NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2019 Year 11 Course Assessment Task 3
Wednesday, 18 September 2019

General instructions

- Working time – 2 hours
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- A NESA Reference Sheet is provided.
- Attempt all questions.
- At the conclusion of the examination, bundle any additional sheets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided.

SECTION II

- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Extra writing space is provided on page 21. If you use this space, clearly indicate which question you are answering.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Class: (please ✓)

○ 11MAT.1 – Mr Lam
○ 11MAT.2 – Mr Sekaran
○ 11MAT.3 – Ms Ham
○ 11MAT.4 – Ms Park
○ 11MAT.5 – Mrs Gan

Marker's use only

QUESTION	1-10	11	12	13	14	15	Total	%
MARKS	/10	/10	/14	/13	/14	/16	/77	

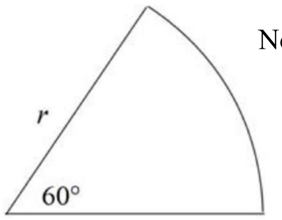
Section I

10 marks
Attempt Questions 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided on page 5.

Questions **Marks**

1. The sector below has an area of 20π square units. **1**



Not to scale

What is the value of r?

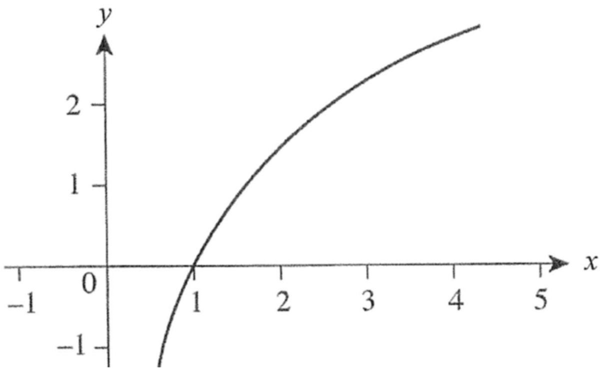
(A) $\sqrt{120}$

(B) $\sqrt{120}\,\pi$

(C) $\sqrt{\frac{\pi}{3}}$

(D) $\sqrt{\frac{1}{3}}$

2. The graph of $y = f(x)$ below has which of the following properties? **1**



(A) $f'(x) > 0$ and $f''(x) > 0$

(B) $f'(x) > 0$ and $f''(x) < 0$

(C) $f'(x) < 0$ and $f''(x) > 0$

(D) $f'(x) < 0$ and $f''(x) < 0$

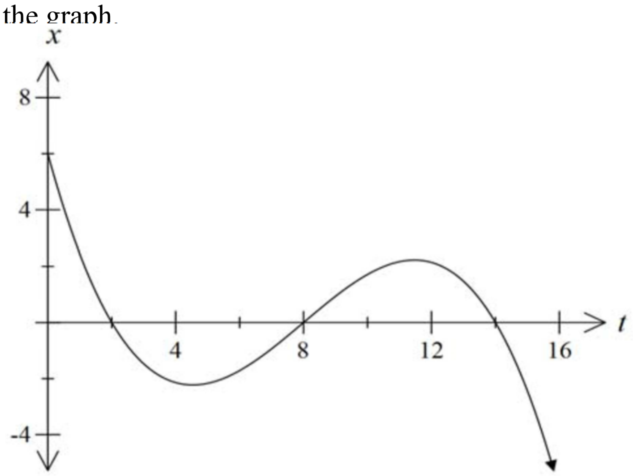
3. Evaluate $\lim\limits_{x \to 3} \frac{x^2-2x-3}{x-3}$? **1**

(A) ∞ (B) 0 (C) 4 (D) 2

4. If $f(x) = a^x + a^{-x}$, which of the following expressions is equal to $[f(x)]^2$? **1**

(A) $1 - f(2x)$ (B) $1 + f(2x)$ (C) $2 - f(2x)$ (D) $2 + f(2x)$

5. The displacement, x metres, from the origin of a particle moving in a straight line at any time t seconds is shown in the graph. **1**



When was the particle first at rest? **1**

(A) $t = 0$ (B) $t = 2$ (C) $t = 4.5$ (D) Never

6. Let $f(x) = k \log_2 x$. If $f^{-1}(1) = 8$, what is the value of k? **1**

(A) 0 (C) 3
(B) $\frac{1}{3}$ (D) 8

7. Let f and g be two functions such that $f(x) = 2x$ and $g(x) = 3x - 5$. **1**

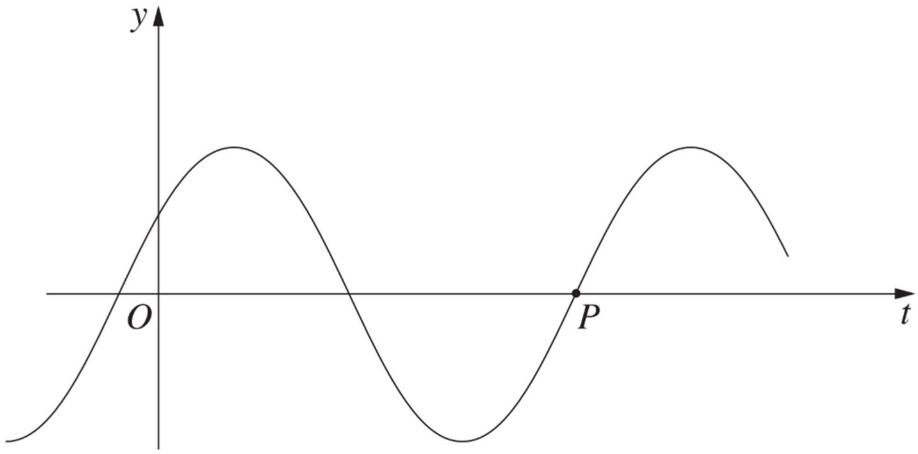
Which of the following is $f(g(x))$?

(A) $6x - 1$ (B) $6x + 1$ (C) $6x^2 + 1$ (D) $6x - 10$

8. Given the curve $y = f(x)$, which of the following is the best representation of $y = f(-|x|)$? **1**

(A) A reflection of $y = f(x)$ for $x \geq 0$ in the y-axis
(B) A reflection of $y = f(x)$ for $x \leq 0$ in the y-axis
(C) A reflection of $y = f(x)$ for $y \geq 0$ in the x-axis
(D) A reflection of $y = f(x)$ for $y \leq 0$ in the x-axis

9. The graph of the function $y = \cos\left(2t - \frac{\pi}{3}\right)$ is shown below. **1**

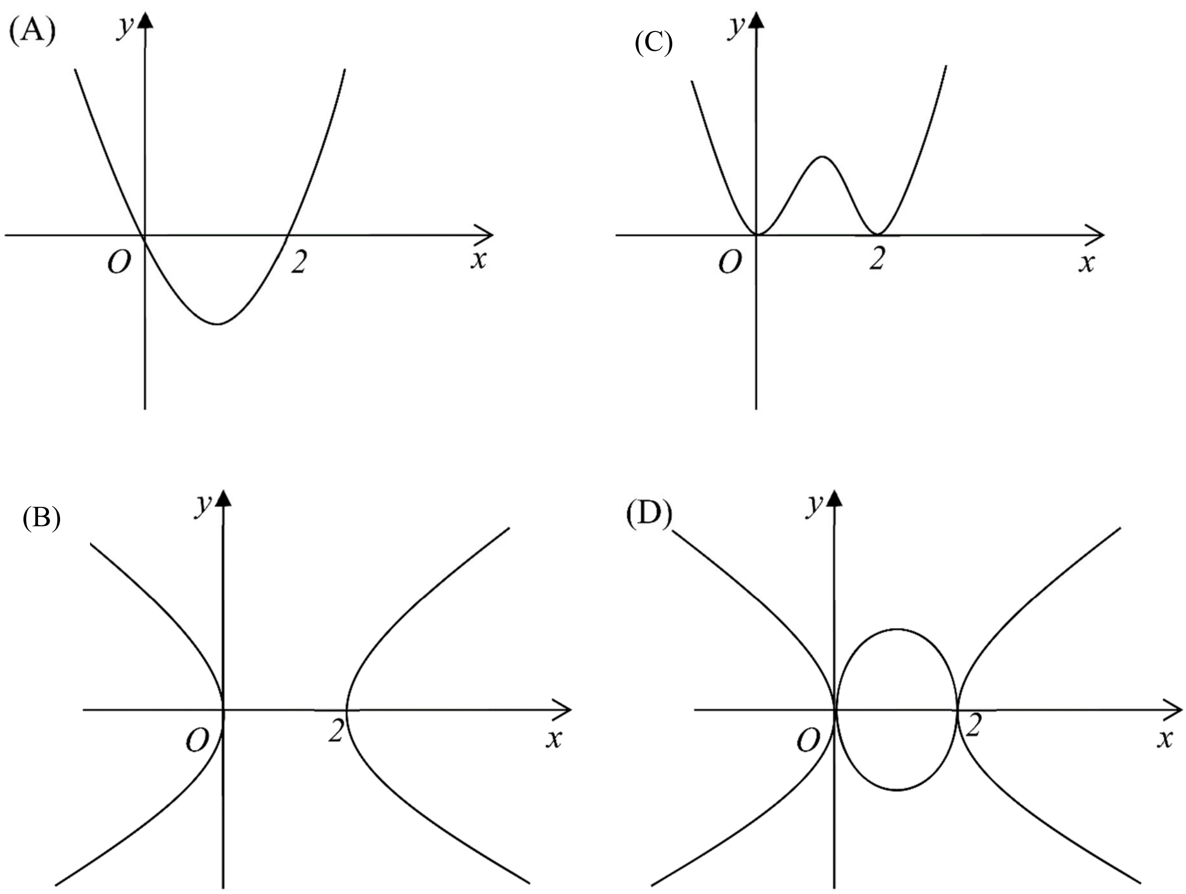


What are the coordinates of the point P?

(A) $\left(\frac{5\pi}{12}, 0\right)$ (C) $\left(\frac{11\pi}{12}, 0\right)$

(B) $\left(\frac{2\pi}{3}, 0\right)$ (D) $\left(\frac{7\pi}{6}, 0\right)$

10. Which graph best represents the curve $y^2 = x^2 - 2x$? **1**



Answer Sheet for Section I

Mark answers to Section I by fully blackening the correct circle

1 —	A	B	C	D
2 —	A	B	C	D
3 —	A	B	C	D
4 —	A	B	C	D
5 —	A	B	C	D
6 —	A	B	C	D
7 —	A	B	C	D
8 —	A	B	C	D
9 —	A	B	C	D
10 —	A	B	C	D

Section II

66 marks
Attempt Questions 11 to 15

Allow approximately 1 hour and 45 minutes for this section.

Answer the questions in the spaces provided.

Extra writing space is provided on page 21. If you use this space, clearly indicate which question you are answering.

Question 11 (10 marks)

(a) Differentiate with respect to x:

i. $y = -3x^4$ **1**

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ii. $f(x) = (3x^2 - 5)^6$ **2**

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iii. $f(x) = \frac{2x-1}{x+4}$ **2**

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iv. $y = 3x^2\sqrt{x-2}$ **2**

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(b) Differentiate $f(x) = x^2$ from the first principles. **2**

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(c) Find the x coordinate of the point on the curve $y = x^2 + 5$ where the tangent has the gradient of -2. **1**

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Question 12 (14 marks) **Marks**

(a) Solve $\frac{x}{x^2-1} \geq 0$, giving the answer in interval notation. **3**

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(b) i. Sketch the graphs $y = \sqrt{4-x}$ and $y = \frac{1}{2}|x-4|$ on the same set of axes, clearly showing the x and y intercepts. **3**

ii. Hence or otherwise solve $\sqrt{4-x} > \frac{1}{2}|x-4|$. **1**

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iii. Find the equation of the inverse function of $y = \sqrt{4-x}$ and state its domain. **2**

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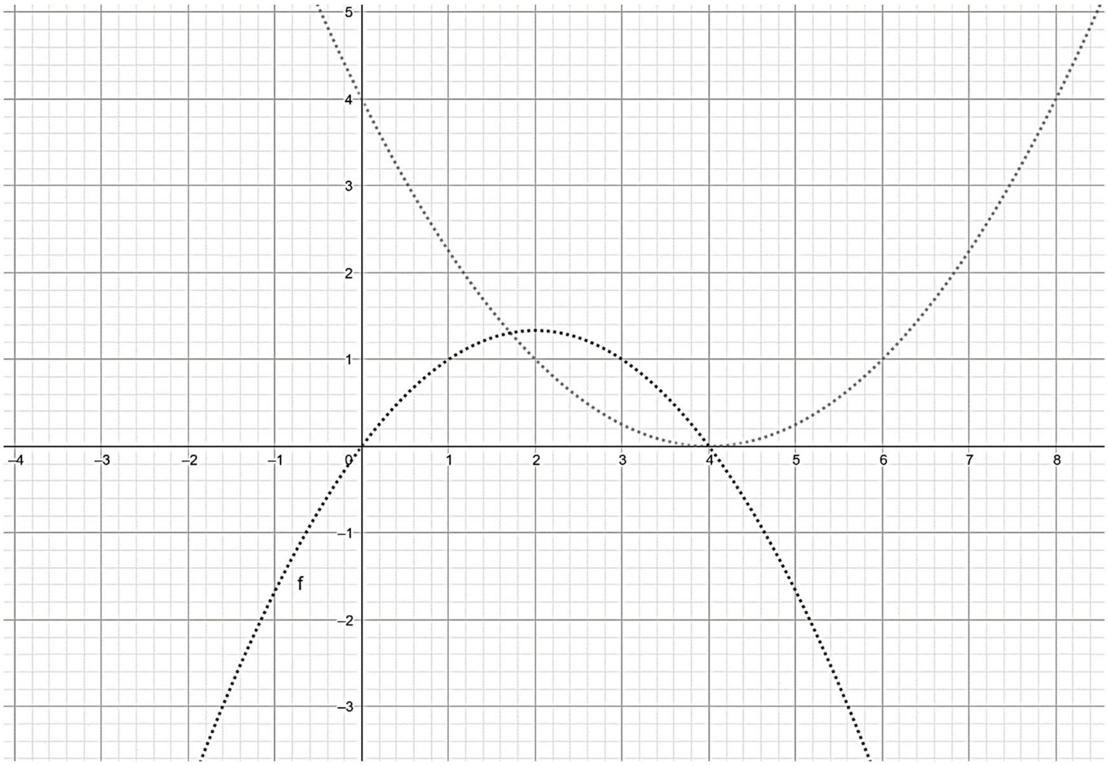
iv. Sketch the graph of the inverse function of $y = \sqrt{4-x}$ showing x and y intercepts. **2**

(c) The graphs of $y = f(x)$ and $y = g(x)$ are given below. **2**

Sketch the graph of $y = f(x) \times g(x)$.



(d) Find the Cartesian equation of the parametric equations $x = 2t + 1$ and $y = \frac{1}{t} - 1$. **1**

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Question 13 (13 marks)

(a) If $\log_a b = 0.3$ and $\log_a c = 0.4$, find the value of $\log_b \left(\frac{a}{c}\right)$. **3**

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(b) i. Show that $\frac{dy}{dx} = \frac{e^{x^2}(2x^2-1)}{2x^2}$ if $y = \frac{e^{x^2}}{2x}$. **2**

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ii. A tangent is drawn from the origin O to the curve $y = \frac{e^{x^2}}{2x}$. **3**

By taking the equation of the tangent as $y = mx$, show that the x coordinates of the points of contact of the tangent and the curve are $x = \pm 1$.

Examination continues overleaf…

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(c) The concentration, C, of good bacteria in a healthy intestine is usually 110 bacteria per gram of intestinal contents. After a bout of food poisoning, this concentration drops to 15 bacteria per gram. The rate of increase of the concentration of good bacteria is proportional to the difference from the normal 110 bacteria per gram, that is,

$$\frac{dC}{dt} = k(110 - C)$$

where t is the number of hours since suffering food poisoning.

i. Show that $C = 110 - Ae^{-kt}$ satisfies the above equation. **1**

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19 hours after suffering food poisoning, David's intestines has a concentration of 20 bacteria per gram.

ii. Show that $k = -\frac{1}{19}\log_e \frac{18}{19}$ **2**

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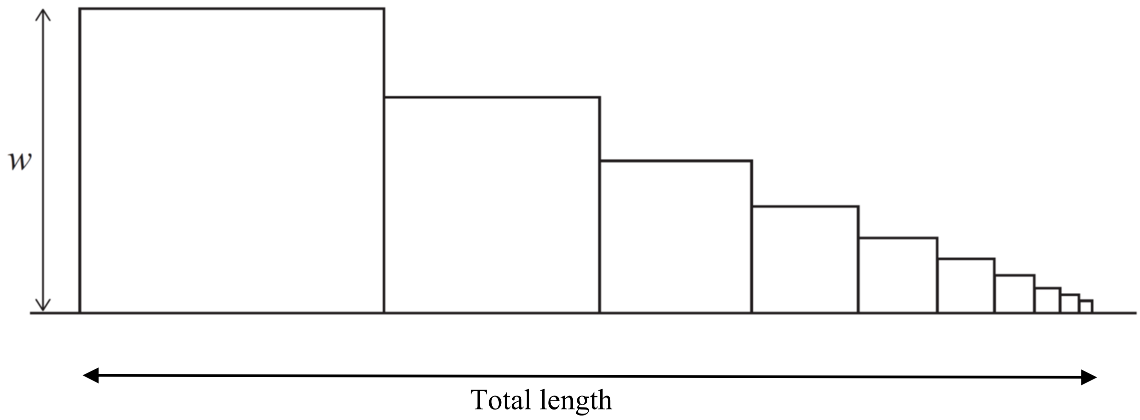
iii. How long after suffering food poisoning will the concentration of good bacteria in his intestines reach 90% of its healthy state? Give your answer correct to the nearest hour. **2**

Question 14 (14 marks)

(a) Ms Park is creating a mosaic pattern by placing square tiles next to each other along a straight line as shown below. The tiles are in contact with adjacent tiles, but do not overlap.



The area of each tile is half the area of the previous tile, and the sides of the largest tile has length w cm.

i. Show that the length of the sides of the second largest tile is $\frac{\sqrt{2}}{2}w$. **1**

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ii. Explain why the lengths of the sides of the tiles forms a geometric sequence. **2**

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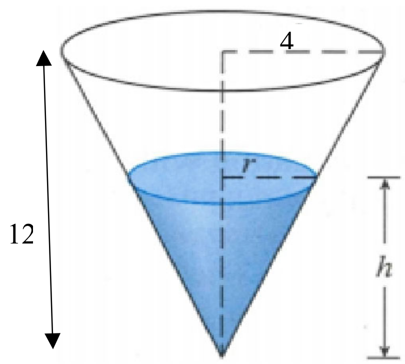
iii. Find, in terms of w, the length of the sides of the n^{th} largest tile. **1**

iv. Using the same pattern, what is the minimum number of tiles needed so that it forms a line of at least length $3w$ cm? **3**

v. Show that, no matter how many tiles are in the pattern, the total length of the series of tiles will be less than $3.5w$. **2**

(b)



A cone of radius 4 cm and height 12 cm is being filled with water at a constant rate of 2 cm^3/s.

Let the depth of the water in the cone after t seconds be h cm and the radius of the surface water be r cm.

i. Show that $r = \frac{h}{3}$. **1**

ii. Show that the volume of the water at any time t is $V = \frac{\pi}{27}h^3$. **1**

iii. Find the rate at which the depth is increasing when the depth is 4 cm. **3**

Question 15 (16 marks)

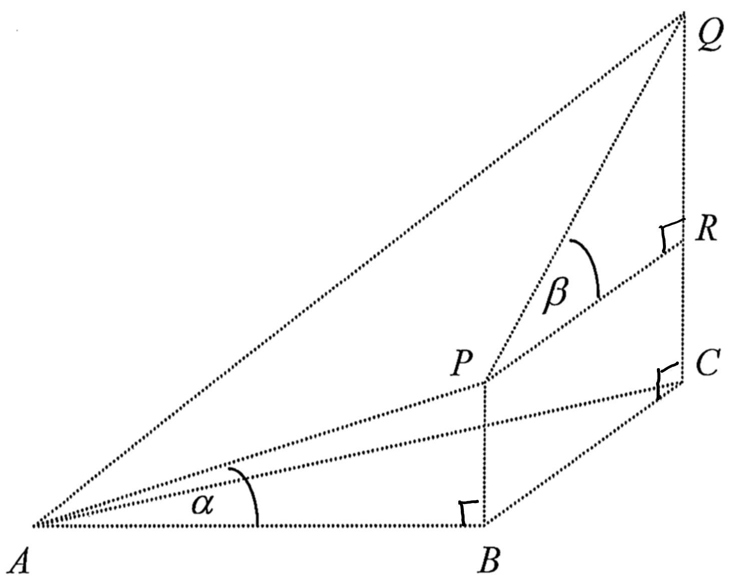
(a) Find the exact value of $(\sin 22.5° + \cos 22.5°)^2$ **2**

(b) If $\sin x = \frac{1}{3}$, $\cos y = -\frac{1}{\sqrt{3}}$ such that $\frac{\pi}{2} < x < \pi$ and $\pi < y < \frac{3\pi}{2}$. **4**

Find the exact value of $\tan(x + y)$.

(c) Prove that $\frac{\sin 3\theta}{\sin \theta} + \frac{\cos 3\theta}{\cos \theta} = 4\cos 2\theta$ **3**

(d)



ABC is a horizontal, right-angled, isosceles triangle where $AB = BC$ and $\angle ABC = 90°$. P is vertically above B; Q is vertically above C and PR is parallel to BC. The angle of elevation of P from A, and Q from P are α and β respectively. The length of AB is x m.

i. Show that $QC = x(\tan\alpha + \tan\beta)$ **2**

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ii. If the angle of elevation of Q from A is θ, prove that **2**

$$\tan\theta = \frac{\tan\alpha + \tan\beta}{\sqrt{2}}$$

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iii. if $\angle APQ = \phi$, prove that $\cos\phi = -\sin\alpha\sin\beta$ **3**

End of examination.

Section II extra writing space

If you use this space, clearly indicate which question you are answering.

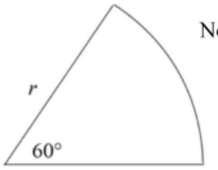
Section I

10 marks
Attempt Questions 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided on page 5.

Questions **Marks**

1. The sector below has an area of 20π square units. **1**



Not to scale

What is the value of r?

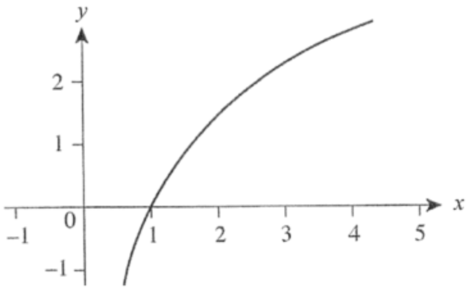
(A) $\sqrt{120}$

(B) $\sqrt{120}\,\pi$

(C) $\sqrt{\frac{\pi}{3}}$

(D) $\sqrt{\frac{1}{3}}$

2. The graph of $y = f(x)$ below has which of the following properties? **1**



(A) $f'(x) > 0$ and $f''(x) > 0$

(B) $f'(x) > 0$ and $f''(x) < 0$

(C) $f'(x) < 0$ and $f''(x) > 0$

(D) $f'(x) < 0$ and $f''(x) < 0$

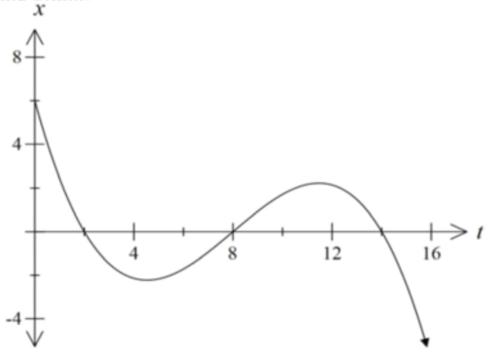
3. Evaluate $\lim_{x \to 3} \frac{x^2-2x-3}{x-3}$? **1**

(A) ∞ (B) 0 (C) 4 (D) 2

4. If $f(x) = a^x + a^{-x}$, which of the following expressions is equal to $[f(x)]^2$? **1**

(A) $1 - f(2x)$ (B) $1 + f(2x)$ (C) $2 - f(2x)$ (D) $2 + f(2x)$

5. The displacement, x metres, from the origin of a particle moving in a straight line at any time t seconds is shown in the graph. **1**



When was the particle first at rest? **1**

(A) $t = 0$ (B) $t = 2$ (C) $t = 4.5$ (D) Never

6. Let $f(x) = k \log_2 x$. If $f^{-1}(1) = 8$, what is the value of k? **1**

(A) 0
(B) $\frac{1}{3}$
(C) 3
(D) 8

7. Let f and g be two functions such that $f(x) = 2x$ and $g(x) = 3x - 5$. **1**

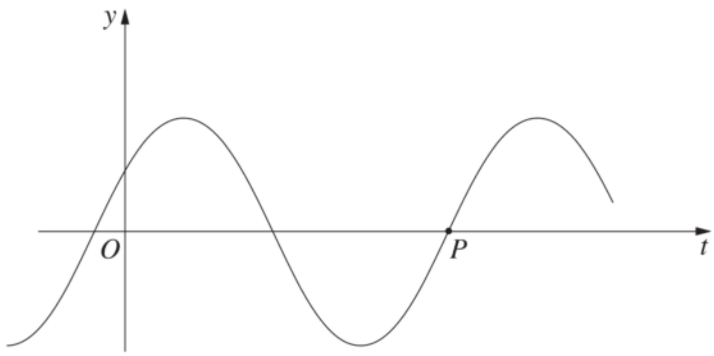
Which of the following is $f(g(x))$?

(A) $6x - 1$ (B) $6x + 1$ (C) $6x^2 + 1$ (D) $6x - 10$

8. Given the curve $y = f(x)$, which of the following is the best representation of $y = f(-|x|)$? **1**

(A) A reflection of $y = f(x)$ for $x \geq 0$ in the y-axis
(B) A reflection of $y = f(x)$ for $x \leq 0$ in the y-axis
(C) A reflection of $y = f(x)$ for $y \geq 0$ in the x-axis
(D) A reflection of $y = f(x)$ for $y \leq 0$ in the x-axis

9. The graph of the function $y = \cos\left(2t - \frac{\pi}{3}\right)$ is shown below. **1**



What are the coordinates of the point P?

(A) $\left(\frac{5\pi}{12}, 0\right)$

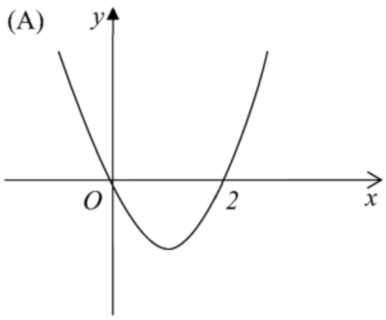
(B) $\left(\frac{2\pi}{3}, 0\right)$

(C) $\left(\frac{11\pi}{12}, 0\right)$

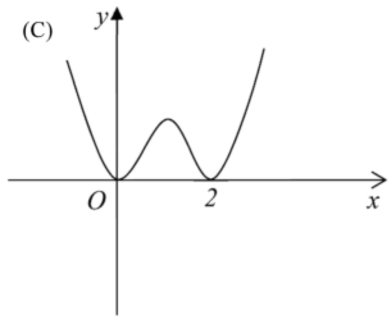
(D) $\left(\frac{7\pi}{6}, 0\right)$

10. Which graph best represents the curve $y^2 = x^2 - 2x$? **1**

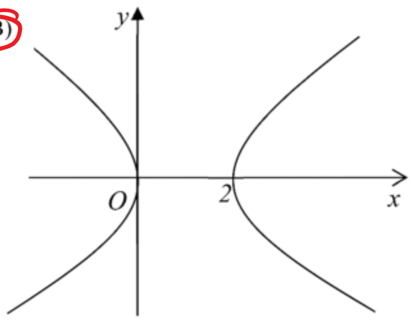
(A)



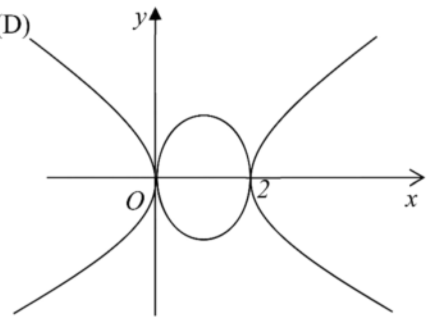
(C)



(B)



(D)



Answer Sheet for Section I

Mark answers to Section I by fully blackening the correct circle

1 —		B	C	D
2 —	A		C	D
3 —	A	B		D
4 —	A	B	C	
5 —	A	B		D
6 —	A		C	D
7 —	A	B	C	
8 —	A		C	D
9 —	A	B		D
10 —	A		C	D

Section II

72 marks
Attempt Questions 11 to 15

Allow approximately 1 hour and 45 minutes for this section.

Answer the questions in the spaces provided.

Extra writing space is provided on page 22. If you use this space, clearly indicate which question you are answering.

Question 11 (10 marks)

(a) Differentiate with respect to x:

i. $y = -3x^4$ **1**

$y' = -12x^3$

ii. $f(x) = (3x^2 - 5)^6$ **2**

$f'(x) = 6(3x^2-5)^5 \times 6x$

$= 36x(3x^2-5)^5$

iii. $f(x) = \frac{2x-1}{x+4}$ **2**

$f'(x) = \dfrac{(x+4)(2) - (2x-1)\cdot 1}{(x+4)^2}$

$= \dfrac{9}{(x+4)^2}$

iv. $y = 3x^2\sqrt{x-2}$ **2**

$y' = \sqrt{x-2}\cdot(6x) + 3x^2 \cdot \frac{1}{2}(x-2)^{-1/2}\cdot 1$

$= \dfrac{(x-2)\cdot 12x + 3x^2}{\sqrt{x-2}}$

$= \dfrac{15x^2 - 24x}{\sqrt{x-2}}$

(b) Differentiate $f(x) = x^2$ from the first principles. **2**

$f'(x) = \lim_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$

$\lim_{h \to 0} \dfrac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h)$

$= 2x$

(c) Find the x coordinate of the point on the curve $y = x^2 + 5$ where the tangent has the gradient of -2. **1**

$y' = 2x = -2$

$x = -1$

Question 12 (14 marks) **Marks**

(a) Solve $\frac{x}{x^2-1} \geq 0$, giving the answer in interval notation. **3**

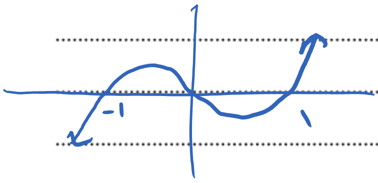
$x \neq \pm 1$

$\times (x^2-1)^2$

$x(x^2-1) \geq 0$

$x(x-1)(x+1) \geq 0$

$x \in (-1, 0] \cup (1, \infty)$



$x \in (-1, 0]$ OR $x \in (1, \infty)$

ok.

(b) i. Sketch the graphs $y = \sqrt{4-x}$ and $y = \frac{1}{2}|x-4|$ on the same set of axes, clearly showing the x and y intercepts. **3**

ii. Hence or otherwise solve $\sqrt{4-x} > \frac{1}{2}|x-4|$. **1**

$0 < x < 4$

iii. Find the equation of the inverse function of $y = \sqrt{4 - x}$ and state its domain. **2**

$x = \sqrt{4-y}$

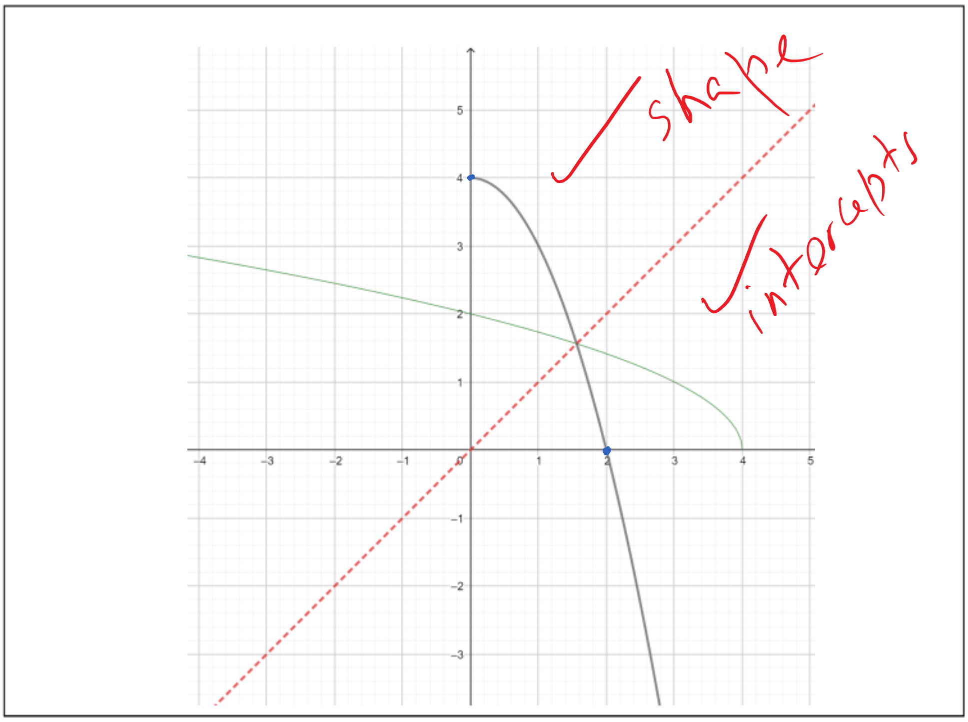
$4 - y = x^2$

$y = 4 - x^2$

$f^{-1}(x) = 4 - x^2$ ✓

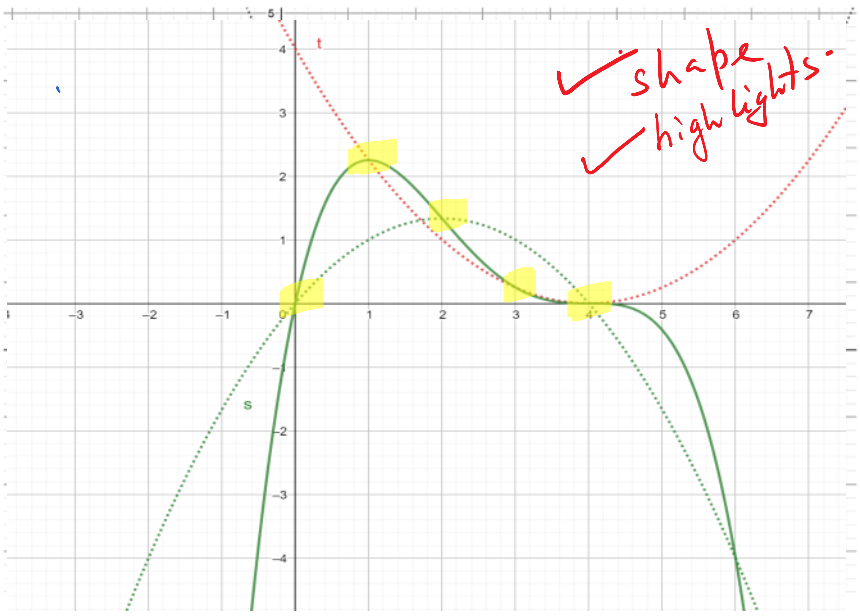
Domain : $x \geq 0$ ✓

iv. Sketch the graph of the inverse function of $y = \sqrt{4 - x}$ showing x and y intercepts. **2**



(c) The graphs of $y = f(x)$ and $y = g(x)$ are given below. **2**

Sketch the graph of $y = f(x) \times g(x)$.



(d) Find the Cartesian equation of the parametric equations $x = 2t + 1$ and $y = \frac{1}{t} - 1$. **1**

$t = \frac{x-1}{2}$

$\frac{1}{t} = \frac{2}{x-1}$

$y = \frac{2}{x-1} - 1$

$= \frac{3-x}{x-1}$

Question 13 (13 marks)

(a) If $\log_a b = 0.3$ and $\log_a c = 0.4$, find the value of $\log_b \left(\frac{a}{c}\right)$. **3**

$$\log_b\left(\frac{a}{c}\right) = \frac{\log_a\left(\frac{a}{c}\right)}{\log_a b}$$

$$= \frac{1 - \log_a c}{\log_a b} = \frac{1 - 0.4}{0.3}$$

$$= 2$$

(b) i. Show that $\frac{dy}{dx} = \frac{e^{x^2}(2x^2-1)}{2x^2}$ if $y = \frac{e^{x^2}}{2x}$. **2**

$$\frac{dy}{dx} = \frac{2x(2xe^{x^2}) - e^{x^2}(2)}{4x^2}$$

$$= \frac{\cancel{2}e^{x^2}(2x^2-1)}{\cancel{4}x^2}$$

$$= \frac{e^{x^2}(2x^2-1)}{2x^2}$$

ii. A tangent is drawn from the origin O to the curve $y = \frac{e^{x^2}}{2x}$. **3**

By taking the equation of the tangent as $y = mx$, show that the x coordinates of the points of contact of the tangent and the curve are $x = \pm 1$.

Let "a" be the x-coordinate at point of contact.

then $ma = \frac{e^{a^2}}{2a}$

$m = \frac{e^{a^2}(2a^2-1)}{2a^2}$

$\therefore \frac{e^{a^2}(2a^2-1)}{2a^2} \cdot a = \frac{e^{a^2}}{2a}$

$2a^2 - 1 = 1$

$a^2 = 1$

$a = \pm 1$

$\therefore$ x coordinate of the points are ± 1

Examination continues overleaf…

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(c) The concentration, C, of good bacteria in a healthy intestine is usually 110 bacteria per gram of intestinal contents. After a bout of food poisoning, this concentration drops to 15 bacteria per gram. The rate of increase of the concentration of good bacteria is proportional to the difference from the normal 110 bacteria per gram, that is,

$$\frac{dC}{dt} = k(110 - C)$$

where t is the number of hours since suffering food poisoning.

i. Show that $C = 110 - Ae^{-kt}$ satisfies the above equation. **1**

$\frac{dc}{dt} = -A(-ke^{-kt})$

$= k(Ae^{-kt})$

$= k(110 - C)$ since $C = 110 - Ae^{-kt}$

$\therefore C = 110 - Ae^{-kt}$ satisfies the equal'

19 hours after suffering food poisoning, David's intestines has a concentration of 20 bacteria per gram.

ii. Show that $k = -\frac{1}{19}\log_e \frac{18}{19}$ **2**

$t = 19, \quad C = 20$

$t = 0 \quad C = 15$

$15 = 110 - Ae^{0}$

$\therefore A = 95$

$20 = 110 - 95e^{-19k}$

$95e^{-19k} = 90$

$-19k = \ln\frac{18}{19}$

$k = \frac{-1}{19}\ln\frac{18}{19}$

iii. How long after suffering food poisoning will the concentration of good bacteria in his intestines reach 90% of its healthy state? Give your answer correct to the nearest hour. **2**

$0.9 \times 110 = 110 - 95e^{-kt}$

$95e^{-kt} = 11$

$-kt = \ln \frac{11}{95}$

$t = \frac{1}{k} \ln \frac{95}{11}$

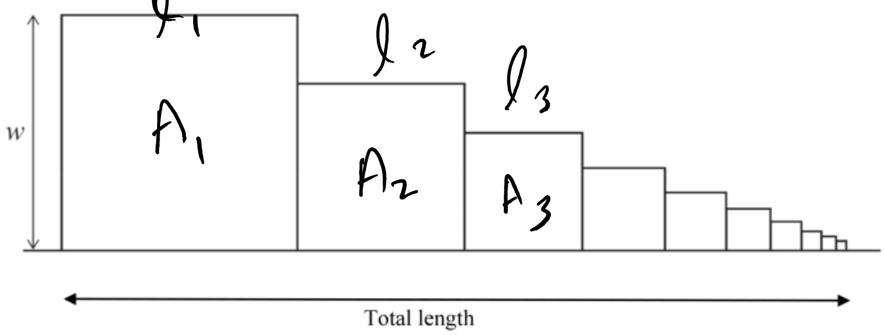
$= \frac{19 \times \ln \frac{95}{11}}{\ln \frac{19}{18}}$

$= 757.6429821$

$\therefore$ takes 758 hours.

Question 14 (14 marks)

(a) Ms Park is creating a mosaic pattern by placing square tiles next to each other along a straight line as shown below. The tiles are in contact with adjacent tiles, but do not overlap.



The area of each tile is half the area of the previous tile, and the sides of the largest tile has length w cm.

i. Show that the length of the sides of the second largest tile is $\frac{\sqrt{2}}{2}w$. **1**

$l_2^2 = \frac{1}{2}w^2$

$l_2 = \frac{w}{\sqrt{2}} = \frac{\sqrt{2}w}{2}$ cm

ii. Explain why the lengths of the sides of the tiles forms a geometric sequence. **2**

$A_3 = \frac{1}{2}A_2 = \frac{1}{2}\cdot\frac{w^2}{2}$ $\therefore l_3 = \frac{w}{2}$

$\frac{l_2}{l_1} = \frac{\sqrt{2}w}{2} \times \frac{1}{w} = \frac{\sqrt{2}}{2}$

$\frac{l_3}{l_2} = \frac{w}{2} \times \frac{2}{\sqrt{2}w} = \frac{\sqrt{2}}{2}$

$\therefore$ GP with $r = \frac{\sqrt{2}}{2}$

iii. Find, in terms of w, the length of the sides of the n^{th} largest tile. **1**

$l_n = \left(\frac{\sqrt{2}}{2}\right)^{n-1} w$ ✓

iv. Using the same pattern, what is the minimum number of tiles needed so that it forms a line of at least length $3w$ cm? **3**

$S_n = \frac{w\left(\left(\frac{\sqrt{2}}{2}\right)^n - 1\right)}{\frac{\sqrt{2}}{2} - 1} = 3w$ ✓

$\left(\frac{\sqrt{2}}{2}\right)^n - 1 = 3\left(\frac{\sqrt{2}}{2}\right) - 3$

$\left(\frac{\sqrt{2}}{2}\right)^n = 3\left(\frac{\sqrt{2}}{2}\right) - 2$ ✓

$n = \frac{\ln\left(3\left(\frac{\sqrt{2}}{2}\right) - 2\right)}{\ln\left(\frac{\sqrt{2}}{2}\right)} = 6.08621\ldots$

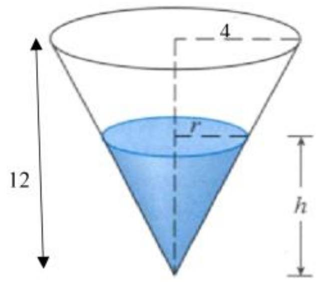
$\therefore$ minimum of tiles needed is 7 ✓

v. Show that, no matter how many tiles are in the pattern, the total length of the series of tiles will be less than $3.5w$. **2**

$$S_\infty = \frac{w}{1-\frac{\sqrt{2}}{2}} = 3.414w < 3.5w$$

$\therefore$ total length always $< 3.5w$

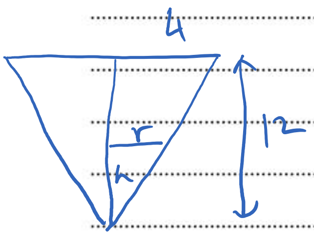
(b)



A cone of radius 4 cm and height 12 cm is being filled with water at a constant rate of 2 cm^3/s.

Let the depth of the water in the cone after t seconds be h cm and the radius of the surface water be r cm.

i. Show that $r = \frac{h}{3}$. **1**



$\frac{r}{h} = \frac{4}{12}$ (matching sides of similar Δs)

$\therefore r = \frac{h}{3}$

ii. Show that the volume of the water at any time t is $V = \frac{\pi}{27}h^3$. **1**

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$

$= \frac{\pi}{27}h^3$

iii. Find the rate at which the depth is increasing when the depth is 4 cm. **3**

$\frac{dv}{dh} = \frac{\pi}{9}h^2$ $\qquad \frac{dv}{dt} = 2 \text{ cm}^3/\text{s}$

$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt} = \frac{9}{\pi h^2} \times 2$

$= \frac{18}{\pi \times 4^2} = \frac{9}{8\pi} \text{ cm/s}$

Question 15 (16 marks)

(a) Find the exact value of $(\sin 22.5° + \cos 22.5°)^2$ **2**

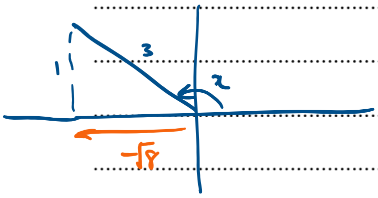
$= \sin^2 22.5° + 2\sin 22.5° \cos 22.5° + \cos^2 22.5°$ ✓

$= 1 + 2\sin 22.5° \cos 22.5°$

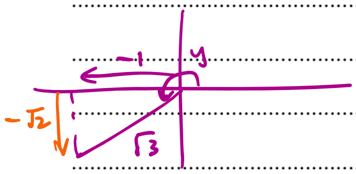
$= 1 + \sin 45° = 1 + \frac{1}{\sqrt{2}}$ ✓

(b) If $\sin x = \frac{1}{3}$, $\cos y = -\frac{1}{\sqrt{3}}$ such that $\frac{\pi}{2} < x < \pi$ and $\pi < y < \frac{3\pi}{2}$. **4**

Find the exact value of $\tan(x + y)$.



$\therefore \tan x = -\frac{1}{\sqrt{8}}$



$\therefore \tan y = \sqrt{2}$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{-\frac{1}{\sqrt{8}} + \sqrt{2}}{1 - \left(-\frac{1}{\sqrt{8}}\right)(\sqrt{2})} \begin{matrix}\times\sqrt{8}\\ \times\sqrt{8}\end{matrix}$$

$$= \frac{\sqrt{16} - 1}{\sqrt{8} + \sqrt{2}} = \frac{4-1}{3\sqrt{2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(c) Prove that $\frac{\sin 3\theta}{\sin \theta} + \frac{\cos 3\theta}{\cos \theta} = 4\cos 2\theta$ **3**

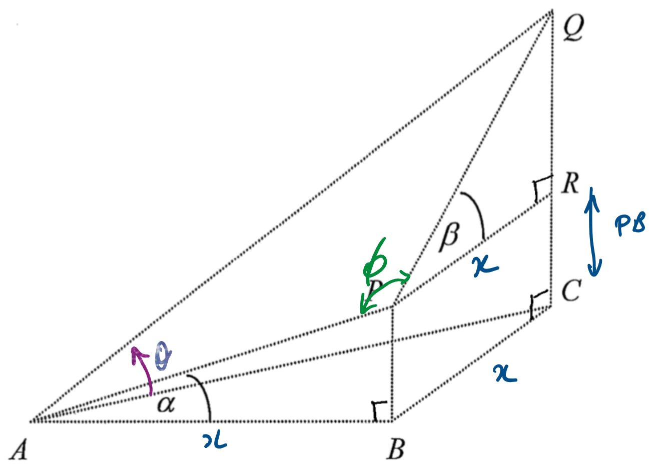
$$\frac{\sin 3\theta \cos\theta + \cos 3\theta \sin\theta}{\sin\theta \cos\theta}$$

$$= \frac{\sin(3\theta + \theta)}{\frac{1}{2} \times 2\sin\theta\cos\theta} = \frac{2\sin 4\theta}{\sin 2\theta}$$

$$= \frac{2 \times 2\cancel{\sin 2\theta}\cos 2\theta}{\cancel{\sin 2\theta}}$$

$$= 4\cos 2\theta$$

(d)



ABC is a horizontal, right-angled, isosceles triangle where $AB = BC$ and $\angle ABC = 90°$. P is vertically above B; Q is vertically above C and PR is parallel to BC. The angle of elevation of P from A, and Q from P are α and β respectively. The length of AB is x m.

i. Show that $QC = x(\tan\alpha + \tan\beta)$ **2**

$QC = QR + RC$.

$= QR + PB$

$\frac{QR}{PR} = \tan\beta$ $\quad \frac{PB}{AB} = \tan\alpha$

$QR = x\tan\beta$ $\quad PB = x\tan\alpha$

$\therefore QC = x\tan\beta + x\tan\alpha$

$= x(\tan\alpha + \tan\beta)$ ✓

ii. If the angle of elevation of Q from A is θ, prove that **2**

$$\tan\theta = \frac{\tan\alpha + \tan\beta}{\sqrt{2}}$$

$\tan\theta = \frac{QC}{AC}$

Also, in $\triangle ABC$,

$AC^2 = AB^2 + BC^2$

$= x^2 + x^2$

$= 2x^2$ ✓

$\therefore AC = \sqrt{2}x$

$\therefore \tan\theta = \frac{\cancel{x}(\tan\alpha + \tan\beta)}{\sqrt{2}\cancel{x}}$

$= \frac{\tan\alpha + \tan\beta}{\sqrt{2}}$ ✓

iii. if $\angle APQ = \phi$, prove that $\cos\phi = -\sin\alpha\sin\beta$ **3**

Apply the cosine rule in ΔAPQ:

$$AQ^2 = AP^2 + PQ^2 - 2(AP)(PQ)\cos\phi$$

$AP = \frac{x}{\cos\alpha}$

$PQ = \frac{x}{\cos\beta}$

$AQ = \frac{AC}{\cos\theta}$

$= \frac{x\sqrt{2}}{\cos\theta}$

$$\frac{2\cancel{x^2}}{\cos^2\theta} = \frac{\cancel{x^2}}{\cos^2\alpha} + \frac{\cancel{x^2}}{\cos^2\beta} - 2\left(\frac{\cancel{x}}{\cos\alpha}\right)\left(\frac{\cancel{x}}{\cos\beta}\right)\cos\phi$$

$$2\sec^2\theta = \sec^2\alpha + \sec^2\beta - 2\cos\phi(\sec\alpha)(\sec\beta)$$

$$2(1+\tan^2\theta) = (1+\tan^2\alpha) + (1+\tan^2\beta) - 2\cos\phi\sec\alpha\sec\beta$$

$$\cancel{2} + \cancel{2}\left(\frac{\tan\alpha+\tan\beta}{\cancel{\sqrt{2}}}\right)^2 = \cancel{2} + \tan^2\alpha + \tan^2\beta - 2\cos\phi\sec\alpha\sec\beta$$

$$\cancel{\tan^2\alpha} + \cancel{2}\tan\alpha\tan\beta + \cancel{\tan^2\beta} = \cancel{\tan^2\alpha} + \cancel{\tan^2\beta} - \cancel{2}\cos\phi\sec\alpha\sec\beta$$

$$\frac{\sin\alpha\sin\beta}{\cancel{\cos\alpha\cos\beta}} = -\frac{1}{\cancel{\cos\alpha\cos\beta}}\cos\phi$$

$$\therefore \cos\phi = -\sin\alpha\sin\beta$$

End of examination.