}{2!2!} = 6$ ways.

Together with other 2 letters it can be arranged in $3! = 6$ ways

$\therefore P = \frac{6 \times 6}{180} = \frac{1}{5}$

2019 Year 11 Mathematics Extension 1 Yearly SOLUTIONS

Suggested Solution (s)	Comments
See below	

Q12 (d)

$P(x) = ax^3 + bx^2 + d$

$P'(x) = 3ax^2 + 2bx$

$P''(x) = 6ax + 2b$

$P'(0) = 0$ & $P'\left(-\frac{2b}{3a}\right) = 0$

Hence both 0 & $-2b/3a$ can be a double root of $P(x) = 0$

Now, let 0 be a double root

Hence $P(0) = 0$, $d = 0$

$\Rightarrow$ If $27a^2d + 4b^3 = 0$

then $b = 0 \Rightarrow P(x) = ax^3$

so 0 is a triple root.

Thus if 0 is a double root, then $27a^2d + 4b^3 \neq 0$.

Let $-\frac{2b}{3a}$ be a double root of $P(x) = 0$

Hence $P\left(-\frac{2b}{3a}\right) = 0$

ie) $a\left(\frac{-2b}{3a}\right)^3 + b\left(\frac{-2b}{3a}\right)^2 + d = 0$

$\therefore 27a^2d + 4b^3 = 0$

Question 13

(a) $E(x) = \Sigma x\,p(x)$

$2 = 1 \times a + 2 \times b + 3 \times 0.2 + 4 \times 0.1$

$2 = a + 2b + 1$

$a + 2b = 1$ — ①

Since the function is a prob. distribution

$a + b + 0.2 + 0.1 = 1$

$a + b = 0.7$ — ②

Solve ① & ② simultaneously

$b = 0.3$

$a = 0.4$

(b) 7, 6, 5, 4, 3, 2, 1 & 0

selecting 4 such that digits are in

(i) Descending order:

selecting any 4 digits & then they can be arranged in the required order.

${}^8C_4 = 70$

(ii) Ascending order:

Cannot start with 0

So, ${}^7C_4 = 35$

Suggested Solution (s)	Comments
Q 13 (c) (i) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ Let $A = 4\theta$ & $B = \theta$ $\tan 3\theta = \dfrac{\tan 4\theta - \tan\theta}{1+\tan 4\theta \tan\theta}$ $\therefore \tan 4\theta - \tan\theta = \tan 3\theta\,(1+\tan 4\theta \tan\theta)$ (ii) LHS $= \tan 4\theta - \tan\theta$ $= \tan 3\theta\,(1+\tan 4\theta\tan\theta)$ part (i) $= \dfrac{\sin 3\theta}{\cos 3\theta}\left(1+\dfrac{\sin 4\theta}{\cos 4\theta}\times\dfrac{\sin\theta}{\cos\theta}\right)$ $= \dfrac{\sin 3\theta}{\cos 3\theta}\left(\dfrac{\cos 4\theta\cos\theta + \sin 4\theta\sin\theta}{\cos 4\theta\cos\theta}\right)$ $= \dfrac{\sin 3\theta}{\cos 3\theta}\left(\dfrac{\cos(4\theta-\theta)}{\cos 4\theta\cos\theta}\right)$ $= \dfrac{\sin 3\theta\,\cancel{\cos 3\theta}}{\cancel{\cos 3\theta}\cos 4\theta\cos\theta}$ $= \dfrac{\sin 3\theta}{\cos 4\theta\cos\theta}$ (iii) $\tan 4\theta = \tan\theta \Rightarrow \tan 4\theta - \tan\theta = 0$ ie) $\dfrac{\sin 3\theta}{\cos 4\theta\cos\theta} = 0$ (part (ii)) $\sin 3\theta = 0 \quad \therefore 3\theta = 0, \pi, 2\pi, 3\pi$	

$\therefore \theta = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ or π

2019 Year 11 Mathematics Extension 1 Yearly SOLUTIONS

Suggested Solution (s)	Comments

Q13 (d)

$V = \frac{\pi}{3}(h^3 - l^3)$

$\frac{dV}{dl} = \frac{-3\pi l^2}{3} = -\pi l^2$

& $\frac{dl}{dt} = 10$ (given)

$\therefore \frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$

$= -\pi l^2 \times 10$

$= -10\pi l^2$

When $l = 2$

$\frac{dV}{dt} = -10\pi(2)^2 = -40\pi \text{ cm}^3/\text{s}$

(ii) The cone has lost $\frac{1}{8}$ of the water when

$\frac{\pi l^3}{3} = \frac{1}{8}\frac{\pi h^3}{3}$

ie) when $h = 2l$

The rate of change of volume is $\frac{dV}{dt} = -10\pi l^2$

$= -\frac{10}{4}\pi h^2$

$= -\frac{5}{2}\pi h^2 \text{ cm}^3/\text{s}$

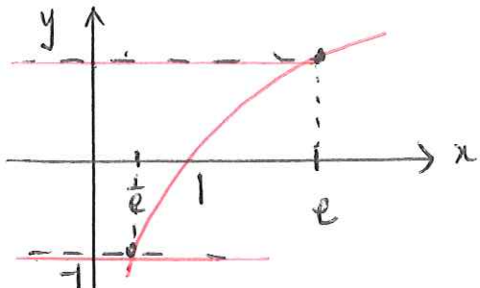
Question 14

(a) $\frac{1}{(n+1)!} - \frac{1}{(n-1)!}$

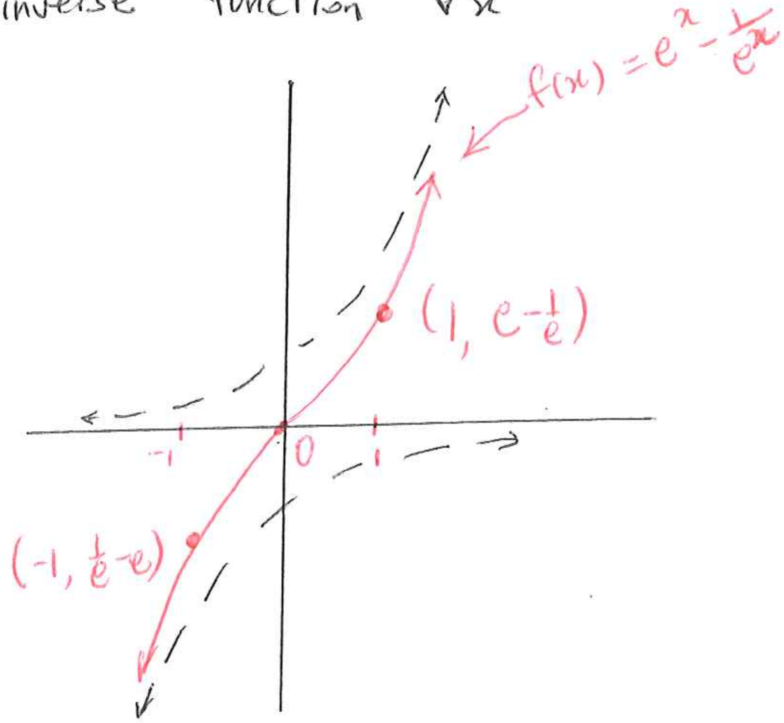
$= \frac{1 - (n+1)n}{(n+1)(n)(n-1)!}$

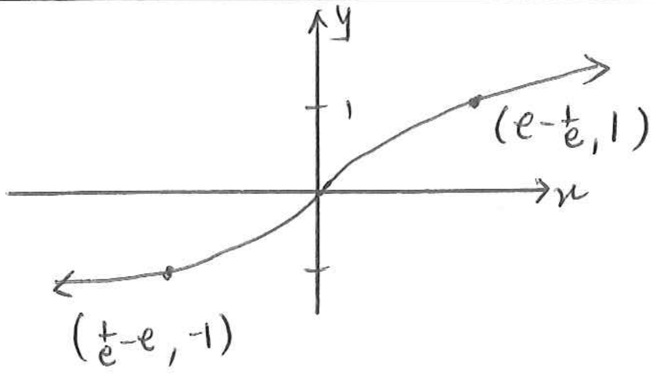
$= \frac{1 - n^2 - n}{(n+1)!}$

(b) The function $g(x)$ exist when $-1 \le \ln x \le 1$



From the graph, we can see that $-1 \le \ln x \le 1$ is valid when $\frac{1}{e} \le x \le e$.

Suggested Solution (s)	Comments
Q14 (c)	
(i) $E(a+bX) = \sum (a+bx)\cdot p(x)$ $= \sum \{ a\cdot p(x) + bx\, p(x) \}$ $= \sum a\cdot p(x) + \sum bx\cdot p(x)$ $= a\sum p(x) + b\sum x\cdot p(x)$ since $\sum p(x) = 1$ $= a + b\cdot E(X)$	
(ii) $E(3X-10) = -10 + 3E(X)$ $= -10 + 3\times 6$ $= 8$ (iii) $Var(3X-10)$ $= 3^2\, Var(X)$ $= 9\times 2 = 18$	
Q 14 (d) Let $f(x) = e^x - \frac{1}{e^x}$ (i) $f'(x) = e^x + e^{-x} = e^x + \frac{1}{e^x}$ $f(x)$ is a monotonic increasing function, so $f(x)$ has an inverse function $\forall x$ (ii) 	

Suggested Solution (s)	Comments
(iii) 	

(iv) $y=e^{x}-\frac{1}{e^{x}}$

$f^{-1}(x)$: $x=e^{y}-\frac{1}{e^{y}}$

ie) $xe^{y}=e^{2y}-1$

$e^{2y}-xe^{y}-1=0$

$e^{y}=\dfrac{x\pm\sqrt{x^2+4}}{2}$

but consider that $e^{y}>0$ for all y

$x^2+4>x^2$

$\sqrt{x^2+4}>x$

$x-\sqrt{x^2+4}<0$

Hence $e^{y}=\dfrac{x-\sqrt{x^2+4}}{2}$ is not a solution

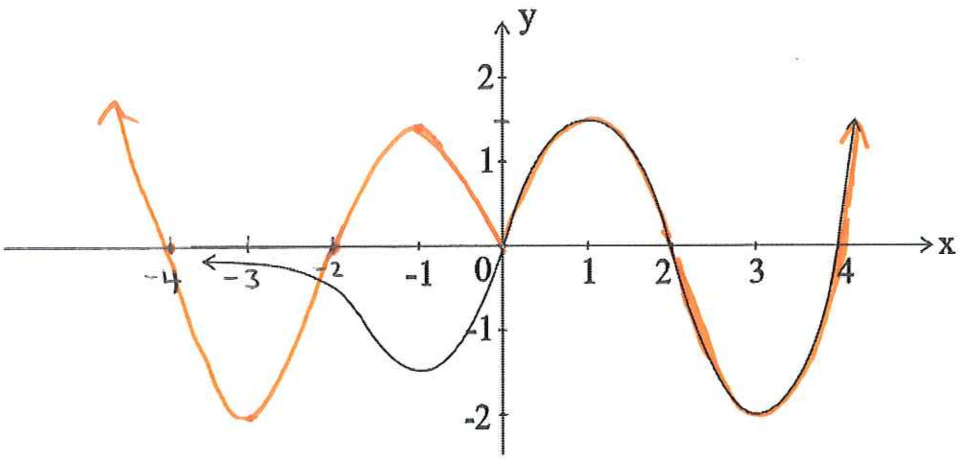
$\therefore e^{y}=\dfrac{x+\sqrt{x^2+4}}{2}$

$y=\ln\left(\dfrac{x+\sqrt{x^2+4}}{2}\right)$

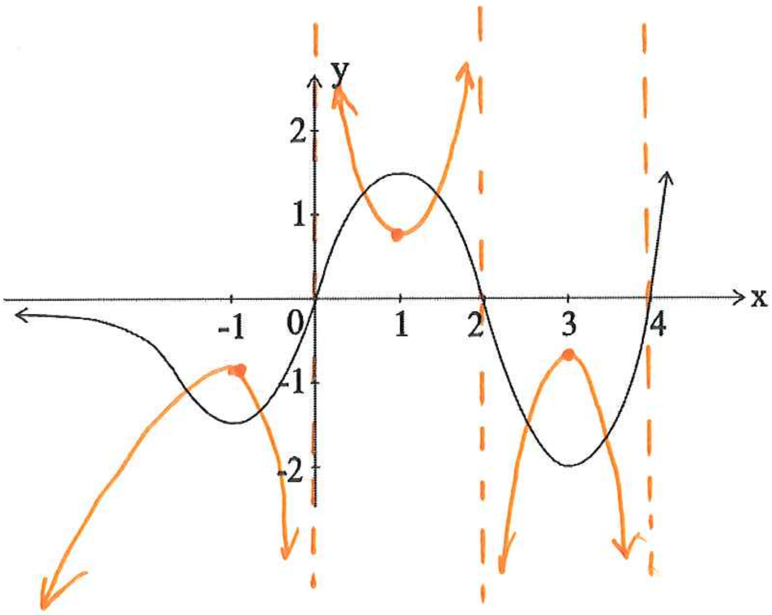
$\therefore f^{-1}(x)=\ln\left(\dfrac{x+\sqrt{x^2+4}}{2}\right)$

2019 Year 11 Ext 1 Yearly Question 12 b

(i)



(ii)



(iii)