Name: ____________________

Teacher: ____________________

KNOX GRAMMAR SCHOOL

2023
Year 11 Yearly Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Section I: 10 marks (pages 3 - 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II: 60 marks (pages 7 -10)

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section

Teachers:
Mr Meli
Mr Willcocks (Examiner)
Mr Menzies
Mr Cheah
Mr Vuletich

Total marks: 70

This paper MUST NOT be removed from the examination room

Write your full name and your teacher's name on the front cover of each writing booklet

Number of Students in Course: 123

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Section I
Questions 1 – 10 (1 mark for each question)

Read each question and choose an answer A, B, C or D.
Record your answers on the Answer Sheet provided.
Allow about 15 minutes for this section.

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1 What is the domain of the function $f(x) = \sin^{-1}\dfrac{x}{\pi}$?

A. $[-\pi , \pi]$

B. $[-1 , 1]$

C. $(-\pi , \pi)$

D. $\left[-\dfrac{\pi}{2} , \dfrac{\pi}{2}\right]$

2 Consider $g(x) = \dfrac{5}{8x^3} - 2$.

What is the value of $g^{-1}(3)$?

A. 2

B. 8

C. $\dfrac{1}{2}$

D. $\dfrac{1}{8}$

3 Given that $\tan\alpha = \dfrac{a}{b}$, find the value of $\tan 2\alpha$.

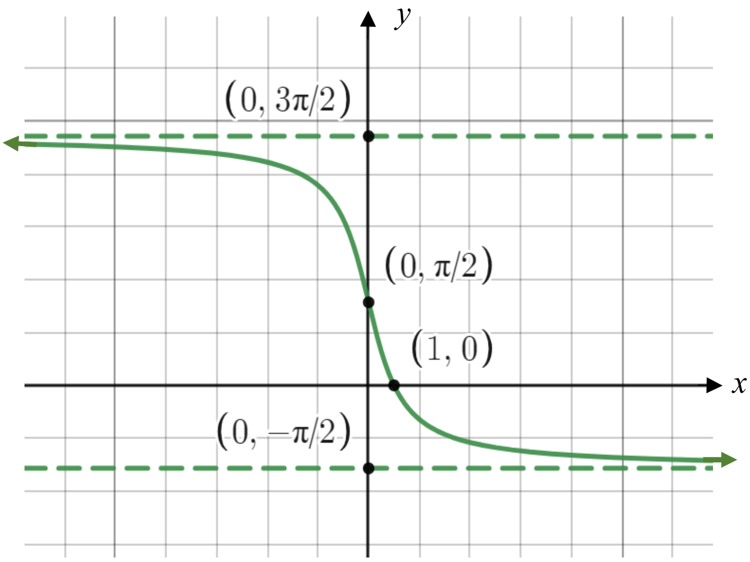
A. $\dfrac{2a}{b}$

B. $\dfrac{2ab}{a^2 - b^2}$

C. $\dfrac{2a}{a^2 - b^2}$

D. $\dfrac{2ab}{b^2 - a^2}$

4 The diagram shows the graph of the function $y = f(x)$.



Which of the following could be the equation of the function $y = f(x)$?

A. $y = \dfrac{\pi}{2} + 2\tan^{-1} x$

B. $y = \pi - 2\tan^{-1} x$

C. $y = \dfrac{\pi}{2} - \tan^{-1} x$

D. $y = \dfrac{\pi}{2} - 2\tan^{-1} x$

5 Given that $\sin^{-1} c = d$ where $-\dfrac{\pi}{2} \le d \le 0$, which of the following could be the value of $\cos^{-1} c$?

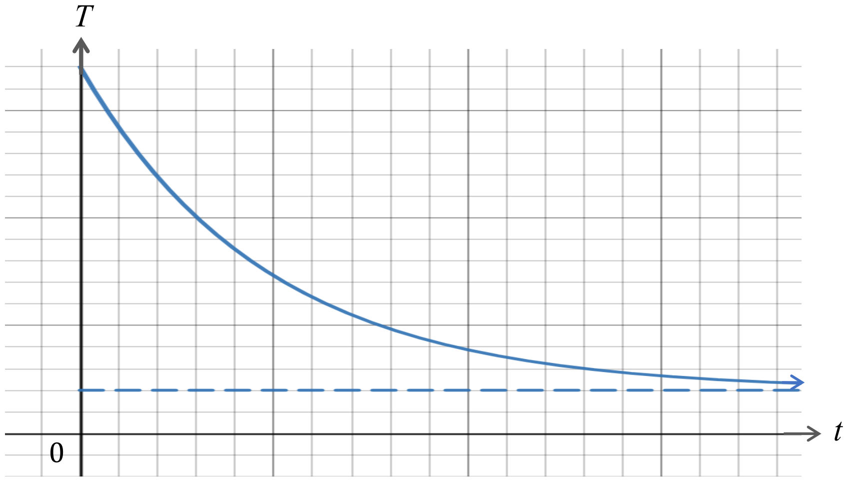
A. $\dfrac{\pi}{8}$

B. $-\dfrac{\pi}{8}$

C. $\dfrac{9\pi}{8}$

D. $\dfrac{7\pi}{8}$

6 A hot iron bar is placed on a table in cold room. Over a period of time the temperature T of the bar decreases at a decreasing rate, as modelled by the graph below.



Which equation best represents this graph?

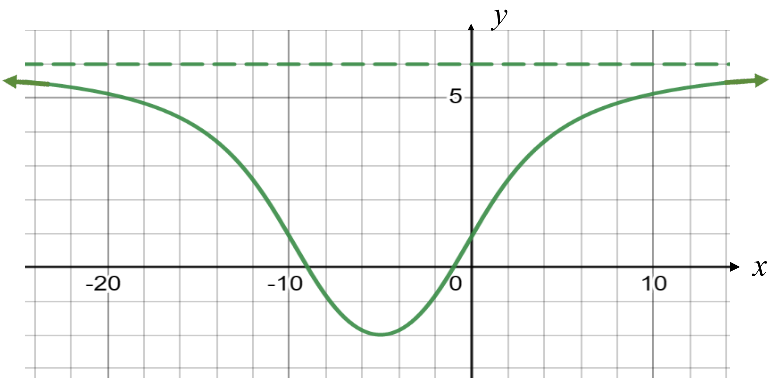
A. $\mathrm{T}(t)=170-20\times 2^{0.03t}$

B. $\mathrm{T}(t)=20+100\times 2^{0.03t}$

C. $\mathrm{T}(t)=170-150\times 2^{-0.03t}\ t$

D. $\mathrm{T}(t)=20+150\times 2^{-0.03t}$

7 The diagram shows the graph of the function $y=f(x)$.



What is the range of $y=\dfrac{1}{f(x)}$?

A. $\left(-\infty,-\dfrac{1}{2}\right]\cup\left(\dfrac{1}{6},\infty\right)$

B. $(-\infty,-2)\cup\left[\dfrac{1}{6},\infty\right)$

C. $(-\infty,-2)\cup[6\ ,\ \infty)$

D. $\left(-\infty,-\dfrac{1}{2}\right)\cup\left(\dfrac{1}{6},\infty\right)$

8 The polynomial $H(x)$ has a degree 5. When $H(x)$ is divided by another polynomial $G(x)$ the remainder is $x^2 - x + 2$. Which of the following is true about the degree of the polynomial $K(x) = H(x)\left(G(x)\right)^2$?

A. The degree could be 7, 8, 9 or 10.

B. The degree could be 11, 13 or 15.

C. The degree could be 9, 11, 13 or 15.

D. The degree could be 13, 15 or 17.

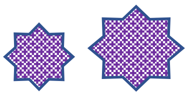
9 The numbers 1, 2, 4, 6, 8, 10, 15, 20, 60, 120 are written on cards and placed in a bag. What is the smallest number of cards we need to take from the bag to make sure we get at least 2 pairs of numbers that have a product of 120?

A. 4

B. 6

C. 8

D. 9

10 Each one of these two stars  is to be each placed at random on a vertex of an octagon.

What is the probability that these two stars will be placed on a diagonal?

A. $\frac{5}{7}$

B. $\frac{1}{4}$

C. $\frac{1}{10}$

D. $\frac{1}{14}$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Solve $|x-3| \le 7$. **1**

(b) The polynomial $P(x) = 2x^3 + ax + 4$ has a zero at $x = -2$. **2**
Find the value of a.

(c) Solve the inequality $\dfrac{1}{x-2} \le 2$. **3**

(d) (i) Sketch the polynomial $y = x(x+3)(x-2)^2$. **2**

(ii) Hence, solve the inequation $x(x+3)(x-2)^2 \ge 0$. **1**

(e) In how many ways can all of the letters of the word PYTHAGORAS be arranged in a circle so that the consonants stay together? The consonants are the letters P, Y, T, H, G, R and S. **2**

(f) Simplify $\dfrac{1-\cos 2x}{2\cos^2 x}$. **2**

(g) Find the exact value of $\dfrac{2\sin 20°}{\cos 110° - \cos 70°}$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) By using $t = \tan\frac{\theta}{2}$, solve the equation $\sin\theta + \sqrt{3}\cos\theta = 1$ for $0 \le \theta \le 2\pi$. **3**

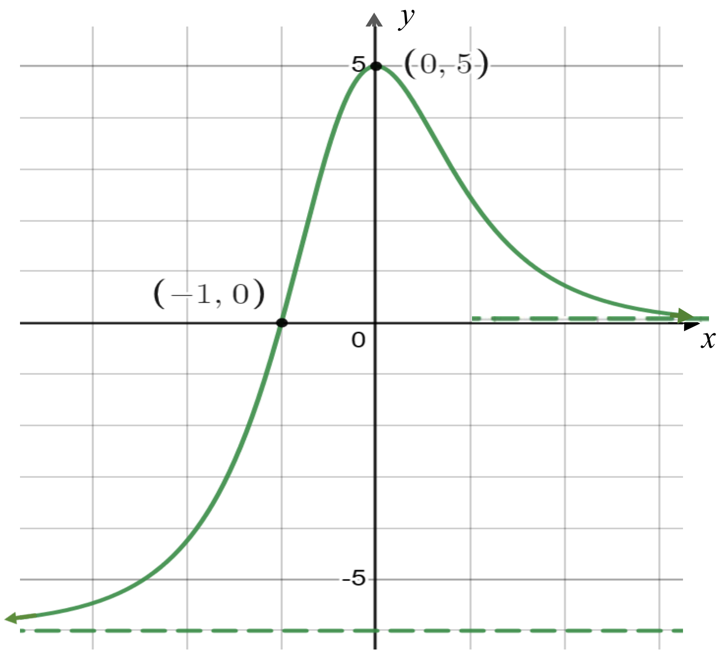
(b) The letters of the word MONKEYS are to be arranged in a row. **2**

Find the number of ways that the letters K and E will be separated by at least 2 letters.

(c) The equation $8x^4 + 28x^3 - 54x^2 + 29x - 5 = 0$ has a root of multiplicity 3. **3**

Find the roots of this equation.

(d) The diagram below shows the graph of $y = f(x)$.



On the separate page provided at the end of this paper, draw the following functions, showing all asymptotes and intercepts and submit it with the rest of your solutions to Question 12.

(i) $y^2 = f(x)$ **2**

(ii) $|y| = f(|x|)$ **2**

(e) The radius of a sphere is increasing at the rate of 0.05 cm s^{-1}. **3**

Find the rate at which its volume is increasing when the volume is 288π cm^3

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the exact value of $\sin\left(\cos^{-1}\dfrac{1}{3}\right)$. **2**

(b) There are n swimmers in a swimming club. **3**

The trainer is to select a group of four or five of these swimmers to swim in a competition.

The number of different groups of five swimmers he can select is twice the number of different groups of four swimmers.

Find n, the number of swimmers in the swimming club.

(c) In a school athletic carnival 77 students are randomly divided into 3 groups, **2**

where each group has a different number of students.

What is the maximum number of students that the **smallest group** can have?

(d) Using at least a third of a page, sketch the graph of the function $f(x)=\dfrac{4}{\pi}\cos^{-1}2x$. **2**

(e) Consider the functions $f(x)=x\sqrt{x}$ and $g(x)=e^{bx}$ in the domain $x>0$. **3**

and b is a positive constant. Let $h(x)=f(g(x))$.

Given that $h^{-1}(e)=6b$, find the value of b.

(f) A curve has parametric equations $x=1+\cos\theta$ and $y=\cos 2\theta$.

(i) Find the Cartesian equation of this curve. **1**

(ii) Determine the co-ordinates of the stationary point of this curve. **1**

(iii) What is the domain of this function. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A class of fifteen mathematics students are seated around a circular table to discuss mathematical problems.

(i) In how many different ways can the students be arranged? **1**

(ii) If the seats are randomly assigned, find the probability that four particular students, Sam, Toby, Euan and Will, are **not** all seated together as a group of four? Give your answer in the simplest form. **2**

.

(b) The displacement x metres of a particle moving along a straight line at a time t seconds is given by $x = \dfrac{10}{e^{t-2} + e^{-t+2}}$ **3**

Find the maximum **positive** displacement of the particle.

(c) (i) Show that $2\sin(k-1)\theta\cos\theta - \sin(k-2)\theta = \sin k\theta$. **2**

(ii) Hence show that $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \sin 5\theta$. **3**

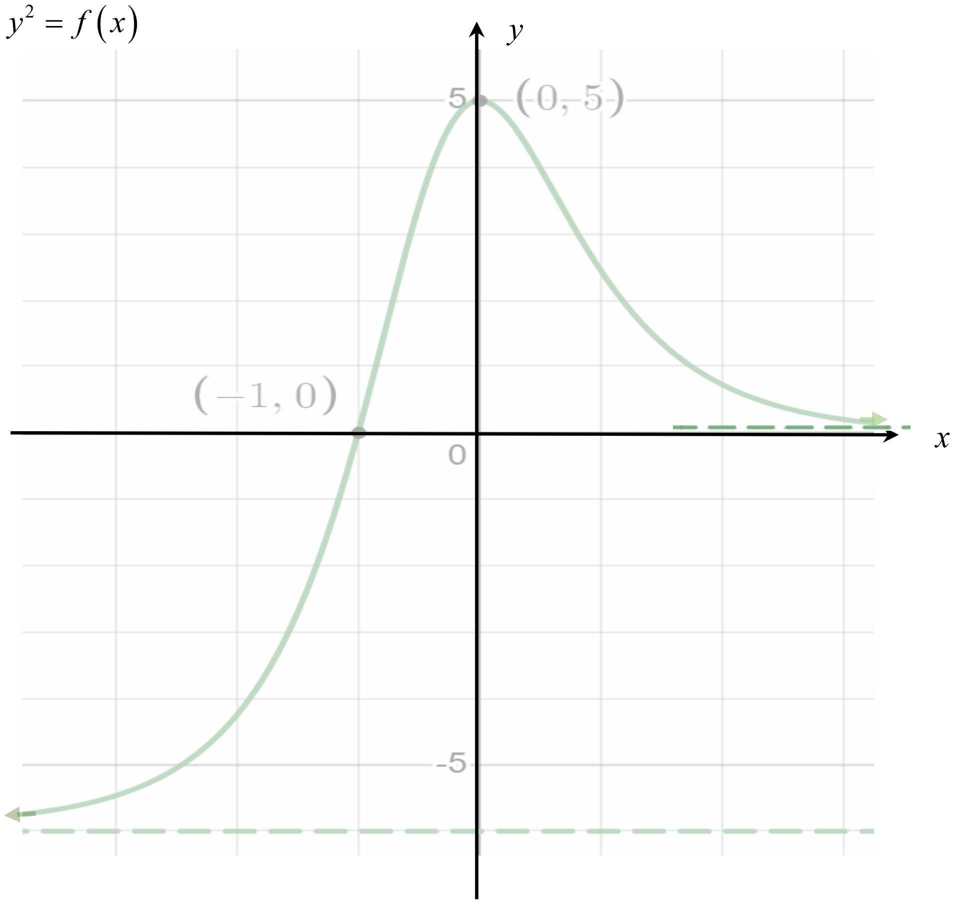
(iii) By letting $x = \sin\theta$ in the equation $16x^4 - 20x^2 + 5 = 0$ and by considering the roots of the equation formed prove that $\sin^2\dfrac{\pi}{5} + \sin^2\dfrac{2\pi}{5} = \dfrac{5}{4}$. **4**

End of Examination

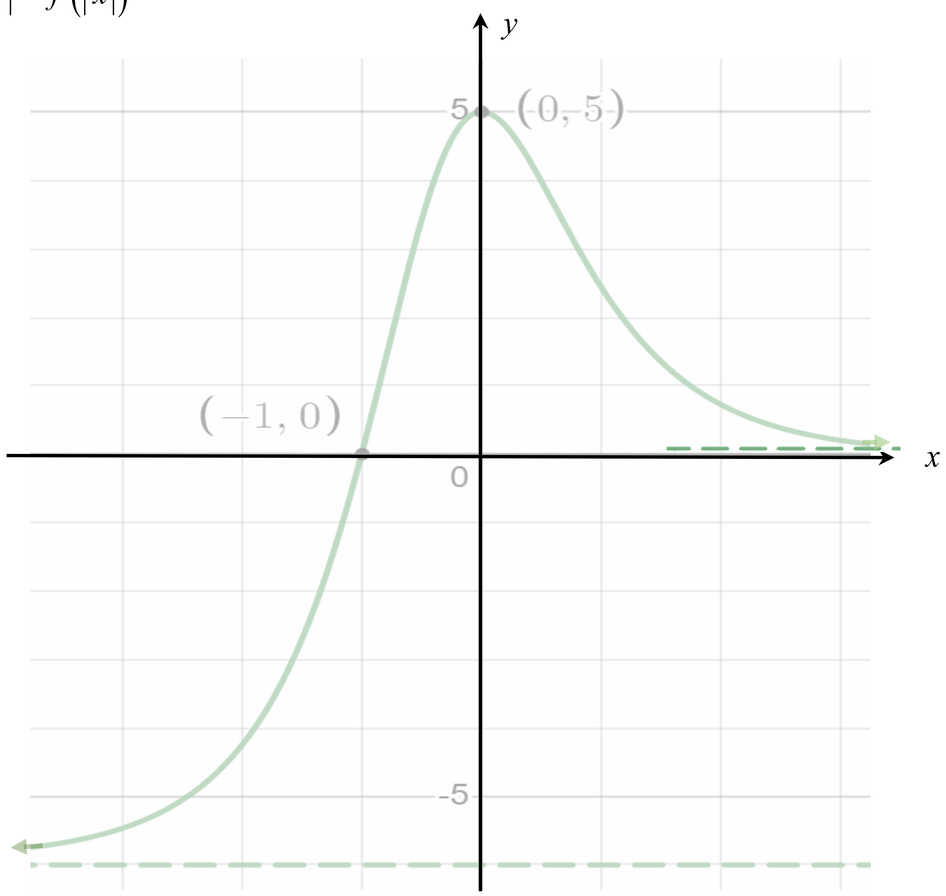
Name: ______________________________ **Teacher:** ______________________________

Answer Sheet for Question 12 (d)

(i) $y^2 = f(x)$



(ii) $|y| = f(|x|)$



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Suggested Solution(s)

1. **A**

The function $f(x)=\sin^{-1}x$ has a domain $-1\le x\le 1$

So $f(x)=\sin^{-1}\dfrac{x}{\pi}$ has a domain of $-1\le\dfrac{x}{\pi}\le 1$ which is $-\pi\le x\le\pi$ which can also be expressed as $[-\pi,\pi]$

Hence, the correct option is **A**.

2. **C**

To find $g^{-1}(3)$, we need to find x that makes $g(x)=3$.

This means $3=\dfrac{5}{8x^3}-2$ that is $\dfrac{5}{8x^3}=5$

Therefore, $8x^3=1$ So, $x=\dfrac{1}{2}$

Hence, $g^{-1}(3)=\dfrac{1}{2}$.

Alternative method

To find the inverse of $y=\dfrac{5}{8x^3}-2$

we interchange x and y, we get

$x=\dfrac{5}{8y^3}-2$ this means

$x+2=\dfrac{5}{8y^3}$ that is $\dfrac{x+2}{5}=\dfrac{1}{8y^3}$

$y^3=\dfrac{5}{8(x+2)}$ So, $y=\sqrt[3]{\dfrac{5}{8(x+2)}}$

Therefore, $g^{-1}(x)=\sqrt[3]{\dfrac{5}{8(x+2)}}$.

$$Now,\ g^{-1}(3)=\sqrt[3]{\frac{5}{8(3+2)}}=\frac{1}{2}$$

Hence, the correct option is **C**.

3. **D**

By substituting $\tan\alpha=\dfrac{a}{b}$ in

$\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}$, we get

$\tan 2\alpha=\dfrac{2\times\dfrac{a}{b}}{1-\dfrac{a^2}{b^2}}$ that is

$$\tan 2\alpha=\frac{\dfrac{2a}{b}}{1-\dfrac{a^2}{b^2}}\times\frac{b^2}{b^2}$$

$$\tan 2\alpha=\frac{2ab}{b^2-a^2}$$

Hence, the correct option is **D**.

4. **D**

An efficient method is to test the equations by substituting $x=1$ into each of the four options.

In option A, we get

$y=\dfrac{\pi}{2}+2\times\dfrac{\pi}{4}=\pi\ne 0$ this shows that A is not the correct equation.

In option B, we get

$y=\pi-2\times\dfrac{\pi}{4}=\dfrac{\pi}{2}\ne 0$ this shows that B is not the correct equation.

In option C, we get

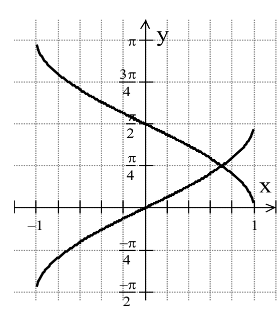
$y=\dfrac{\pi}{2}-\dfrac{\pi}{4}=\dfrac{\pi}{4}\ne 0$ this shows that C is not the correct equation.

In option D, we get

$y=\dfrac{\pi}{2}-2\times\dfrac{\pi}{4}=0$ this shows that D is the correct equation.

Hence, the only possible correct option is **D**.

5. **D**



When $x\to\pm\infty$, $f(x)\to 6$ and $y=\dfrac{1}{f(x)}\to\dfrac{1}{6}$

As $-\frac{\pi}{2} \le d \le 0$ then $-1 \le c \le 0$ and

from the graph we can see that for this range of values of c that $\frac{\pi}{2} < \cos^{-1} c < \pi$.

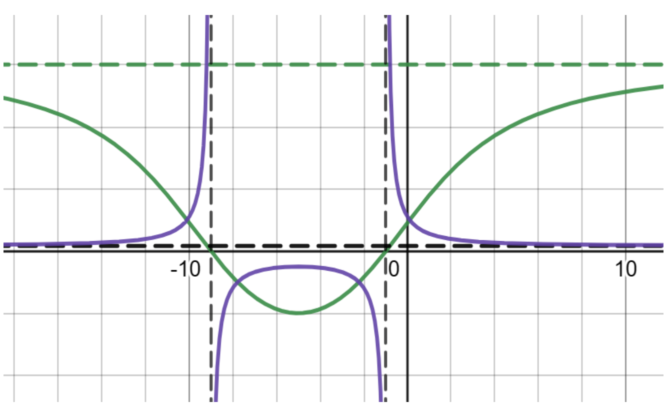
This indicates that the only valid option is $\frac{7\pi}{8}$.

Hence, the correct option is **D**.

6. **D**

The shape of the graph represents a decreasing exponential graph concaving up.
In option A, the graph of the equation will be a decreasing exponential graph concaving down.
So, it is invalid.
In option B, the graph of the equation will be an increasing exponential graph concaving up.
So, it is invalid.
In option C, the graph of the equation will be an increasing exponential graph concaving down.
So, it is invalid.
In option D, the graph of the equation will be a decreasing exponential graph concaving up.
So, it is valid.
Hence, the correct option is D.

7. **A**



When $x = 5$, $f(x) = -2$ *and* $y = \frac{1}{f(x)} = -\frac{1}{2}$.

When $x = -1$ *or* -9, $f(x) = 0$ *and* $y = \frac{1}{f(x)} \to \pm\infty$

and this indicates $x = -1$ *and* -9 are vertical asymptotes.

The purple curve shown in the graph above is $y = \frac{1}{f(x)}$

Its range is $y \le -\frac{1}{2}$, $y > \frac{1}{6}$ this can be expressed as $\left(-\infty, -\frac{1}{2}\right] \cup \left(\frac{1}{6}, \infty\right)$. Hence, the correct option is **A**.

8. **B**

When a polynomial $H(x)$ is divided by another polynomial $G(x)$, the degree of the remainder must be less than the degree of $G(x)$. As the degree of the remainder is 2, the degree of $G(x)$ must be 3, 4, or 5.

So, the degree of $\left[G(x)\right]^2$ must be 6, 8 or 10 and therefore the degree of $H(x)\left[G(x)\right]^2$ would be $5+6=11$, $5+8=13$ *or* $5+10=15$.

Hence, the correct option is **B**.

9. **C**

The pairs that produce a product of 120 are 1 and 120, 2 and 60, 6 and 20, 8 and 15. So, of the ten numbers eight can be used to create a product of 120 and the two remaining that are 4 and 10 do not. If these numbers are selected first, and then 1 of each pair are selected, there would be 6 selected without a pair creating a product of 120. The next 2 selected would have to combine with one of the previously selected numbers to create a product of 120. This means that the least number of cards that would need to be selected to get at least 2 pairs that have a product of 120 is 8.
Hence, the correct option is **C**.

10. **A**

There are $\binom{8}{2} = 28$ different ways to arrange the two stars onto the 8 available vertices of an octagon.
Also, there are 20 diagonals in an octagon.

This can be determined by drawing the diagonals or by subtracting 8 from the total 28 ways as they are sides. Therefore, the probability that the two stars will lie on a diagonal is $\frac{20}{28} = \frac{5}{7}$.

Hence, the correct option is **A**.

Question 11

(a) $|x-3| \le 7$ this means $-7 \le x-3 \le 7$

that is $-4 \le x \le 10$.✓

(b) As $P(x) = 2x^3 + ax + 4$ has a zero at

$x = -2$ *then* $P(-2) = 0$. ✓ *or showing substitution*

By substituting $x = -2$, we get

$-16 - 2a + 4 = 0$

that is $-12 = 2a$ Hence, $a = -6$. ✓

(c) To solve $\dfrac{1}{x-2} \le 2$, we multiply both sides by the positive factor $(x-2)^2$ and, a s it is positive, then inequality holds, we get

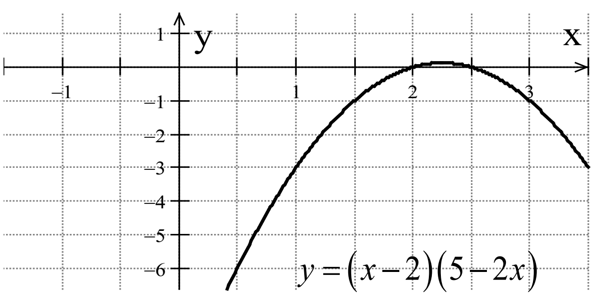
$x - 2 \le 2(x-2)^2$ where $x \ne 2$

$(x-2) - 2(x-2)^2 \le 0$ that is

$(x-2)(1-2(x-2)) \le 0$

$(x-2)(5-2x) \le 0$ ✓

By graphing the parabola $y = (x-2)(5-2x)$,



we can see that the solution of $\dfrac{1}{x-2} \le 2$ occurs when the y values on the graph are negative or zero, except

$x = 2$, that is $x < 2$ ✓ or $x \ge 2\frac{1}{2}$. ✓ *one each*

Alternative method

$\dfrac{1}{x-2} \le 2$ this means $\dfrac{1}{x-2} - 2 \le 0$ that is

$\dfrac{1-2(x-2)}{x-2} \le 0$ So, $\dfrac{5-2x}{x-2} \le 0$.

		2		$2\frac{1}{2}$	
$5-2x$	+	+	+	0	–
$x-2$	–	0	+	+	+
$\dfrac{2-x}{x-1}$	–	undefined	+	0	–

✓ *correct table*

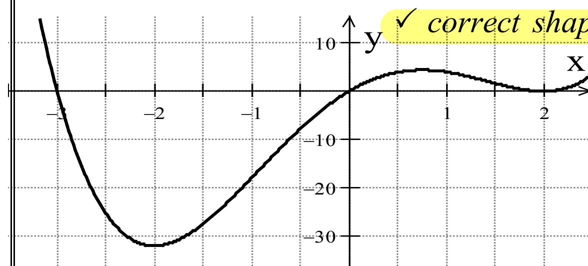
From the table we can see that the solution is

is $x < 2$ ✓ or $x \ge 2\frac{1}{2}$. ✓ *one each*

(d) (i) $y = x(x+3)(x-2)^2$ will cut the x axis at $x = 0, -3$ *or* 2. It has a turning point on the x axis at $x = 2$.

✓ *x − intercepts & turning pt*

✓ *correct shape*



(ii) From the graph the solution of

$x(x+3)(x-2)^2 \ge 0$ occurs when $y \ge 0$

that is $x \le -3$ or $x \ge 0$.✓ *both*

(e) 7 consonants, 3 vowels equates to 4 objects $= 3!$ ✓

7! ways of arranging the consonants and dividing by 2.

15120✓ *only one bald answer*

(f) $\dfrac{1-\cos 2x}{2\cos^2 x} = \dfrac{1-(2\cos^2 x - 1)}{2\cos^2 x}$ ✓ *identity*

$= \dfrac{2-2\cos^2 x}{2\cos^2 x}$

$= \dfrac{2(1-\cos^2 x)}{2\cos^2 x}$

$= \dfrac{2\sin^2 x}{2\cos^2 x}$

$= \tan^2 x$ ✓

(g) $\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$

that is $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$.

Consider $\cos 110°-\cos 70°$.

Let $A+B=110°$ (1) and $A-B=70°$ (2)

By adding (1) and (2), we get

$2A=180°$ that is $A=90°$ and $B=20°$.

So, $\cos 110°-\cos 70°=-2\sin 90°\sin 20°$ ✓

$=-2\sin 20°$ (3)

By substituting (3), in the given fraction we get

$$\frac{2\sin 20°}{\cos 110°-\cos 70°}=\frac{2\sin 20°}{-2\sin 20°}=-1$$ ✓ *only one mark for a bald answer*

Question 12

(a) Consider $\sin\theta+\sqrt{3}\cos\theta=1$.

Let $t=\tan\frac{\theta}{2}$ this means

$\sin\theta=\frac{2t}{1+t^2}$ and $\cos\theta=\frac{1-t^2}{1+t^2}$

So, the equation will be

$$\frac{2t}{1+t^2}+\sqrt{3}\times\frac{1-t^2}{1+t^2}=1$$ ✓

Multiplying both sides by $1+t^2$, we get

$$2t+\sqrt{3}\left(1-t^2\right)=1+t^2$$

$$2t+\sqrt{3}-\sqrt{3}t^2-1-t^2=0$$

$$\left(1+\sqrt{3}\right)t^2-2t+1-\sqrt{3}=0$$

$$t=\frac{2\pm\sqrt{4-4\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}}{2\left(1+\sqrt{3}\right)}$$

$$t=\frac{2\pm\sqrt{12}}{2\left(1+\sqrt{3}\right)}=\frac{2\pm2\sqrt{3}}{2\left(1+\sqrt{3}\right)}$$

$$t=\frac{2\left(1+\sqrt{3}\right)}{2\left(1+\sqrt{3}\right)}\text{ or }\frac{2\left(1-\sqrt{3}\right)}{2\left(1+\sqrt{3}\right)}$$

So, $t=1$ *or* $t=\frac{1-\sqrt{3}}{1+\sqrt{3}}=\frac{4-2\sqrt{3}}{-2}=\sqrt{3}-2$.

When $t=1$ and when $t=\sqrt{3}-2$ ✓ *both*

$\tan\frac{\theta}{2}=\tan\frac{\pi}{4}$ $\qquad$ $\tan\frac{\theta}{2}=\tan\left(-\frac{\pi}{12}\right)$

$\frac{\theta}{2}=\frac{\pi}{4}+n\pi$ $\qquad$ $\frac{\theta}{2}=-\frac{\pi}{12}+n\pi$

$=\frac{\pi}{2}+2n\pi$ $\qquad$ $=-\frac{\pi}{6}+2n\pi$

when $n=0$, $=\frac{\pi}{2}$ $\qquad$ *when* $n=1$, $=\frac{11\pi}{6}$

Hence, $\theta=\frac{\pi}{2}$ *or* $\frac{11\pi}{6}$ for $0\le\theta\le2\pi$. ✓ *both only* $\frac{\pi}{2}$ *give 2/3*

(b) The table below shows the different possible arrangements, with K before E.

✓ *indicate a method that could lead to a correct solution*

K	5	4	E	3	2	1	$=5!=120$
K	5	4	3	E	2	1	$=5!=120$
K	5	4	3	2	E	1	$=5!=120$
K	5	4	3	2	1	E	$=5!=120$
5	K	4	3	E	2	1	$=5!=120$
5	K	4	3	2	E	1	$=5!=120$
5	K	4	3	2	1	E	$=5!=120$
5	4	K	3	2	E	1	$=5!=120$
5	4	K	3	2	1	E	$=5!=120$
5	4	3	K	2	1	E	$=5!=120$

There are 10 ways of positioning the 7 letters with K before E and at least 2 letters between them. Each of these positions can have the other letters arranged in 5! ways. So, there are $10\times120=1200$ ways of doing this. There will also be 1200 ways of organising the E before the K, with at least 2 letters between them. As there are

2400 ways of organising the letters with at least 2 letters between the K and the E. ✓ *only one bald answer*

(c) As $P(x)=8x^4+28x^3-54x^2+29x-5$

has a root of multiplicity 3 then $P'(x)$ has the same root with multiplicity 2 and $P''(x)$ has the same root as a single root. So, we should look at the roots of $P''(x)$.

$$P'(x)=32x^3+84x^2-108x+29$$

$$P''(x)=96x^2+168x-108$$

Now, to find the required root we let $P''(x)=0$, we get $96x^2+168x-108=0$ that is $8x^2+14x-9=0$

$x=\dfrac{-14\pm\sqrt{14^2-4\times8\times-9}}{2\times 8}$ So, $x=-\dfrac{9}{4}$ *or* $\dfrac{1}{2}$. ✓ *both*

The correct root should also satisfy $P'(x)=0$ and as $P'\left(-\dfrac{9}{4}\right)\neq 0$ *and* $P'\left(\dfrac{1}{2}\right)=0$ then $x=\dfrac{1}{2}$ is the root of multiplicity 3. ✓

As the polynomial is degree 4 we expect that there will be 4 zeros. We know that $x=\dfrac{1}{2}$ is a root of multiplicity 3, let the remaining root be α.

The product of the roots $=-\dfrac{5}{16}$ this means

$\dfrac{1}{2}\times\dfrac{1}{2}\times\dfrac{1}{2}\times\alpha=-\dfrac{5}{8}$. So, $\alpha=-5$ is the fourth root. ✓

Hence, the four roots will be $\dfrac{1}{2},\dfrac{1}{2},\dfrac{1}{2}$ and -5.

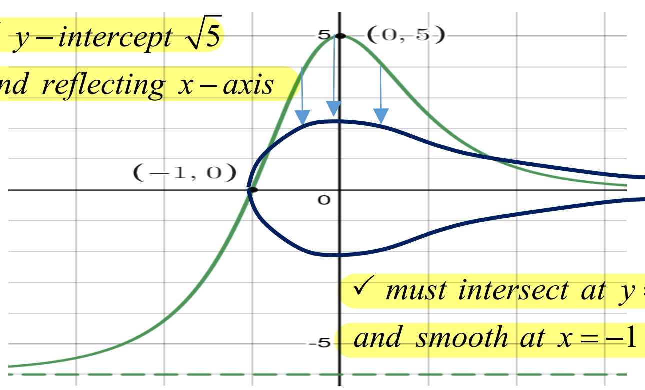
(d) (i) To sketch the graph of $y^2=f(x)$ we actually sketch the graph of $y=\pm\sqrt{f(x)}$.

When sketching $y=+\sqrt{f(x)}$ remember that if $f(x)<0$ then $\sqrt{f(x)}$ does not exist. If $0<f(x)<1$ $0<f(x)<1$ then $\sqrt{f(x)}>f(x)$.

If $f(x)=1$ then $\sqrt{f(x)}=1$

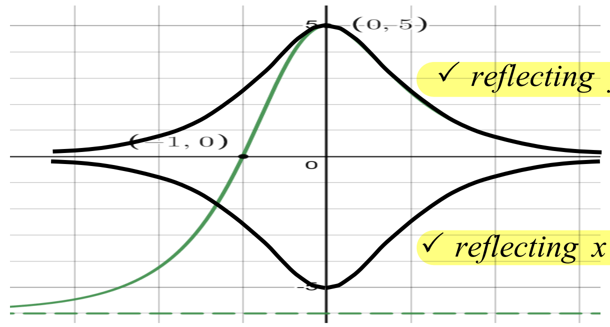
and if $f(x)>1$, then $\sqrt{f(x)}<f(x)$.

✓ *$y-$intercept $\sqrt{5}$ and reflecting $x-$axis*



✓ *must intersect at $y=1$ and smooth at $x=-1$*

(ii) $f(|x|)$ means that $f(-x)=f(x)$, hence we keep the part of the curve to the right of the y axis and we reflect it across the y axis to obtain the other part. To finalise the graph to produce $|y|=f(|x|)$ we reflect the parts of the previous graph, which are above the x-axis, across the x -axis.



✓ *reflecting $y-$axis*

✓ *reflecting $x-$axis*

(e) As the volume of a sphere is $V=\dfrac{4}{3}\pi r^3$ then $\dfrac{dv}{dr}=4\pi r^2$. Also, we are given the radius is increasing at a rate of 0.05 cm/s this means $\dfrac{dr}{dt}=0.05$.

The rate at which the volume is increasing is $\dfrac{dv}{dt}$ then by using the chain rule, we can write

$$\frac{dv}{dt}=\frac{dv}{dr}\times\frac{dr}{dt}=4\pi r^2\times0.05=0.2\pi r^2. \checkmark$$

Now, when the volume is $288\pi\ cm^3$ its radius can be calculated by solving $288\pi=\dfrac{4}{3}\pi r^3$ that is

$\dfrac{3}{4}\times288=r^3$ So, $r=6$. ✓

By substituting $r=6$ in $\dfrac{dv}{dt}$, we get

$$\frac{dv}{dt}=0.2\pi\times6^2=\frac{36\pi}{5}\text{ cm}^3\text{ s}^{-1} \checkmark$$

no penalty for incorrect or no units

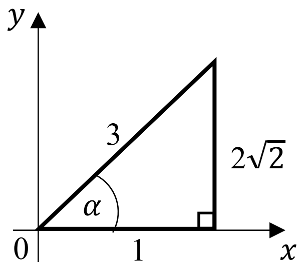
Hence, the volume of the sphere is increasing at $\dfrac{36\pi}{5}\text{ cm}^3\text{ s}^{-1}$ when its volume is $288\pi\, cm^3$.

Question 13

(a) Let $\alpha = \cos^{-1}\frac{1}{3}$

that is $\cos\alpha = \frac{1}{3} > 0$

this means $0 < \alpha < \frac{\pi}{2}$.



As α is an acute angle we construct a right-angled triangle in the first quadrant and we use Pythagoras' theorem, to find a the missing third side.

So, $a = \sqrt{3^2 - 1^2} = 2\sqrt{2}$. ✓ or $\sqrt{8}$

Now, $\sin\alpha = \frac{2\sqrt{2}}{3}$ and $\alpha = \cos^{-1}\frac{1}{3}$ then

$$\sin\left(\cos^{-1}\frac{1}{3}\right) = \frac{2\sqrt{2}}{3}. ✓$$

(b) As order is not important the number of groups will be combinations. This indicates that the number of different groups containing five swimmers and four swimmers are $\binom{n}{5}$ and $\binom{n}{4}$ respectively.

We are given that $\binom{n}{5} = 2\times\binom{n}{4}$

this means $\frac{n!}{5!(n-5)!} = 2\times\frac{n!}{4!(n-4)!}$ ✓

Dividing both sides by $\frac{n!}{4!(n-5)!}$, we get

$\frac{1}{5} = \frac{2}{n-4}$ ✓ this means $n - 4 = 10$ So, $n = 14$.

Hence, the number of swimmers in the club is 14. ✓

(c) To find the maximum number of students in the smallest group we must make the larger groups as small as possible. As $77 \div 3 = 25\frac{2}{3}$ ✓ *must show remainder of 2*

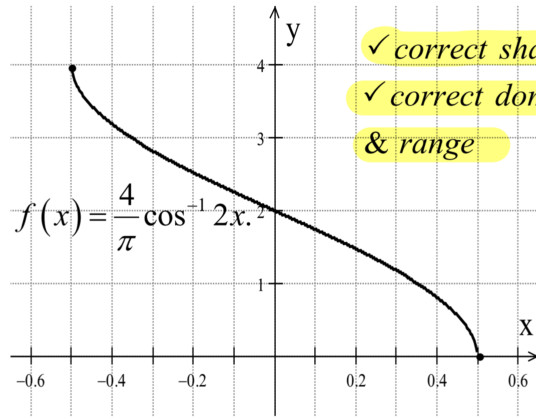
the groups could be $25 + 26 + 27 = 78$ which is more than 77. The next choice would be $24 + 26 + 27 = 77$

Hence, the maximum number of students in the smallest group would be 24. ✓

(d) Consider the function $f(x) = \frac{4}{\pi}\cos^{-1}2x$.

To find its domain we let $-1 \le 2x \le 1$ and dividing by 2, we get its domain to be $\frac{-1}{2} \le x \le \frac{1}{2}$.

To find its range we let $0 \le \cos^{-1}2x \le \pi$ and multiplying by $\frac{4}{\pi}$, we get its range to be $0 \le y \le 4$.



✓ *correct shape*
✓ *correct domain & range*

(e) Given that $f(x) = x\sqrt{x}$ and $g(x) = e^{bx}$ then

$$h(x) = f(g(x)) = f(e^{bx}) = e^{bx}\sqrt{e^{bx}}. ✓$$

To find $h^{-1}(x)$ we interchange x and y, we get

$x = e^{by}\sqrt{e^{by}}$ and to make y the subject we make

$x = (e^{by})^{\frac{3}{2}}$ and by applying ln on both sides, we get

$\ln x = \frac{3}{2}\ln e^{by}$ that is $\ln x = \frac{3by}{2}$ So, $y = \frac{2}{3b}\ln x$ ✓

Therefore, $h^{-1}(x) = \frac{2}{3b}\ln x$. By substituting $h^{-1}(e) = 6b$

we get $6b = \frac{2}{3b}\ln e$ that is $6b = \frac{2}{3b}$ so, $b^2 = \frac{1}{9}$

Hence, $b = \pm\frac{1}{3}$ and as b is positive then $b = \frac{1}{3}$. ✓

(f) (i) The parametric equations of the curve are $x = 1 + \cos\theta$ and $y = \cos 2\theta$ and in order to find its cartesian equation we need to eliminate the parameter θ.

By rearranging the given equations, we get

$\cos\theta = x - 1$ (1) and $y = 2\cos^2\theta - 1$ (2)

By substituting (1) in (2), we get

$y = 2(x-1)^2 - 1$ or $y = 2x^2 - 4x + 1$ ✓ *either*

(ii) This equation shows that the curve is a parabola with vertex at $(1,-1)$. ✓

(iii) But as $-1 \le \cos\theta \le 1$ then $0 \le 1+\cos\theta \le 2$

So, $0 \le x \le 2$ ✓ is the restricted domain.

Question 14

(a) (i) $(15-1)! = 14!$ ✓

(ii) Let E be the event where Sam, Toby, Euan and Will are seated together.

$$P(E) = \frac{(12-1)! \times 4!}{14!}$$ ✓ *correct number of arrangements*

$$= \frac{1}{91}$$

$$P(\bar{E}) = 1 - \frac{1}{91}$$

$$= \frac{90}{91}$$ ✓

(b) $x = \dfrac{10}{e^{t-2} + e^{-t+2}}$

The displacement will be maximum when velocity $\dfrac{dx}{dt} = 0$.

Using the quotient rule from the reference sheet.

Let $u = 10$ and $v = e^{t-2} + e^{-t+2}$

$u' = 0$ and $v' = e^{t-2} - e^{-t+2}$

$$\frac{dx}{dt} = \frac{0 - 10\left(e^{t-2} - e^{-t+2}\right)}{\left(e^{t-2} + e^{-t+2}\right)^2} \text{ or}$$

$$\frac{dx}{dt} = \frac{10\left(e^{-t+2} - e^{t-2}\right)}{\left(e^{t-2} + e^{-t+2}\right)^2}$$

Let $\dfrac{dx}{dt} = 0$ to find the time the displacement is maximum, we get

$$\frac{-10\left(e^{t-2} - e^{-t+2}\right)}{\left(e^{t-2} + e^{-t+2}\right)^2} = 0 \text{ this means}$$

$e^{t-2} - e^{-t+2} = 0$ that is $e^{t-2} = e^{-t+2}$

So, $t - 2 = -t + 2$

$2t = 4$. Hence, $t = 2$ seconds. ✓

To show that is displacement is maximum we find the sign of $\dfrac{dx}{dt}$ on both sides $t = 2$ as shown in the table below.

t	1	2	3
$\dfrac{dx}{dt}$	2.47	0	-2.47
x	↗	max	↘

✓ *must have values*

From the table we can see that the maximum displacement occurs when $t = 2$ and to find this maximum we substitute $t = 2$ in

$x = \dfrac{10}{e^{t-2} + e^{-t+2}}$, we get $x = \dfrac{10}{e^0 + e^0} = 5$.

Hence, the maximum displacement is 5 metres. ✓

(c) (i) $LHS = 2\sin(k-1)\theta\cos\theta - \sin(k-2)\theta$

$$= 2\sin(k\theta - \theta)\cos\theta - \sin(k\theta - 2\theta)$$

$$= 2\cos\theta(\sin k\theta\cos\theta - \cos k\theta\sin\theta) - (\sin k\theta\cos 2\theta - \cos k\theta\sin 2\theta)$$ ✓

$$= 2\sin k\theta\cos^2\theta - 2\cos k\theta\cos\theta\sin\theta - \sin k\theta\cos 2\theta + \cos k\theta\sin 2\theta$$

$$= 2\sin k\theta\cos^2\theta - 2\cos k\theta\cos\theta\sin\theta - \sin k\theta(2\cos^2\theta - 1) + \cos k\theta(2\sin\theta\cos\theta)$$

$$-2\cos^2\theta\sin k\theta + \sin k\theta + 2\cos k\theta\sin\theta\cos\theta$$ ✓

$$= \sin k\theta$$

Alternate Method

$$LHS = \sin((k-1)\theta + \theta) + \sin((k-1)\theta - \theta) - \sin(k-2)\theta$$ ✓

$$= \sin(k\theta) + \sin(k-2)\theta - \sin(k-2)\theta$$ ✓

(ii) let $k = 5$ in the formula from part (i), we get

$2\sin(5-1)\theta\cos\theta - \sin(5-2)\theta = \sin 5\theta$

that is $\sin 5\theta = 2\sin 4\theta\cos\theta - \sin 3\theta$ ✓

$= 4\sin 2\theta\cos 2\theta\cos\theta - \sin(2\theta + \theta)$

$= 4 \times 2\sin\theta\cos\theta\left(1 - 2\sin^2\theta\right)\cos\theta -$

$(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)$

$= 8\sin\theta\cos^2\theta\left(1-2\sin^2\theta\right) - \sin 2\theta\cos\theta - \cos 2\theta\sin\theta$

$= 8\sin\theta\cos^2\theta - 16\sin^3\theta\cos^2\theta - 2\sin\theta\cos\theta\cos\theta$

$-\left(1-2\sin^2\theta\right)\sin\theta$ ✓

$= 8\sin\theta\left(1-\sin^2\theta\right) - 16\sin^3\theta\left(1-\sin^2\theta\right)$

$-2\sin\theta\left(1-\sin^2\theta\right) - \left(1-2\sin^2\theta\right)\sin\theta$

$= 8\sin\theta - 8\sin^3\theta - 16\sin^3\theta + 16\sin^5\theta$

$-2\sin\theta + 2\sin^3\theta - \sin\theta + 2\sin^3\theta$ ✓

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Hence, $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ this means $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta - \sin 5\theta = 0$.

(iii) Let $x = \sin\theta$ so the equation $16x^4 - 20x^2 + 5 = 0$ becomes

$16\sin^4\theta - 20\sin^2\theta + 5 = 0$ (1) and from part (ii) we know that $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \sin 5\theta$ that is

$\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right) = \sin 5\theta$

So $16\sin^4\theta - 20\sin^2\theta + 5 = \dfrac{\sin 5\theta}{\sin\theta}$ (2) ✓

By substituting (2) in (1), we get

$\dfrac{\sin 5\theta}{\sin\theta} = 0$ this indicates that $\sin 5\theta = 0$ that is

$\sin 5\theta = \sin 0$ and $5\theta = 2n\pi$ *or* $5\theta = \pi + 2n\pi$

$\theta = \dfrac{2n\pi}{5}$ *or* $\theta = \dfrac{\pi}{5} + \dfrac{2n\pi}{5}$ ✓ *students dont need this notation just show multiple solutions*

when $n = 0,\ \theta = 0$ *or when* $n = 0,\ \theta = \dfrac{\pi}{5}$

when $n = 1,\ \theta = \dfrac{2\pi}{5}$ *when* $n = 1,\ \theta = \dfrac{3\pi}{5}$

when $n = 2,\ \theta = \dfrac{4\pi}{5}$ *when* $n = 2,\ \theta = \pi$

when $n = 3,\ \theta = \dfrac{6\pi}{5}$ *when* $n = 3,\ \theta = \dfrac{7\pi}{5}$

There is an infinite number of possible values for the angle θ. The four lines above show a few of these possible values.

As the equation $16x^4 - 20x^2 + 5 = 0$ has a degree 4 it should have at maximum four real roots which mean four possible values for $\sin\theta$ regardless of the possible values of θ. Hence, we need to find four different values for $\sin\theta$ using the possible values for angle θ found above as they are the four roots of this equation.

The four different values of $\sin\theta$ or the four roots of the quartic equation are:

$\sin\dfrac{\pi}{5}$, $\sin\dfrac{2\pi}{5}$, $\sin\dfrac{6\pi}{5}$ and $\sin\dfrac{7\pi}{5}$.

But as $\sin\dfrac{6\pi}{5} = -\sin\dfrac{\pi}{5}$ and $\sin\dfrac{7\pi}{5} = -\sin\dfrac{2\pi}{5}$ ✓

Then the roots are $\sin\dfrac{\pi}{5}$, $\sin\dfrac{2\pi}{5}$, $-\sin\dfrac{\pi}{5}$ and $-\sin\dfrac{2\pi}{5}$.

Now, the sum of these roots 2 at the time is

$-\dfrac{20}{16} = -\dfrac{5}{4}$. This means

$\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5} - \sin^2\dfrac{\pi}{5} - \sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5} - \sin\dfrac{2\pi}{5}\sin\dfrac{\pi}{5}$

$-\sin^2\dfrac{2\pi}{5} + \sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5} = -\dfrac{5}{4}$ ✓

Hence, $\sin^2\dfrac{\pi}{5} + \sin^2\dfrac{2\pi}{5} = \dfrac{5}{4}$ as required.

Alternate method

$16x^4 - 20x^2 + 5 = 0$ and using $\sin\theta = x$

$x^2 = \dfrac{20 \pm 4\sqrt{5}}{32}$ or $x^2 = \dfrac{5 \pm \sqrt{5}}{8}$ ✓

$\sin\theta = \sqrt{\dfrac{5+\sqrt{5}}{8}}, -\sqrt{\dfrac{5+\sqrt{5}}{8}}, \sqrt{\dfrac{5-\sqrt{5}}{8}}, -\sqrt{\dfrac{5-\sqrt{5}}{8}}$ ✓

$\theta = \dfrac{2\pi}{5}, -\dfrac{2\pi}{5}, \dfrac{\pi}{5}, -\dfrac{\pi}{5}$ ✓

✓ *either method*

Using either sum in pairs of the quartic in $x^4 = \dfrac{-20}{16}$ or

the sum of the roots in the quadratic in $x^2 = \dfrac{20}{16}$ to give

$\sin^2\dfrac{\pi}{5} + \sin^2\dfrac{2\pi}{5} = \dfrac{5}{4}$.