NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2020 Year 11 Course Assessment Task 4

Friday, 11 September 2020

General instructions

- Working time – 2 hours (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- A NESA Reference Sheet is provided.
- Attempt all questions.
- At the conclusion of the examination, arrange the booklets used in the correct order after this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided

SECTION II

- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** ....

Class: (please ✓)

- ○ 11MAX.1 – Ms Madden
- ○ 11MAX.2 – Mr Siu
- ○ 11MAX.3 – Ms Ham
- ○ 11MAX.4 – Mr Sekaran
- ○ 11MAX.5 – Mr Ho

Marker's use only

QUESTION	**1-5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**Total**	**%**
MARKS	$\overline{}$ 5	$\overline{}$ 15	$\overline{}$ 12	$\overline{}$ 10	$\overline{}$ 9	$\overline{}$ 11	$\overline{}$ 11	$\overline{}$ 12	$\overline{}$ 85	

Section I

5 marks
Attempt Questions 1 to 5
Allow approximately 8 minutes for this section

Mark your answers on the answer grid provided.

Questions **Marks**

1. Which of the following expressions is equal to $\frac{\sin\theta}{\sin\theta+\cos\theta} - \frac{\sin\theta}{\sin\theta-\cos\theta}$? **1**

(A) $\cot 2\theta$ (C) $\sec 2\theta$

(B) $\tan 2\theta$ (D) $\operatorname{cosec} 2\theta$

2. Let $g(x) = x^2 + 2x - 3$ and $f(x) = e^{2x+3}$

Which one of the following represents $f(g(x))$?

(A) $e^{4x+6} + 2e^{2x+3} - 3$

(B) e^{2x^2+4x+9}

(C) e^{2x^2+4x-3}

(D) e^{2x^2+4x-6}

3 Two of the roots of $4x^3 + 8x^2 + kx - 18 = 0$ are equal in magnitude but have opposite sign. What is the value of k? **1**

(A) -9 (B) 9 (C) -2 (D) 2

4. For the function $f(x) = e^{|x|} - 1$, which of the following statements is true? **1**

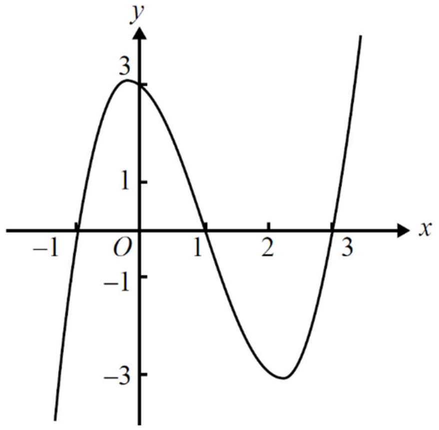
(A) The function is increasing for all x.

(B) The function has an asymptote at $y = -1$.

(C) The function is not differentiable at $x = 0$.

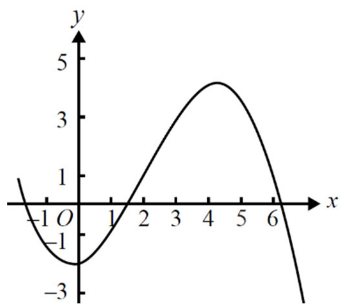
(D) The function has a stationary point at $x = 0$.

5. The graph of a function $y = f(x)$ is shown below. **1**

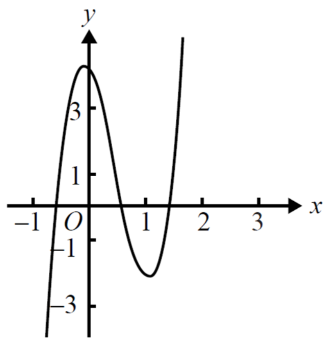


Which graph best represents the curve $y = 1 - f(2x)$?

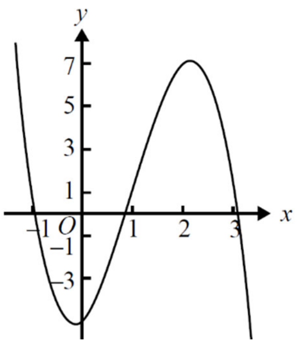
(A)



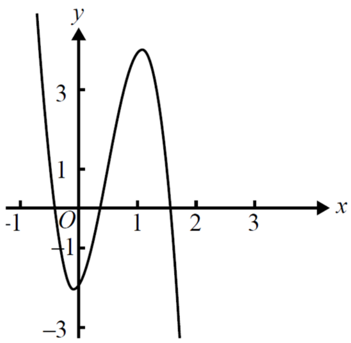
(C)



(B)



(D)



Section II

80 marks
Attempt Questions 6 to 12

Allow approximately 1 hour and 52 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 6-12, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks)

(a) Differentiate $f(x) = x^2 - 3x + 5$ using the first principles. **3**

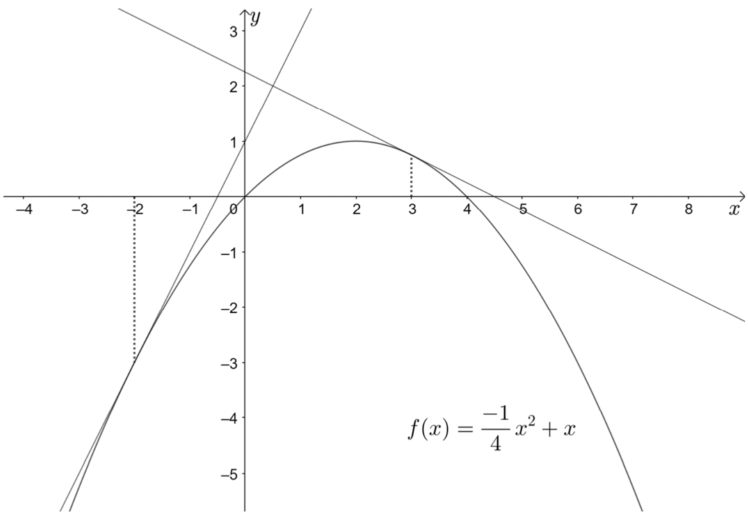
(b) Differentiate the following:

i. $x^2\sqrt{2x+1}$ **3**

ii. $\frac{e^{5x-x^3}}{1-x}$ **3**

(c) Find the equation of the tangent to the curve $y = 3x^3 - 2x^2 + 1$ at the $x = 1$. **3**

(d) The graph of $f(x) = -\frac{1}{4}x^2 + x$ is given below:



i. Find the expression for $f'(x)$. **1**

ii. Hence, show that the illustrated tangents are perpendicular. **2**

Question 7 (12 marks)

(a) Solve $\frac{1}{x} + \frac{x}{(x-2)} < 0$ for x, giving the answer in interval notation. **3**

(b) i. Sketch the graphs : $y = \sqrt{4 - x^2}$ and $y = |x + 2|$ on the same set of axes, clearly showing the x and y intercepts. **3**

ii. Hence or otherwise solve $\sqrt{4 - x^2} < |x + 2|$ **1**

(c) A particle moves in a straight line so that its displacement from a fixed point O is given by $x = t^4 - \frac{3}{2}t^2$ metres, t is the time in seconds after passing through O.

i. Find the expression for the velocity and acceleration of the particle at time t. **2**

ii. Find the time when the particle is at rest after leaving O. **1**

iii. Find the total distance travelled by the particle in the first 2 seconds **2**

Question 8 (10 marks)

(a) A spherical balloon leaks air such that the radius decreases at a rate of 0.5 cms^{-1}. **2**

Calculate the rate of change of the volume of the balloon in terms of π when the radius is 10cm.

(b) Newton's law of cooling states that when an object at temperature T°C is placed in an environment at temperature P°C, the rate of the temperature loss is given by the equation

$$\frac{dT}{dt} = -k(T - P)$$

where t is the time in seconds and k is a positive constant.

i. Verify that $T = P + Ae^{-kt}$ satisfies the above equation. **1**

ii. A soup with initial temperature at 24°C is placed in a freezer in which the internal temperature is maintained at -40°C. After 5 seconds, the temperature of the soup is 19°C. How long (to the nearest second) will it take for the soup's temperature to reduce to 0°C? **3**

(c) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t if $x = 2t^2 + 1$ and $y = \frac{1}{t}$. **4**

Question 9 (9 marks)

(a) If $\sin\theta = \frac{2}{3}$ and $\cos\theta < 0$, find the exact value of $\tan 2\theta$. **2**

(b) Solve $\frac{1}{\sec^2 x} + 3\sin\frac{x}{2}\cos\frac{x}{2} = 0$ for $0° \leq x \leq 360°$. **3**

(c) i. Prove that $\cos^2\left(\frac{\alpha+\beta}{2}\right) - \cos^2\left(\frac{\alpha-\beta}{2}\right) = -\sin\alpha\sin\beta$ **2**

[Hint: $a^2 - b^2 = (a-b)(a+b)$]

ii. Hence show that $\cos^2 75° - \cos^2 15° = -\frac{\sqrt{3}}{2}$. **2**

Question 10 (11 marks)

(a) Consider the function $f(x) = \frac{x}{x+3}$.

i. Show that $f'(x) > 0$ for all x in the domain. **1**

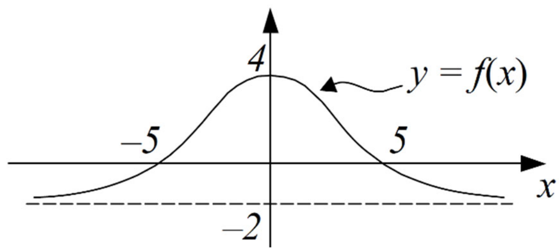
ii. State the equation of the horizontal and vertical asymptote of $y = f(x)$. **2**

iii. Without using any further calculus, sketch the graph of $y = f(x)$, showing important features. **2**

iv. Explain why $y = f(x)$ has an inverse function $y = f^{-1}(x)$. **1**

v. What is the domain of $f^{-1}(x)$? **1**

(b) The following diagram represent the graph of $y = f(x)$.



Draw separate sketches of the graphs of the following showing asymptote and intercepts:

(Each sketch should be roughly one-third of a page)

i. $y = \sqrt{f(x)}$ **2**

ii. $y = x + f(x)$ **2**

Question 11 (11 marks)

(a) Solve $3x^3 - 17x^2 - 8x + 12 = 0$, given that the product of two of the roots is 4. **3**

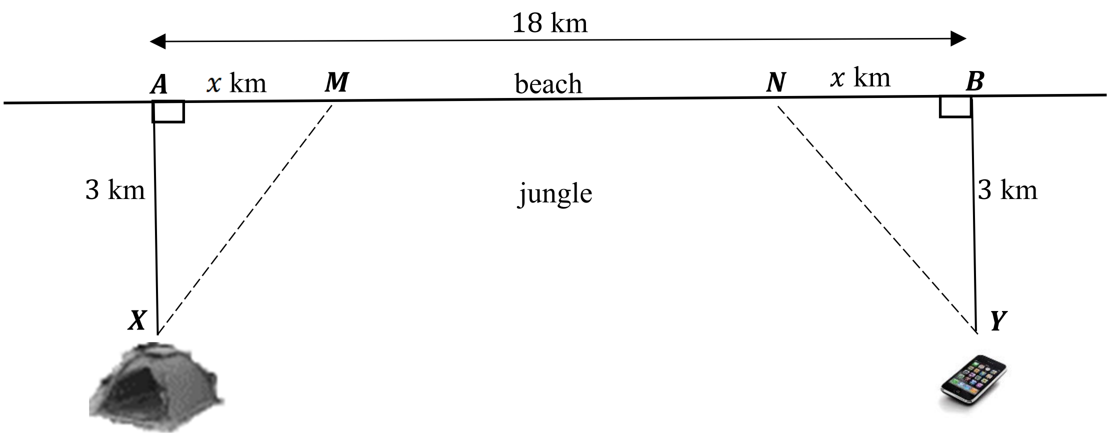
(b) Let $P(x) = x^3 - ax^2 + x$ be a polynomial, where a is a real number. When $P(x)$ is divided by $x - 3$ the remainder is 12. Find the remainder when $P(x)$ is divided by $x + 1$. **2**

(c) i. Prove that if α is a root of multiplicity r, where $r > 1$, of a polynomial equation $P(x) = 0$ then α is also a root of multiplicity $r - 1$ of $P'(x) = 0$. **3**

ii. Find all of the roots of the equation $x^4 - 6x^2 - 8x - 3 = 0$, given that it has a root of multiplicity 3. **3**

Question 12 (13 marks)

Remy loses his mobile phone while he is on a Duke of Edinburgh adventurous journey in a jungle. Remy's camp is at a location X which is 3 km south from a point A. Point A is on the straight beach.
He finds his phone at a location Y which is 3 km south from a point B. Point B, due east of A, is also on the straight beach as shown in the diagram below.
It is given that $AB = 18$ km and $AM = NB = x$ km and $AX = BY = 3$ km.



When Remy finds his phone, he is bitten by a snake which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50 \log_e(1 + 2t)$$

where y is the concentration, and t is the time in hours after the snake bites him. The toxin will kill him if its concentration reaches 100 units.

i. Find the time, to the nearest minute, that Remy has to find an antidote (that is, a cure for the toxin). **2**

Remy has an antidote to the toxin at his camp at X. He can run through the jungle at 5 km/h and he can run along the beach at 13 km/h.

ii. Show that he will not get the antidote in time if he runs directly to his camp parallel to AB through the jungle. **1**

In order to get the antidote, Remy runs through the jungle to N on the beach, runs along the beach to M and then runs through the jungle to the camp at X.

iii. Show that the time taken to reach the camp, T hours, is given by **2**

$$T = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$$

iv. His team manager informed Remy that the time T can be minimised when $x = \frac{5}{4}$ km. **2**

Can Remy's life be saved if he acts according to his team manager instruction?
Justify your answer.

At his camp, Remy takes a capsule containing 50 g of antidote to the toxin. After taking the capsule the quantity of drug in his body decreases over time. At exactly the same time on successive days, he takes another capsule of the same and again the quantity of antidote decreases in his body.

Only 40% of the total amount of the antidote present in Remy's body at the beginning of each day remains at the end of that day. Let A_n be the amount antidote present in his body at the end of n^{th} day before taking $(n+1)^{th}$ capsule.

v. Show that $A_3 = 20(1 + 0.4 + 0.4^2)$. **2**

vi. Remy will no longer be affected by the snake toxin when he first has 33 g of the antidote in his body at the end of the day just before taking the next capsule. **3**
How many days does he need to take a capsule?

End of examination

Answer Sheet for Section I

Mark answers to Section I by fully blackening the correct circle

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1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

1 — A ● C D
2 — A B ● D
3 — ● B C D
4 — A B ● D
5 — A B C ●

Q6

(a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) + 5 - (x^2+3x+5)}{h}$ ✓

$= \lim_{h \to 0} \frac{2xh + 3h + h^2}{h}$ ✓

$= \lim_{h \to 0} (2x + 3 + h) = 2x+3$ ✓

b) i) $y = x^2\sqrt{2x+1}$

$= \sqrt{2x^5+x^4}$ ✓

$y' = \frac{1}{2}(2x^5+x^4)^{-\frac{1}{2}}(10x^4+4x^3)$ ✓

$= \frac{x^2(5x^2+2x)}{x^2\sqrt{2x+1}} = \frac{5x^2+2x}{\sqrt{2x+1}}$ ✓ ✓

OR $y' = \sqrt{2x+1}\,(2x) + x^2\,\frac{1}{2}(2x+1)^{-1/2}.2$ ✓

$= 2x\sqrt{2x+1} + \frac{x^2}{\sqrt{2x+1}}$

$= \frac{2x(2x+1)+x^2}{\sqrt{2x+1}} = \frac{5x^2+2x}{\sqrt{2x+1}}$ ✓

ii) $y = \frac{e^{5x-x^3}}{1-x}$ ✓

$y' = \frac{(1-x)e^{5x-x^3}(5-3x^2) - e^{5x-x^3}(-1)}{(1-x)^2}$ ✓

$= \frac{e^{5x-x^3}}{(1-x)^2}(3x^3 - 3x^2 - 5x + 6)$ ✓

c) $y = 3x^3 - 2x^2 + 1$ ✓

$y' = 9x^2 - 4x$ ✓

At $x=1$ $y' = 5$ ✓, $y = 2$

∴ Equation of the tangent $y - 2 = 5(x-1)$ ✓

$y = 5x - 3$

d)

(i) $f'(x) = -\frac{1}{2}x + 1$ ✓

(ii) $f'(-2) = 2$ and $f'(3) = -\frac{3}{2} + 1 = -\frac{1}{2}$ ✓ both correct.

$f'(-2) \times f'(3) = -1$ ✓

∴ tangents are perpendicular to each other

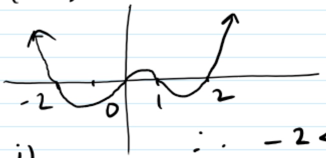
Q7

a) $\frac{1}{x} + \frac{x}{x-2} < 0$

$$\frac{x-2+x^2}{x(x-2)} < 0$$

$$\frac{(x+2)(x-1)}{x(x-2)} < 0$$

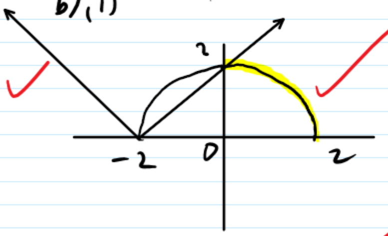
$\times\, x^2(x-2)^2 \qquad x(x-2)(x-1)(x+2) < 0$



$\therefore\ -2 < x < 0$ OR $1 < x < 2$

both correct

b) (i)



intercepts

(ii) $0 < x \leq 2$

b) $x = t^4 - \frac{3}{2}t^2$

i) $\dot{x} = 4t^3 - 3t$ and $\ddot{x} = 12t^2 - 3$

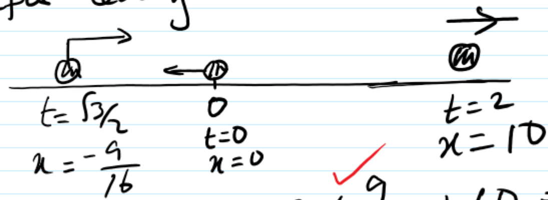
ii) Let $\dot{x} = 0$

then $t(4t^2 - 3) = 0$

$t = 0,\ t = \frac{\sqrt{3}}{2}$ since $t \geq 0$

$\therefore$ Instantaneously at rest when $t = \frac{\sqrt{3}}{2}$ s after leaving

iii)



Distance travelled $= 2 \times \frac{9}{16} + 10 = 10\frac{9}{8}$ m

Q8

(a)  $V = \frac{4}{3}\pi r^3$ $\quad \frac{dr}{dt} = -0.5$ cm/s

$$\frac{dv}{dr} = 4\pi r^2$$

$$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times (-0.5) = -2\pi r^2$$

when $r = 10$ $\quad \frac{dv}{dt} = -2\pi (10)^2 = -200\pi$ cm³/s

(b) i) $T = P + Ae^{-kt}$

$$L.H.S = \frac{dT}{dt}$$
$$= -k(Ae^{-kt})$$
$$= -k(T-P)$$
$$= R.H.S$$

$\therefore T = P + Ae^{-kt}$ is a solution.

ii) $t = 0 \quad T = 24°c$

$t = 5 \quad T = 19°c$

$t \to \infty \quad T \to -40°c$

$\therefore \quad P = -40°c$

$24 = -40 + A \Rightarrow A = 64$

$T = -40 + 64e^{-kt}$

$19 = -40 + 64e^{-5k}$

$e^{-5k} = \frac{59}{64}$

$k = -\frac{1}{5}\ln\frac{59}{64} = \frac{1}{5}\ln\left(\frac{64}{59}\right)$

$0 = -40 + 64e^{-kt}$

$e^{-kt} = \frac{40}{64} = \frac{5}{8}$

$t = -\frac{1}{k}\ln 5/8 = \frac{5\ln(5/8)}{\ln\left(\frac{59}{64}\right)}$

$= 28.90$ sec

$= 29$ sec

c) $\frac{dx}{dt} = 4t \quad \frac{dy}{dt} = -\frac{1}{t^2}$ both

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{1}{t^2} \times \frac{1}{4t} = -\frac{1}{4t^3}$

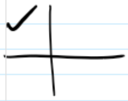
$\frac{d^2y}{dx^2} = \frac{d\left(-\frac{1}{4t^3}\right)}{dx} = \underline{d\left(-\frac{1}{4t^3}\right)}$

$t^4 \times \frac{1}{4t}$

$= \frac{3}{16t^5}$

Q_9 (a) $\sin\theta = 2/3$

✓



$\therefore \tan\theta = -\frac{2}{\sqrt{5}}$ ✓

$$\tan 2\theta = \frac{2\left(-\frac{2}{\sqrt{5}}\right)}{1-\left(-\frac{2}{\sqrt{5}}\right)^2} = \frac{-\frac{4}{\sqrt{5}}}{1/5}$$

$$= -4\sqrt{5}$$ ✓

(b) $\cos^2 x + \frac{3}{2}\sin x = 0$

$2 - 2\sin^2 x + 3\sin x = 0$

$2\sin^2 x - 3\sin x - 2 = 0$ ✓

$(2\sin x + 1)(\sin x - 2) = 0$ ✓

$\sin x = -1/2$ since $\sin x \leq 1$

$x = 210°, 330°$ ✓ both

c) (i) $L.H.S = \cos^2\left(\frac{\alpha+\beta}{2}\right) - \cos^2\left(\frac{\alpha-\beta}{2}\right)$

$= \left(\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\left(\frac{\alpha-\beta}{2}\right)\right)\left(\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)\right)$ ✓

$= -2\sin\frac{\alpha}{2}\sin\frac{\beta}{2} \cdot 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$ ✓

$= -\sin\alpha\sin\beta = R.H.S$ ✓

ii) Let $\alpha = 90°$, $\beta = 60°$

then $\frac{\alpha+\beta}{2} = 75°$, $\frac{\alpha-\beta}{2} = 15°$

$\therefore$ By (i) $\cos^2 75° - \cos^2 15°$

$= -\sin 90° \sin 60°$

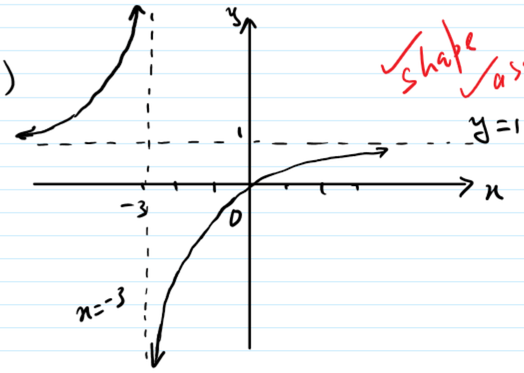
$= -\frac{\sqrt{3}}{2}$ ✓

Q10 (a)(i) $f(x) = \frac{x}{x+3} = 1 - \frac{3}{x+3}$

$f'(x) = \frac{3}{(x+3)^2} > 0 \quad \forall x$ except -3 ✓

ii) Horizontal asymptote : $y = 1$ ✓ vertical asymptote $x = -3$ ✓

iii)



✓shape
✓asymptote with x, y-int

iv) $y = f(x)$ has inverse function since for every y value in the range of f, there is one corresponding x-value in the domain. OR 1-1 function ✓

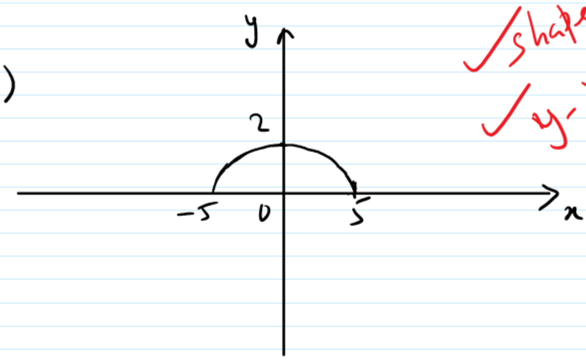
OR equivalent statement.

Note No mark be awarded if "passes the horizontal line test" stated without further explanation

v) Domain of f^{-1} = Range of f

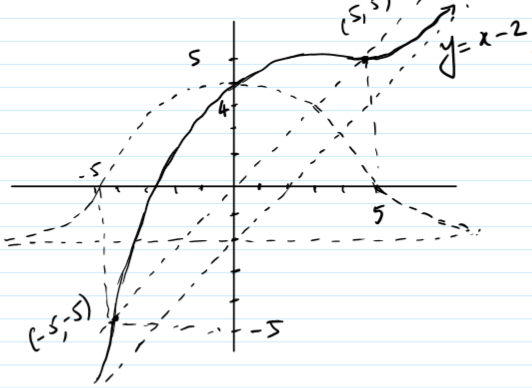
= all real x except $x = 1$ ✓

(b) i)



✓shape
✓y-intercept

ii)



✓shape
✓y-intercept and obliqus asymptote

Q11 Let the root be $\alpha, \frac{4}{\alpha}, \beta$

a) then $\alpha \cdot \frac{4}{\alpha} \cdot \beta = 4\beta = -\frac{12}{3} = -1$

$$3x^3 - 17x^2 - 8x + 12 = (x+1)(3x^2 - 20x + 12)$$
$$= (x+1)(3x-2)(x-6)$$

$\therefore x = -1, \frac{2}{3}, 6$

Must state the solutions of x

b) $P(3) = 27 - 9a + 3 = 12$

$9a = 18$

$a = 2$

remainder $= P(-1) = -1 - a - 1$

$= -2 - 2 = -4$

Common mistake: $(-1)^3 \neq +1$

c) Let α be a root of multiplicity r of $P(x) = 0$

(i) then $P(x) = (x-\alpha)^r Q(x)$ and $Q(\alpha) \neq 0$

Many struggled to select an appropriate expression for $P(x)$

$P'(x) = (x-\alpha)^r Q'(x) + m(x-\alpha)^{r-1} Q(x)$

Many added $R(x)$ to $P(x)$ but did not specify that $R(x) = 0$

$= (x-\alpha)^{r-1}[(x-\alpha)Q'(x) + rQ(x)]$

Many failed to use product rule to differentiate

$= (x-\alpha)^{r-1} H(x)$

where $H(x) = (x-\alpha)Q'(x) + rQ(x)$

and $H(\alpha) = rQ(\alpha) \neq 0$

Factor theorem is insufficient in showing that $x = \alpha$ has multiplicity $(r-1)$ in $P'(x)$

$\therefore x = \alpha$ is a root of multiplicity $r-1$ of $P'(x) = 0$

Write a conclusion

(ii) Let $P(x) = x^4 - 6x^2 - 8x - 3$

then $P'(x) = 4x^3 - 12x - 8$

$P''(x) = 12x^2 - 12$

$P'''(x) = 24x$

Let $P''(x) = 0$ then $x = \pm 1$

$\left.\begin{array}{l} P'(-1) = -4 + 12 - 8 = 0 \\ P(-1) = 1 - 6 + 8 - 3 = 0 \end{array}\right\}$ both

and $P'''(-1) = -24 \neq 0$

$\therefore x = -1$ is the triple root

$P(x) = (x+1)^3(x-3)$ $\therefore$ roots are $x = -1, 3$

Show substitution for $P(-1)$ etc.

Must show $P(-1) = 0$, $P'(-1) = 0$ AND $P''(-1) = 0$ to justify that $x+1$ is a triple root.
A brief explanation would have also been acceptable.

Answer the question

Q12 i) $y = 50\log_e(1+2t)$

$50\log_e(1+2t) = 100$

$1+2t = e^2$

$t = \frac{e^2-1}{2} = 3.19$ hrs

$\doteqdot$ 3hrs 12min

ii) $t = \frac{D}{S} = \frac{18}{5} = 3$ hrs 36 min

> 3 hrs 11 min

$\therefore$ Remy will not get antidote on time if he runs directly from y to x.

Must have a conclusion

iii) Distance $MX = NY = \sqrt{9+x^2}$

and $MN = 18 - 2x$

$\therefore$ Total time $T = \frac{2\sqrt{9+x^2}}{5} + \frac{18-2x}{13}$

$= 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$

iv) When $x = \frac{5}{4}$ km $T = 2\left(\frac{\sqrt{9+\frac{25}{16}}}{5} + \frac{9-5/4}{13}\right)$

$= 2.49 \doteqdot 2$ hrs 30 mins

< 3 hrs 11 min

$\therefore$ Remy's life can be saved.

Must have a conclusion

iv) $A_1 = 50 \times 0.4 = 20$

$A_2 = (A_1 + 50) \times 0.4$

$= 20 \times 0.4 + 20$

$= 20(1+0.4)$

How pattern for A_3 is developed must be shown clearly

$A_3 = (A_2 + 50) \times 0.4$

$= 20(0.4 + 0.4^2) + 20$

$= 20(1 + 0.4 + 0.4^2)$

v) $A_n = 20(1 + 0.4 + \dots + 0.4^{n-1})$

$= 20\frac{(1-0.4^n)}{1-0.4} = \frac{100}{3}(1-0.4^n) = 33$

$1 - 0.4^n = 0.99$

$0.4^n = 0.01$

$n = \frac{\ln 0.01}{\ln 0.4} = 5.025$

Large number of students used trial-error method rather than using GP

$\therefore$ Remy needs to take the capsule for 6 days.