Name: ____________________

KNOX GRAMMAR SCHOOL

Teacher: ____________________

2022
Year 11 Yearly Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I: 10 marks (pages 3 – 7)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II: 60 marks (pages 8 – 16)

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

Teachers:
Mrs Dempsey
Mr Willcocks
Mr Lemaire
Mr Naidoo
Mr Menzies/Mr Vuletich

Setter: LEP

This paper MUST NOT be removed from the examination room

Write your full name and your teacher's name on the front cover of each writing booklet

Number of Students in Course: 127

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Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 How many distinct ways are there to arrange the letters of the word EMBEZZLED?

A. 3024

B. 8640

C. 30 240

D. 362 880

2 Consider the equation $2x^4 + 8x^3 - 5x - 6 = 0$.

What is the product of the roots of this equation?

A. – 4

B. – 3

C. 3

D. 4

3 Which of the following shows the domain of the function $f(x) = \cos^{-1}(4x)$?

A. $\left[0, \frac{\pi}{4}\right]$

B. $[0, 4\pi]$

C. $[-4, 4]$

D. $\left[-\frac{1}{4}, \frac{1}{4}\right]$

4 What is the remainder when $P(x) = -x^3 - 2x^2 - 3x + 8$ is divided by $x + 2$?

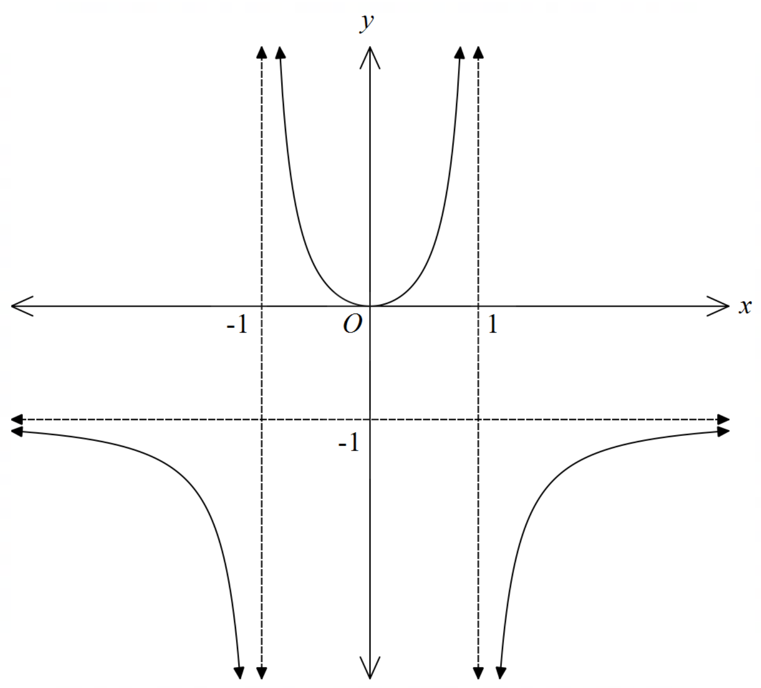
A. –14

B. –2

C. 2

D. 14

5 The diagram shows the graph of $y = f(x)$.



Which equation best describes the graph?

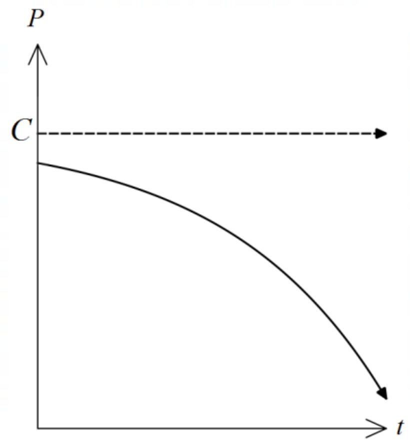
A. $y = \dfrac{x}{x^2 - 1}$

B. $y = \dfrac{x^2}{x^2 - 1}$

C. $y = \dfrac{x}{1 - x^2}$

D. $y = \dfrac{x^2}{1 - x^2}$

6 The population of rabbits on an island is modelled using the equation $P = C + Ae^{kt}$, for certain constants C, A and k. The graph of P over time is shown below.



Which of the following is correct about the constants A and k?

A. $A < 0, k > 0$

B. $A > 0, k > 0$

C. $A < 0, k < 0$

D. $A > 0, k < 0$

7 Out of 12 contestants, eight are to be selected for the final round of a competition. Five of those eight will be placed 1st, 2nd, 3rd, 4th and 5th.

In how many ways can this process be carried out?

A. $\dfrac{12!}{4!5!}$

B. $\dfrac{12!}{4!3!}$

C. $\dfrac{12!}{8!}$

D. $\dfrac{12!}{8!5!}$

8 The polynomial $3x^3+9x^2-4x-15$ has zeros α, β and γ.

What is the value of $\alpha\beta\gamma(\alpha+\beta+\gamma)$?

A. –60

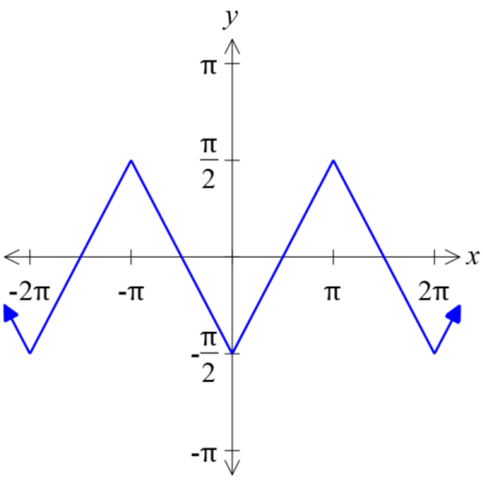
B. –15

C. 15

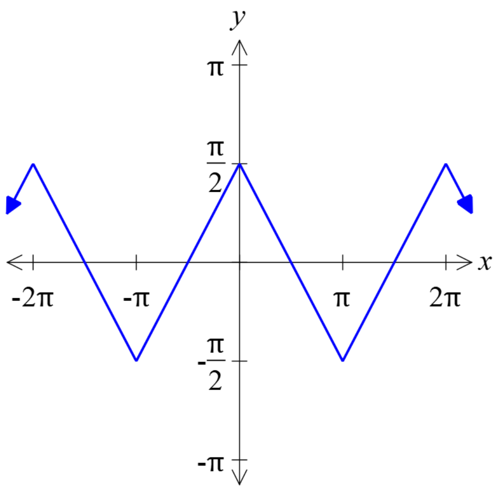
D. 60

9 Which graph represents the function $y=\sin^{-1}(-\cos x)$?

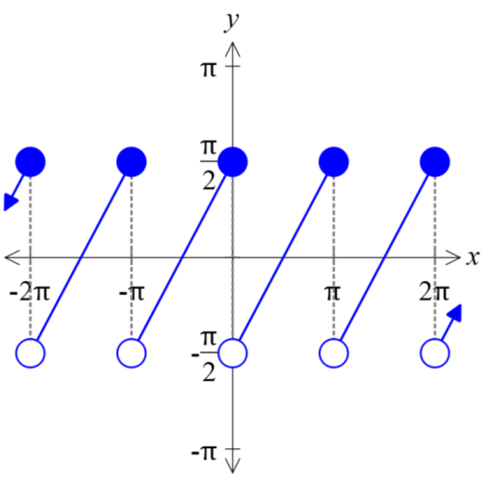
A.



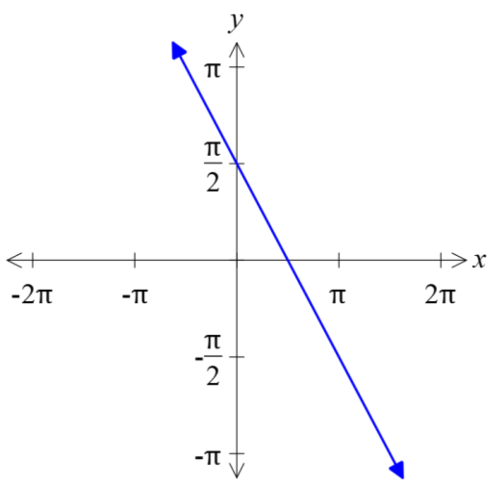
B.



C.



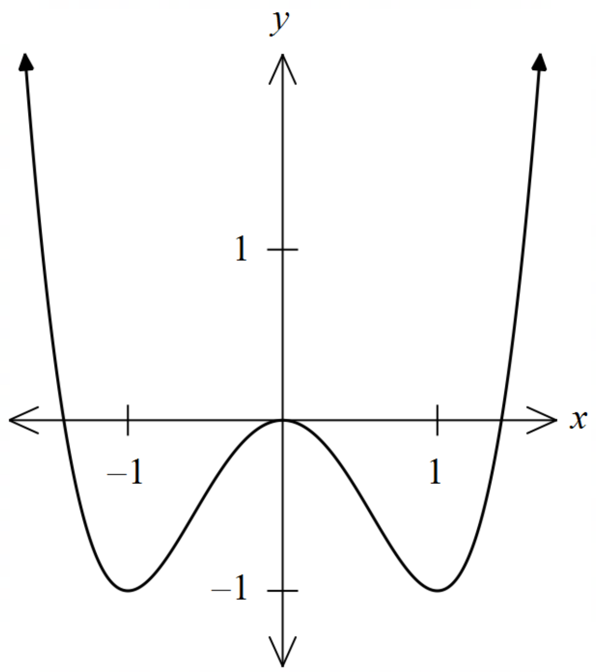
D.



10 The function $f(x) = -\sqrt{1-\sqrt{1+x}}$ has an inverse $f^{-1}(x)$.

The graph of $y = f^{-1}(x)$ forms part of the curve $y = x^4 - 2x^2$.

The diagram shows the curve $y = x^4 - 2x^2$.



How many points do the graphs of $y = f(x)$ and $y = f^{-1}(x)$ have in common?

A. 1

B. 2

C. 3

D. 4

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

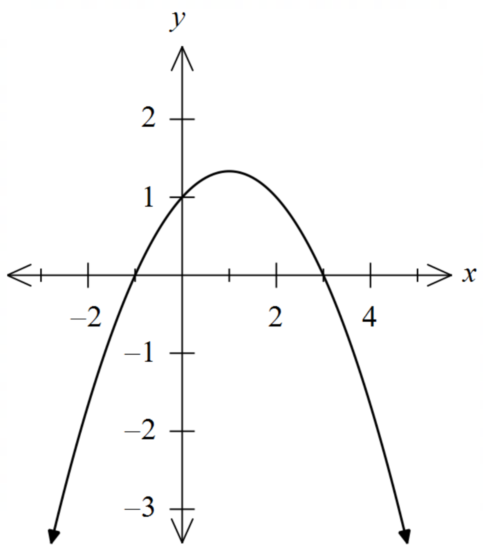
Answer each question in a SEPARATE writing booklet.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the polynomial $Q(x)$ that satisfies $x^3+2x^2-3x-7=(x-2)Q(x)+3$. **2**

(b) The diagram shows the graph of $y=f(x)$.



On pages 15 and 16 provided, draw the following functions, showing all asymptotes and intercepts.

(i) $y=\left|f(x)\right|$ **1**

(ii) $y^2=f(x)$ **2**

(iii) $y=\dfrac{1}{f(x)}$ **3**

Question 11 continues on page 9

Question 11 continued

(c) Solve $\frac{-x}{x-4} \le 1$ **3**

(d) Find the exact value of $\sin 15°$. **2**

(e) A family of 10 are seated randomly around a circular table. **2**

What is the probability that the oldest and youngest members of the family are sitting next to each other?

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Using a t-substitution, prove that $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$, provided that $\cos 2\theta \neq -1$. **2**

(ii) Hence, find the exact value of $\tan\dfrac{\pi}{8}$. **1**

(b) To test some forensic science students, an object was left in a park. At 10 am the temperature of the object is measured to be 30°C. The temperature in the park is a constant 22°C. The object is then immediately moved to a room where the temperature is a constant 5°C.

(i) The temperature of the object in the room can be modelled by the equation **2**

$$T = 5 + 25e^{-kt}$$

where T is the temperature of the object in degrees Celsius, t is the time in hours since the object was placed in the room and k is a constant.

After one hour, the temperature of the object is 20°C.

Show that $k = \ln\left(\dfrac{5}{3}\right)$.

(ii) In a similar manner, if the object were not moved to the room, the temperature of the object in the park before it was discovered can be modelled by an equation of the form **3**

$$T = A + Be^{-kt}$$

where $t = 0$ corresponds to 10am and $k = \ln\left(\dfrac{5}{3}\right)$ as in part (i).

Find the time of day before 10am when the object had a temperature of 37°C, to the nearest minute.
(*Assume the long-term average temperature in the park is 22°C.*)

(c) Find the exact value of $\cos\left[\tan^{-1}(2) + \sin^{-1}\left(-\dfrac{3}{4}\right)\right]$. **3**

Question 12 continues on page 11

Question 12 continued

(d) The polynomial $P(x)$ has a root of multiplicity 4 at $x = a$. **2**

Prove that $P'(x)$ has a root of multiplicity 3 at $x = a$.

(e) A committee of 2 men and 4 women is to be formed from a group of 8 men and 12 women. **2**

How many different committees can be formed that include Alan or Diane, but not both of them?

End of Question 12

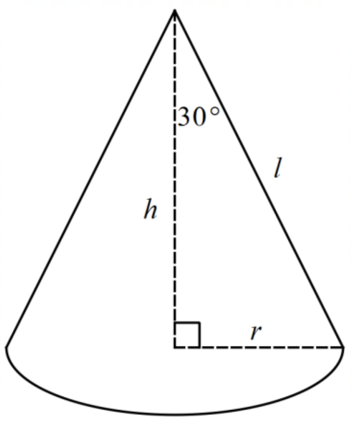
Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let $P(x) = x^{n+1} - (n+1)x + n$, where n is a positive integer.

(i) Show that $P(x)$ has a double zero at $x = 1$. **2**

(ii) Factorise $P(x)$ when $n = 3$. **2**

(b) Sand is being poured onto the top of a conical pile from a conveyor belt. The pile has a height h, base radius r and a slant height l, as shown in the diagram. The semi-vertical angle of the pile is 30°.



(i) Show that $V = \dfrac{\sqrt{3}}{3}\pi r^3$. **1**

(ii) On a particular day, the sand is being poured onto the top of the pile at a constant rate of 60π m^3/min. **2**

Find the rate at which r is changing with respect to time when $r = 15$ metres.

(iii) The site manager wants the curved surface area of the pile to increase at a constant rate of 96 m^2/min. **3**

Find the rate, in terms of r, at which the sand needs to be poured.
(The area of the curved surface of a cone is given by $A = \pi r l$.)

(c) Prove that $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$ for all $x \in [-1, 1]$. **2**

Question 13 continues on page 13

Question 13 continued

(d) Let α, β and γ be the roots of the polynomial equation $x^3 - 2x + 3 = 0$.

(i) Find the value of $\alpha^2 \beta^2 \gamma^2$. **1**

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A 3×3 grid is to be filled with numbers from the set $\{-1, 0, 1\}$, allowing for repetition of numbers. **2**

Prove that among the sums, by rows, columns and diagonals, there are at least two of these sums that are equal.

(b) A curve is defined parametrically by:

$$\begin{cases} x = 2\sin t \\ y = 1 - 2\cos t \end{cases} \qquad \text{for } -\frac{\pi}{2} \le t \le \pi$$

(i) Find the Cartesian equation of the curve. **2**

(ii) By considering the domain and range of each parametric equation above, sketch the curve. **2**

(iii) Find the gradient of the curve at the point where $t = \dfrac{2\pi}{3}$. **2**

(You are given that $\dfrac{d}{d\theta}\sin\theta = \cos\theta$ and $\dfrac{d}{d\theta}\cos\theta = -\sin\theta$.)

(c) (i) Using compound angle results, show that $\sin 3x = 3\sin x - 4\sin^3 x$. **3**

(ii) Consider the equation $\sin 2x = \sin 3x$. You are given that $x = \dfrac{\pi}{5}$ is a solution to this equation. **1**

Show that $x = \dfrac{3\pi}{5}$ is another solution to this equation.

(iii) Hence, or otherwise, find the exact value of $\cos\dfrac{3\pi}{5}$. **3**

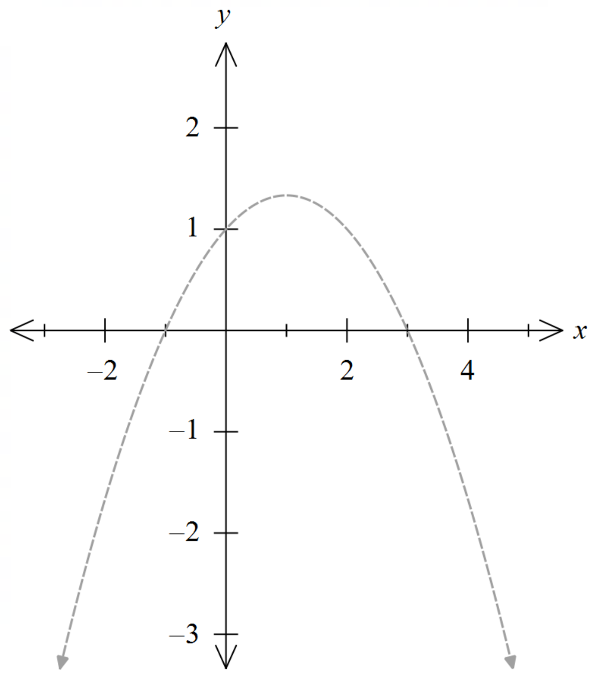
End of Examination

Name: ______________________________ **Teacher: ______________________________**

Answer Sheet for Question 11 b)

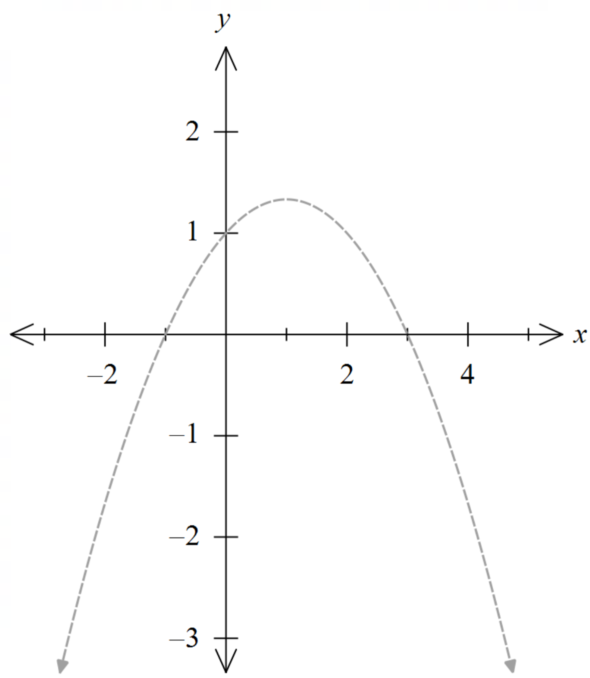
(i)

$$y = |f(x)|$$



(ii)

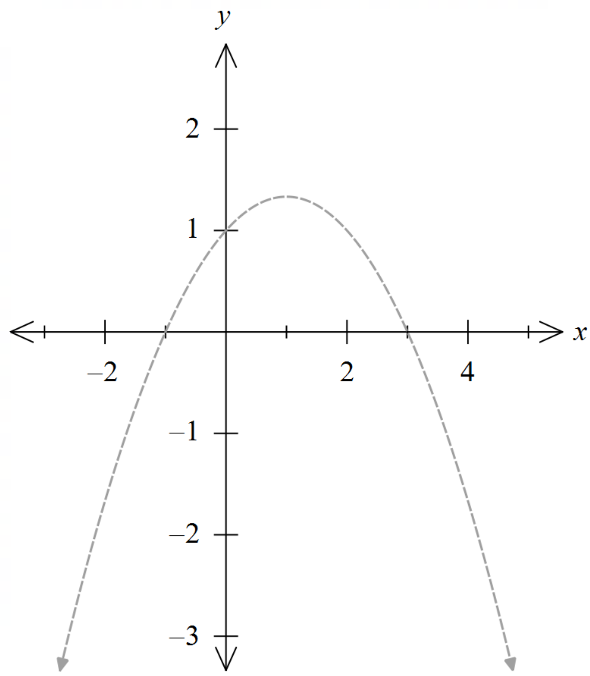
$$y^2 = f(x)$$



Name: ______________________________ **Teacher:** ______________________________

(iii)

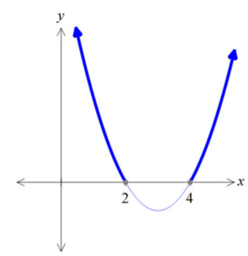
$$y = \frac{1}{f(x)}$$



2022 Year 11 Mathematics Extension 1 Yearly AT3 SOLUTIONS

Suggested Solution (s)	Comments
1) E M B Z L D E Z E $Arrangements = \dfrac{9!}{3!2!} = 30240$ => **C**	
2) $\alpha\beta\gamma\delta = +\dfrac{e}{a} = \dfrac{-6}{2} = -3$ => **B**	
3) $D: -1 \le 4x \le 1$ $-\dfrac{1}{4} \le x \le \dfrac{1}{4} = \left[-\dfrac{1}{4}, \dfrac{1}{4}\right]$ => **D**	
4) By the remainder theorem, $P(-2) = -(-2)^3 - 2(-2)^2 - 3(-2) + 8 = 14$ => **D**	
5) $f(x)$ is even (symmetrical about the y-axis) $\therefore$ either B or D. Also, vertical asymptotes at $x = \pm 1$ tell us that the denominator is either $x^2 - 1$ or $1 - x^2$. Checking the point $x = 2$, we see that $y < -1$. Subbing $x = 2$ into D, we get $y = \dfrac{2^2}{1 - 2^2} = -\dfrac{4}{3} < -1$ => **D**	
6) As the curve is getting steeper from L→R, $\therefore k > 0$. As the curve is decreasing, $A < 0$ => **A**	
7) ${}^{12}C_8 \times {}^{8}P_5 = \dfrac{12!}{8!4!} \times \dfrac{8!}{3!} = \dfrac{12!}{4!3!}$ => **B**	
8) $\alpha\beta\gamma(\alpha+\beta+\gamma) = -\dfrac{d}{a} \times -\dfrac{b}{a} = -\dfrac{-15}{3} \times \dfrac{9}{3} = -15$ => **B**	
9) $y = \sin^{-1}(-\cos x)$ $= -\sin^{-1}(\cos x)$ *as* $\sin^{-1} x$ *is odd* $= -\sin^{-1}\left(\sin\left(\dfrac{\pi}{2} - x\right)\right)$ $= -\left(\dfrac{\pi}{2} - x\right)$ *for* $0 \le x \le \pi$ $= x - \dfrac{\pi}{2}$ $\therefore$ *A or C* At $x = 0$, $y = \sin^{-1}(-\cos 0) = \sin^{-1}(-1) = -\dfrac{\pi}{2}$ => **A**	NB: fn is periodic with period of 2π, so **A** only
10) For the domain of $f(x)$, both $1 + x \ge 0$ and $1 - \sqrt{1+x} \ge 0$ $\sqrt{1+x} \le 1$ $x \ge -1$ and $0 \le 1 + x \le 1$ $-1 \le x \le 0$ $\therefore D: -1 \le x \le 0$ $R: -1 \le y \le 0$ for $f(x)$	

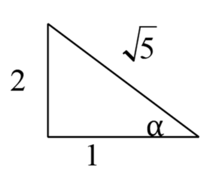
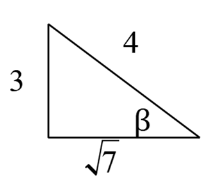
2022 Year 11 Mathematics Extension 1 Yearly AT3 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Swapping D and R for the inverse function, we get $\therefore D: -1 \le x \le 0$ $R: -1 \le y \le 0$ for $f^{-1}(x)$ $f(x)$ and $f^{-1}(x)$ intersect only along the line $y = x$ in this domain, which can be seen 3 times if $y = x$ is drawn on the graph. => **C** <u>Note</u>: if we were asked to find the actual solutions, rather than just the number, it would involve solving $y = x$ and $y = x^4 - 2x^2$ simultaneously, giving $x^4 - 2x^2 - x = 0$ $x\left(x^3 - 2x - 1\right) = 0 \quad \therefore x = 0$ *Let* $g(x) = x^3 - 2x - 1$ $g(-1) = 0 \quad \Rightarrow (x+1)$ *is a factor* $\therefore g(x) = (x+1)\left(x^2 - x - 1\right)$ Solving for the last bracket gives $x^2 - x - 1 = 0$ $x = \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$ $= \dfrac{1 \pm \sqrt{5}}{2}$ $\simeq 1.618...$ *or* $-0.618...$ Thus, for the domain $D: -1 \le x \le 0$ we get the solutions of $x = 0, -1$ *or* $\dfrac{1 - \sqrt{5}}{2}$. **Section II** 11) (a) $x^3 + 2x^2 - 3x - 7 = (x-2)Q(x) + 3$ $Q(x) = \dfrac{x^3 + 2x^2 - 3x - 10}{x - 2}$ $= x^2 + 4x + 5$	 ✓ ✓	(*Part (a) can also be done using polynomial long division*) (b) See attached page. (c) $\dfrac{-x}{x-4} \le 1 \qquad x \ne 4$ $-x(x-4) \le (x-4)^2$ $(x-4)^2 + x(x-4) \ge 0$ $(x-4)(x-4+x) \ge 0$ $(x-4)(2x-4) \ge 0$ $(x-4)(x-2) \ge 0$  $\left.\begin{array}{l}\therefore x \le 2 \text{ or } x > 4 \\ (-\infty, 2] \cup (4, \infty)\end{array}\right\}$ *either format* (d) <u>Method 1</u>: Difference of angles $\sin 15 = \sin(45 - 30)$ $= \sin 45 \cos 30 - \cos 45 \sin 30$ $= \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2}$ $= \dfrac{\sqrt{3} - 1}{2\sqrt{2}} \quad \left(= \dfrac{\sqrt{6} - \sqrt{2}}{4} \right)$ <u>Method 2</u>: Double angle $\cos 2\theta = 1 - 2\sin^2\theta$ $2\sin^2\theta = 1 - \cos 2\theta$ $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$ $\sin\theta = \pm\sqrt{\dfrac{1 - \cos 2\theta}{2}}$	 ✓ for excluding 4 ✓ for multiplying by $(x-4)^2$ ✓ for simplifying down ✓ final answer ✓ ✓ (award mark with irrational denom) ✓

2022 Year 11 Mathematics Extension 1 Yearly AT3 SOLUTIONS

Suggested Solution (s)	Comments
Let $\theta = 15°$. *As* $\sin 15° > 0$	
$\sin 15° = +\sqrt{\dfrac{1-\cos 30°}{2}}$	
$= \sqrt{\dfrac{1-\frac{\sqrt{3}}{2}}{2}}$	
$= \sqrt{\dfrac{2-\sqrt{3}}{4}}$	
$= \dfrac{1}{2}\sqrt{2-\sqrt{3}}$	✓
(e)	
$P(\textit{oldest and youngest together})$	1 for either ways of arranging in a circle or total ways,
$= \dfrac{2\times 8!}{9!}$	
$= \dfrac{2}{9}$	1 for answer
12)	
(a)	
(i)	
$LHS = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$	
Let $t = \tan\theta$	
$LHS = \dfrac{1-\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$	✓
$= \dfrac{1+t^2-(1-t^2)}{1+t^2+1-t^2}$	
$= \dfrac{2t^2}{2}$	
$= t^2$	
$= \tan^2\theta = RHS$	✓

Suggested Solution (s)	Comments
(ii)	
Let $\theta = \dfrac{\pi}{8}$	
$\tan^2\dfrac{\pi}{8} = \dfrac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$	
$= \dfrac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$	
$= \dfrac{\sqrt{2}-1}{\sqrt{2}+1}$	
$= \dfrac{\left(\sqrt{2}-1\right)^2}{2-1}$	
$\tan\dfrac{\pi}{8} = \sqrt{2}-1 \qquad as\ \tan\dfrac{\pi}{8} > 0$	✓ Also accept $\sqrt{3-2\sqrt{2}}$
(b)	
(i)	
$T = 5 + 25e^{-kt}$	
When $t = 1, T = 20$	
$20 = 5 + 25e^{-k}$	✓
$25e^{-k} = 15$	
$e^{-k} = \dfrac{15}{25}$	
$-k = \ln\left(\dfrac{3}{5}\right)$	
$k = \ln\left(\dfrac{5}{3}\right)$	✓
(ii)	
$T = A + Be^{-kt}$	
As $t \to \infty, T \to 22$	
$22 = A + Be^{-\infty}$	
$A = 22$	✓
When $t = 0, T = 30$	

Suggested Solution (s)	Comments
$30 = 22 + Be^0$ $B = 8$ $\therefore T = 22 + 8e^{-kt}$	
At $T = 37$ $37 = 22 + 8e^{-kt}$ $8e^{-kt} = 15$ $e^{-kt} = \frac{15}{8}$ $-kt = \ln\left(\frac{15}{8}\right)$ $t = -\frac{\ln\left(15/8\right)}{\ln\left(5/3\right)}$ $= -1.23057...$ $= -1\ hr\ 13\ \text{min}\ 50\ \text{sec}$	✓
$\therefore 10am - 1\ hr\ 13\ \text{min}\ 50\ \text{sec}$ $= 8:46am$ $\therefore$ at 8:46 am, the body had a temperature of 37°C.	✓
(c) Let $\alpha = \tan^{-1} 2$ and $\beta = \sin^{-1}\left(-\frac{3}{4}\right)$. $\alpha = \tan^{-1} 2 \qquad \beta = \sin^{-1}\left(-\frac{3}{4}\right)$ $\tan\alpha = 2 \qquad \sin\beta = -\frac{3}{4}$ $\therefore \alpha$ is in the first quadrant and β is in the 4th quadrant.	
	
$\therefore \sin\alpha = \frac{2}{\sqrt{5}} \qquad \therefore \sin\beta = -\frac{3}{4}$ $\cos\alpha = \frac{1}{\sqrt{5}} \qquad \cos\beta = \frac{\sqrt{7}}{4}$	✓

Suggested Solution (s)	Comments
$\therefore \cos\left[\tan^{-1}(2) + \sin^{-1}\left(-\frac{3}{4}\right)\right]$ $= \cos(\alpha + \beta)$ $= \cos\alpha\cos\beta - \sin\alpha\sin\beta$	✓
$= \frac{1}{\sqrt{5}} \times \frac{\sqrt{7}}{4} - \frac{2}{\sqrt{5}} \times \frac{-3}{4}$ $= \frac{\sqrt{7}+6}{4\sqrt{5}} \quad \left(= \frac{\sqrt{35}+6\sqrt{5}}{20}\right)$	✓ (award mark for irrational denom)
(d) As $P(x)$ has a root of multiplicity 4 at $x = a$, $P(x) = (x-a)^4 Q(x)$ for some function $Q(x)$ with $Q(a) \neq 0$.	
$P(x) = (x-a)^4 Q(x)$ $P'(x) = 4(x-a)^3 Q(x) + (x-a)^4 Q'(x)$ $= (x-a)^3 \left(4 \times Q(x) + (x-a)Q'(x)\right)$ $= (x-a)^3 Q_1(x)$ where $Q_1(x) = 4 \times Q(x) + (x-a)Q'(x)$ and note that $Q_1(a) \neq 0$.	✓
But here, $(x-a)^3$ is a factor of $P'(x)$ $\therefore P'(x)$ has a root of multiplicity 3 at $x = a$.	✓
(e) Committees including Alan or Diane but not both = committees with A only + committees with D only. $= {}^1C_1 \times {}^7C_1 \times {}^{11}C_4 + {}^7C_2 \times {}^1C_1 \times {}^{11}C_3$	1 either expression
$= 5775$	✓

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

13)

(a)

(i)

$$P(x) = x^{n+1} - (n+1)x + n$$

$$P(1) = 1^{n+1} - (n+1) + n$$
$$= 1 - n - 1 + n$$
$$= 0$$

$\therefore P(x)$ has a root at $x = 1$. ✓

Also,

$$P(x) = x^{n+1} - (n+1)x + n$$
$$P'(x) = (n+1)x^{n+1} - (n+1)$$
$$P'(1) = (n+1)1^{n+1} - (n+1)$$
$$= (n+1) - (n+1)$$
$$= 0$$

$\therefore P(x)$ has a double root at $x = 1$ ✓

(ii)

When $n = 3$, $P(x) = x^4 - 4x + 3$

From (i), $(x-1)^2$ is a factor of $P(x)$

$$\begin{array}{r} x^2 + 2x + 3 \\ x^2 - 2x + 1 \overline{\smash{)}\, x^4 + 0x^3 + 0x^2 - 4x + 3} \\ -\quad x^4 - 2x^3 + x^2 \quad\quad\;\; \\ \hline 2x^3 - x^2 - 4x \quad\; \\ -\quad 2x^3 - 4x^2 + 2x \quad\; \\ \hline 3x^2 - 6x + 3 \\ -\quad 3x^2 - 6x + 3 \\ \hline 0 \end{array}$$

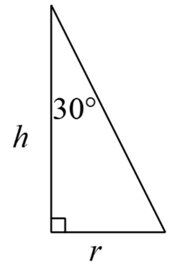
1 dividing by $(x-1)^2$

$\therefore P(x) = (x-1)^2(x^2 + 2x + 3)$ ✓

(Note: this doesn't factorise further)

(b)

(i)



$$\tan 30 = \frac{r}{h}$$

$$h = \frac{r}{\tan 30}$$
$$= r\sqrt{3}$$

$$V_{cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi r^2 \times r\sqrt{3}$$
$$= \frac{\sqrt{3}}{3}\pi r^3$$ ✓

(ii)

$$V = \frac{\sqrt{3}}{3}\pi r^3$$

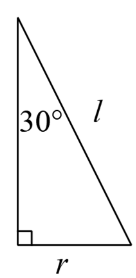
$$\frac{dV}{dr} = \sqrt{3}\pi r^2$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{\sqrt{3}\pi r^2} \times 60\pi$$ ✓
$$= \frac{20\sqrt{3}}{r^2}$$

When $r = 15$,

$$\frac{dr}{dt} = \frac{20\sqrt{3}}{15^2}$$
$$= \frac{4\sqrt{3}}{45} m/\min$$ ✓

(iii)



$$\sin 30 = \frac{r}{l}$$

$$l = \frac{r}{\sin 30}$$
$$= 2r$$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
$A = \pi r l$ $= \pi r \times 2r$ $= 2\pi r^2$ $\frac{dA}{dr} = 4\pi r$	✓	$\alpha^3 - 2\alpha + 3 = 0$. Similarly, we get $\beta^3 - 2\beta + 3 = 0$ and $\gamma^3 - 2\gamma + 3 = 0$. Summing these gives:	1 for seeing that the roots satisfy the equation
$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dA} \times \frac{dA}{dt}$ $= \sqrt{3}\pi r^2 \times \frac{1}{4\pi r} \times 96$	1 progress	$\alpha^3 - 2\alpha + 3 = 0$ $\beta^3 - 2\beta + 3 = 0$ $+ \quad \gamma^3 - 2\gamma + 3 = 0$ $\alpha^3 + \beta^3 + \gamma^3 - 2(\alpha+\beta+\gamma) + 9 = 0$ $\alpha^3 + \beta^3 + \gamma^3 = 2(\alpha+\beta+\gamma) - 9$ $= 2(0) - 9$	
$= 24\sqrt{3}r$	✓	$= -9$	✓
(c) Let $\alpha = \sin^{-1} x$ $x = \sin\alpha \qquad with -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ $= \cos\left(\frac{\pi}{2} - \alpha\right) \quad and\ 0 \le \frac{\pi}{2} - \alpha \le \pi$	✓ (no penalty if restrictions not noted)	Method 2: Algebraic approach $\alpha^3 + \beta^3 + \gamma^3$ $= (\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\alpha\gamma) + 3\alpha\beta\gamma$	1 algebraic manipulat-ion
$\cos^{-1} x = \frac{\pi}{2} - \alpha$		$= (0)^3 - 3(0)(-2) + 3(-3)$ $= -9$	✓
$\alpha + \cos^{-1} x = \frac{\pi}{2}$ $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$	✓	14) (a) Considering the possible sums, we can get any value from -3 to +3 inclusive. Thus, there are 7 possible sum values (our pigeonholes). On the 3×3 grid, there are 3 rows, 3 columns and 2 diagonals, a total of 8 (our pigeons).	1 for either the pigeons or holes clearly identified
Note: you could equivalently begin with "Let $\alpha = \cos^{-1} x$" etc (d) (i) $\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2$ $= \left(\frac{-3}{1}\right)^2$		$\therefore$ by the pigeonhole principle, among the rows, columns and diagonals, there must be two sums that are the equal.	✓
$= 9$	✓		
(ii) Method 1: Roots as solutions As α, β and γ are roots of the polynomial, we get that		(b) (i) $x = 2\sin t \qquad y = 1 - 2\cos t$ $\sin t = \frac{x}{2} \qquad \cos t = \frac{1-y}{2}$	

2022 Year 11 Mathematics Extension 1 Yearly AT3 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

$$\sin^2 t = \frac{x^2}{4} \qquad \cos^2 t = \frac{(1-y)^2}{4}$$

1 for either equation

$$\sin^2 t + \cos^2 t = 1$$

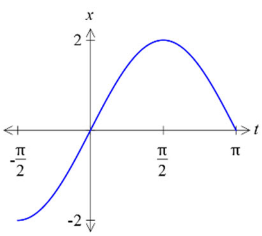
$$\frac{x^2}{4} + \frac{(1-y)^2}{4} = 1$$

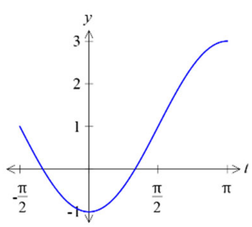
$$x^2 + (y-1)^2 = 4$$

✓

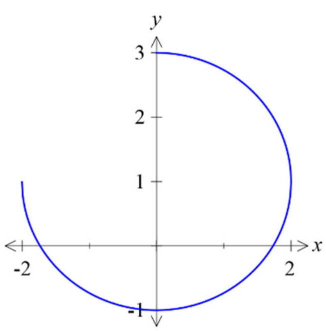
(i)
From (i), the curve is part of a circle, centre $(0,1)$, radius 2.

$x = 2\sin t$ $\qquad$ $y = 1 - 2\cos t$





$\therefore$ the curve starts at $(-2,1)$ at $t = -\dfrac{\pi}{2}$, then moves to $(0,-1)$ at $t = 0$, then $(2,1)$ at $t = \dfrac{\pi}{2}$, and stops at $(0,3)$ when $t = \pi$.



1 correct base curve

1 correct section of the curve

(ii)

$x = 2\sin t$ $\qquad$ $y = 1 - 2\cos t$

$\dfrac{dx}{dt} = 2\cos t$ $\qquad$ $\dfrac{dy}{dt} = 2\sin t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin t}{2\cos t} = \tan t$$

✓

When $t = \dfrac{2\pi}{3}$,

$$\frac{dy}{dx} = \tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}$$

✓

(c)

(i)

$$\begin{aligned} LHS &= \sin 3x \\ &= \sin(2x + x) \\ &= \sin 2x \cos x + \sin x \cos 2x \quad ✓ \\ &= 2\sin x \cos^2 x + \sin x(1 - 2\sin^2 x) \quad ✓ \\ &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x \quad ✓ \\ &= RHS \end{aligned}$$

(ii)
<u>Method 1</u>: Numerical Evaluation

$$\sin\left(2 \times \frac{3\pi}{5}\right) \approx -0.587785252...$$

$$\sin\left(3 \times \frac{3\pi}{5}\right) \approx -0.587785252...$$

$\therefore x = \dfrac{3\pi}{5}$ is a solution of $\sin 2x = \sin 3x$ ✓

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
<u>Method 2</u>: Simplifying expressions $\sin\left(2\times\frac{3\pi}{5}\right)=\sin\frac{6\pi}{5}$ $=\sin\left(\pi+\frac{\pi}{5}\right)$ $=-\sin\left(\frac{\pi}{5}\right)$ $\sin\left(3\times\frac{3\pi}{5}\right)=\sin\frac{9\pi}{5}$ $=\sin\left(2\pi-\frac{\pi}{5}\right)$ $=-\sin\left(\frac{\pi}{5}\right)$ $\therefore x=\frac{3\pi}{5}$ is a solution of $\sin 2x=\sin 3x$	✓ (showing both equal)	Now consider $2x+3x$ when $x=\frac{3\pi}{5}$. $2\left(\frac{3\pi}{5}\right)+3\left(\frac{3\pi}{5}\right)$ $=\frac{6\pi}{5}+\frac{9\pi}{5}$ $=3\pi$ $\therefore x=\frac{3\pi}{5}$ is a solution of $\sin 2x=\sin 3x$ *(iii) is over the page.*	✓
<u>Method 3</u>: Generalising solutions The sines of two angles are equal if the angles are supplementary. $\sin x=\sin y$ *where* $y=\pi-x$ Also, if $\sin x=\sin y$, then $\sin x=\sin(y+2\pi k)$ for some $k\in\mathbb{Z}$, and in particular $\sin x=\sin(\pi-y+2\pi k)$ using the above. Therefore, $x=\pi-y+2\pi k$, or $x+y=(2k+1)\pi$. Thus, if the two angles sum to an odd integer multiple of π, their sines will be the same.	✓		

Suggested Solution (s)	Comments
(iii) $$\sin 2x = \sin 3x$$ $$2\sin x\cos x = 3\sin x - 4\sin^3 x \quad \textit{from } (i)$$ $$4\sin^3 x + 2\sin x\cos x - 3\sin x = 0$$ $$\sin x\left(4\sin^2 x + 2\cos x - 3\right) = 0$$ We know that $x = \frac{3\pi}{5}$ is not a solution of $\sin x = 0$, so it must be a solution of the bracket.	
$$4\sin^2 x + 2\cos x - 3 = 0$$	✓
$$4\left(1-\cos^2 x\right) + 2\cos x - 3 = 0$$ $$4 - 4\cos^2 x + 2\cos x - 3 = 0$$ $$4\cos^2 x - 2\cos x - 1 = 0$$ $$\cos x = \frac{2 \pm \sqrt{2^2 - 4(4)(-1)}}{8}$$ $$= \frac{2 \pm \sqrt{20}}{8}$$ $$= \frac{2 \pm 2\sqrt{5}}{8}$$	
$$= \frac{1 \pm \sqrt{5}}{4}$$	✓
Now, we know from (ii) that $x = \frac{\pi}{5}$ and $x = \frac{3\pi}{5}$ are solutions of this equation, and hence correspond to the two above solutions. But $\cos\frac{\pi}{5} > 0$ as $\frac{\pi}{5}$ is in the first quadrant, and $\cos\frac{3\pi}{5} < 0$ as $\frac{3\pi}{5}$ is in the second quadrant.	
$$\therefore \cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$$	✓

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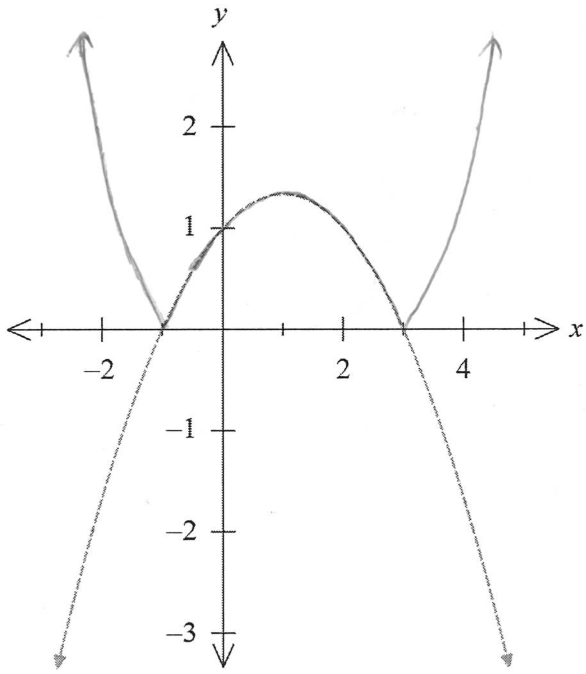
Answer Sheet for Question 11 b)

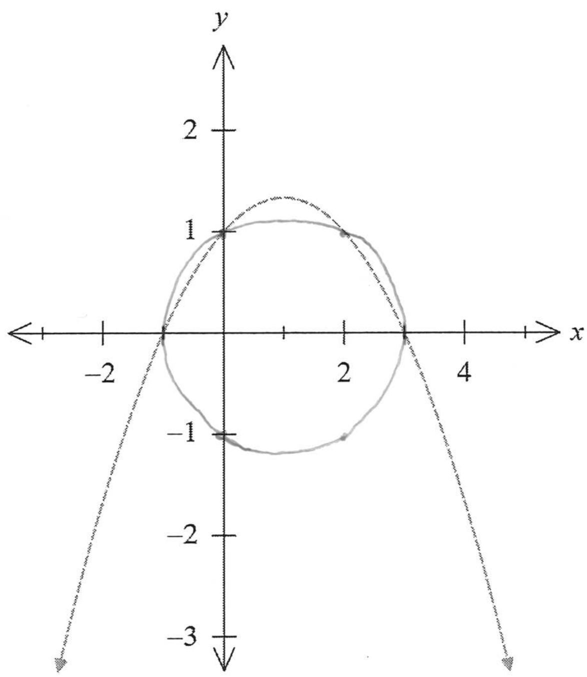
(i)

$y = |f(x)|$



(ii)

$y^2 = f(x)$



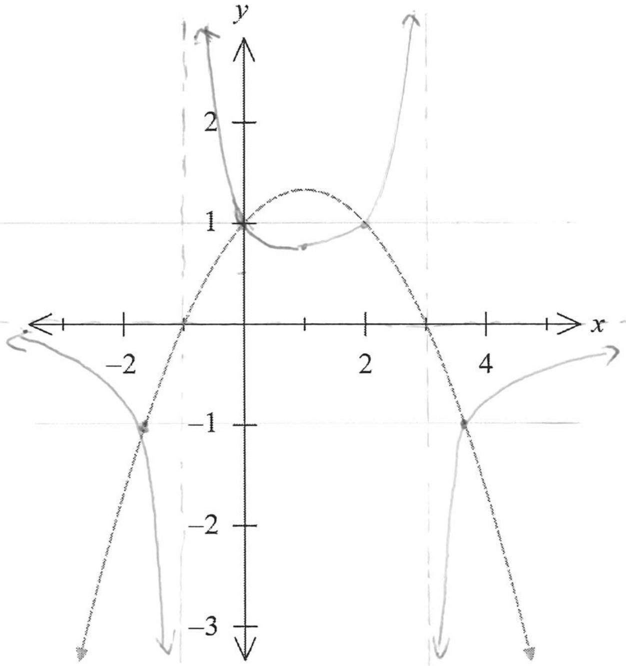
① Domain of function correct

① shape, including below x-axis.

Name: ______________________________ Teacher: ______________________________

(iii)

$$y = \frac{1}{f(x)}$$



① correct asymptotes

① shape

① correct points at $y = \pm 1$