NORTH SYDNEY BOYS HIGH SCHOOL

100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 1)

2013 Preliminary Course Final Examination

Thursday August 29, 2013

General instructions

- Working time – 2 hours.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 11M3A – Mr Ireland

◯ 11M3B – Mr Berry

◯ 11M3C – Ms Ziaziaris

◯ 11M3D – Mr Lam

◯ 11M3E – Mr Lowe

◯ 11M3F – Mr Choy

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	%
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	

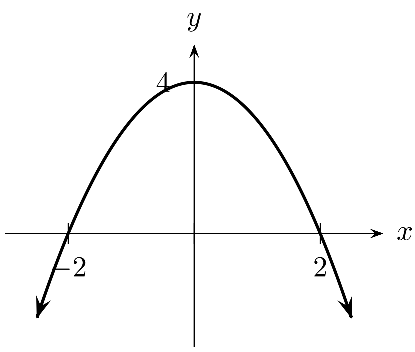
Section I

10 marks
Attempt Question 1 to 10
Allow approximately 10 minutes for this section

Mark your answers on the answer sheet provided.

Questions **Marks**

1. Which equation corresponds to the graph below? **1**



(A) $y = 2 - x^2$ (B) $y = 4 - x^2$ (C) $y = x + 4$ (D) $y = x^2 + 4$

2. The point P divides the interval $A\left(\frac{17}{3}, 2\right)$ and $B(-3, 4)$ externally in the ratio $2 : 3$. Which of the following are the coordinates of P? **1**

(A) $(-23, 2)$ (B) $(-9, -12)$ (C) $(9, 0)$ (D) $(23, -2)$

3. Which of the following conditions for $\dfrac{dP}{dt}$ and $\dfrac{d^2P}{dt^2}$ describe the slowing growth of a variable P? **1**

(A) $\dfrac{dP}{dt} > 0$ and $\dfrac{d^2P}{dt} > 0$.

(B) $\dfrac{dP}{dt} < 0$ and $\dfrac{d^2P}{dt} < 0$.

(C) $\dfrac{dP}{dt} > 0$ and $\dfrac{d^2P}{dt} < 0$.

(D) $\dfrac{dP}{dt} < 0$ and $\dfrac{d^2P}{dt} > 0$.

4. Which of the following best describes the function $f(x) = \dfrac{x^2 - 1}{x}$? **1**

(A) Even function

(B) Odd function

(C) Neither odd nor even function

(D) Zero function

5. Which of the following is an expression for $\dfrac{d}{dx}\left(x\sqrt{x^2+2}\right)$? **1**

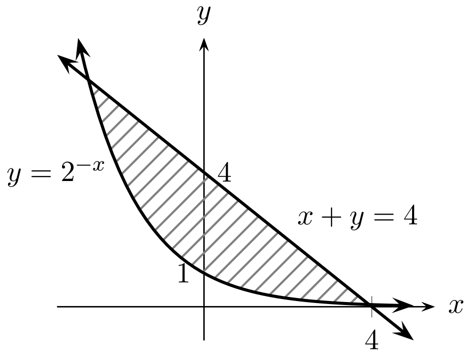
(A) $\dfrac{x}{\sqrt{x^2+2}}$

(B) $1+\dfrac{x}{x^2+2}$

(C) $\dfrac{2x^2+2}{\sqrt{x^2+2}}$

(D) $\dfrac{2x^2+x+4}{\sqrt{x^2+2}}$

6. Which pair of inequalities defines the shaded region? **1**



(A) $\begin{cases} x+y \geq 4 \\ y \geq 2^{-x} \end{cases}$

(B) $\begin{cases} x+y \leq 4 \\ y \geq 2^{-x} \end{cases}$

(C) $\begin{cases} x+y \geq 4 \\ y \leq 2^{-x} \end{cases}$

(D) $\begin{cases} x+y \leq 4 \\ y \leq 2^{-x} \end{cases}$

7. If $3^a = 7^b$, what is the value of $9^{\frac{a}{b}}$? **1**

(A) 0.56 (B) 1.77 (C) 3.46 (D) 49

8. Which of the following exists at the point $(1,1)$ on the graph of $y=(x-1)^3+1$? **1**

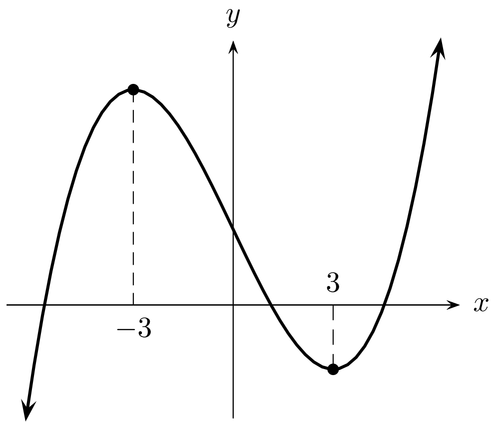
(A) A local minimum.

(B) A local maximum.

(C) A stationary point of inflexion.

(D) None of the above.

9. From the graph of $y = f(x)$, when is $f'(x)$ negative? **1**



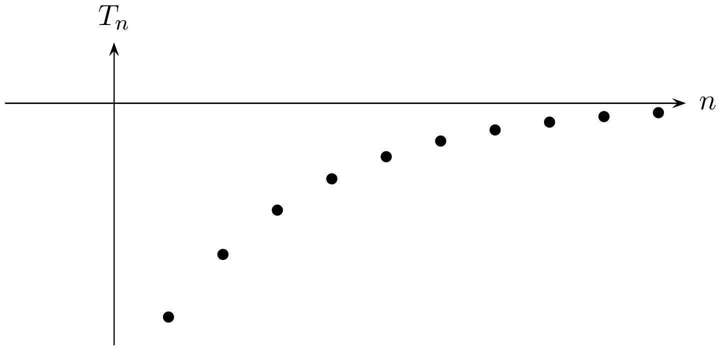
(A) $x < -3$ or $x > 3$

(B) $-3 < x < 3$

(C) $x \le -3$ or $x \ge 3$

(D) $-3 \le x \le 3$

10. The graph shows consecutive terms of a sequence. Which of the following statements best describes the sequence? **1**



(A) geometric, $|r| \ge 1$.

(B) geometric, $|r| < 1$.

(C) arithmetic, $|d| \ge 1$.

(D) arithmetic, $|d| < 1$.

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 1 hour and 50 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Fully factorise: $x^2 - 2x + 1 - 4y^2$ **2**

(b) Fully simplify: $\sqrt{\dfrac{3^{5k+2}}{27^k}}$. **2**

(c) Solve $x^2 + 2x - 8 > 0$. **2**

(d) Solve $|x + 3| < 2$. **2**

(e) Evaluate $\displaystyle\lim_{x \to -1} \frac{x^2 - 2x - 3}{2x^2 - x - 3}$. **2**

(f) Fully simplify: $\dfrac{\log\left(a^2b^2\right) - \log\left(ab^2\right)}{\log\sqrt{a}}$. **2**

(g) Solve the following inequation: $\dfrac{-1}{x(x-2)} < 1$. **3**

Question 12 (15 Marks) Commence a NEW page. **Marks**

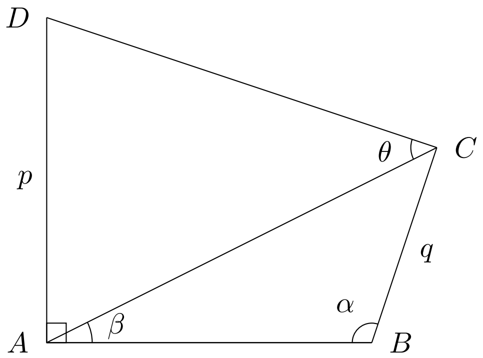
(a) Solve $\tan^2\theta = \dfrac{1}{3}$ for $0^\circ \le \theta \le 360^\circ$. **3**

(b) If $\tan\theta = -\dfrac{4}{5}$ and $\cos\theta > 0$, find the exact value of $\sin\theta$. **2**

(c) Prove that $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{1+\cos\theta}{\sin\theta}$. **2**

(d) $ABCD$ is a quadrilateral with AD perpendicular to AB.

Given that $\angle CAB = \beta$, $\angle ABC = \alpha$, $\angle ACD = \theta$, $AD = p$ and $BC = q$,

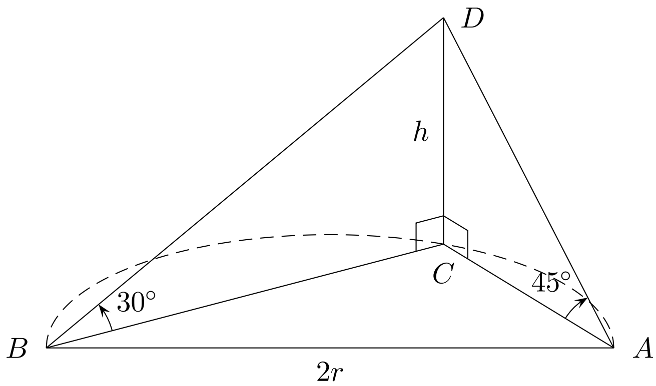


i. Show that $\angle ADC = 90^\circ - (\theta - \beta)$. **2**

ii. Using the sine rule, prove that **3**

$$q = \frac{p\sin\beta\cos(\theta-\beta)}{\sin\theta\sin\alpha}$$

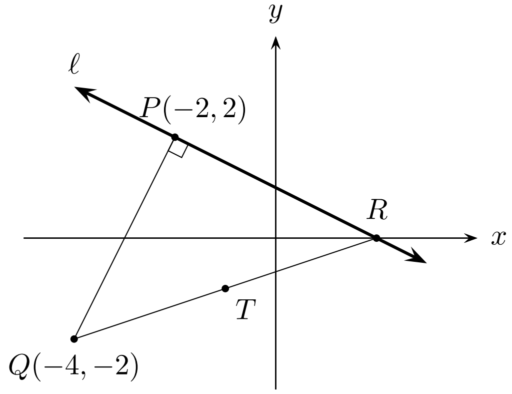
(e) AB is the diameter of a semicircular block of land with radius r metres. CD is a vertical flagpole of height h metres standing on its base with C on the arc AB. From A and B, the angles of elevation to the top of the flagpole (D) are 45° and 30° respectively. **3**



Show that $h = r$.

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) In the diagram, $P(-2,2)$ and $Q(-4,-2)$ are marked.



i. Show that the equation of the line ℓ through P, perpendicular to PQ is given by **1**

$$x + 2y - 2 = 0$$

ii. The line ℓ intersects the x axis at R. Find the coordinates of R. **1**

iii. Show that the midpoint T, of QR is $(-1,-1)$. **1**

iv. Find the perpendicular distance from T to the interval PR. **2**

(b) Consider the series 721, 708, 695...

i. Show that this forms an arithmetic series. **1**

ii. Show that $T_n = 734 - 13n$, where T_n is the n-th term of the series. **2**

iii. Find the value of the first negative term in the series. **2**

(c) Given x and $y > 0$,

- x, -2 and y forms a geometric series
- -2, y and x forms an arithmetic series.

i. Find the value of xy. **1**

ii. Find the value(s) of x and y. **2**

iii. Find the limiting sum (where it exists) of the geometric series **2**

$$x, -2, y \cdots$$

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) Differentiate the following functions with respect to x.

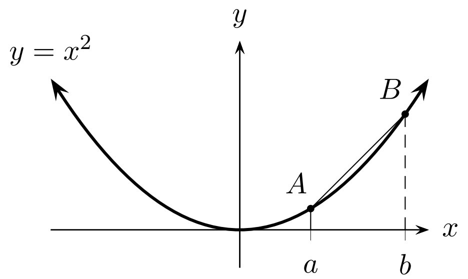
i. $y = x^5 + 5x^2 - 6x + 7$. **1**

ii. $y = (3x-1)^5$. **1**

iii. $y = \dfrac{x-1}{x^2+1}$. **2**

iv. $y = \dfrac{1}{2\sqrt{x}}$. **2**

(b) In the diagram, A and B are points on the parabola $y = x^2$. The x coordinate of A is a, and the x coordinate of B is b.

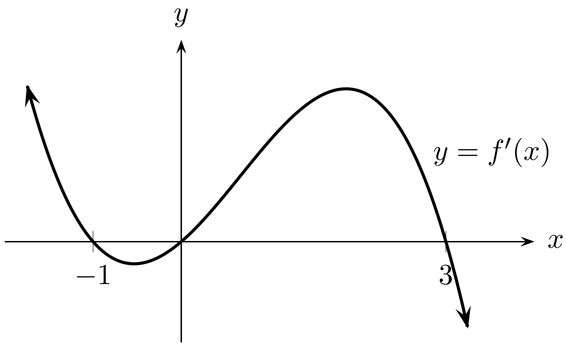


i. Find the gradient of the chord AB in terms of a and b. **1**

ii. What is the limiting value of the gradient of the chord AB as b approaches a? **1**

iii. What is the geometric significance of this limiting value? **1**

(c) The diagram shows the graph of $y = f'(x)$ for a certain function $y = f(x)$. **3**



If $f(x)$ has exactly two zeros, sketch a possible graph of $y = f(x)$. Do *not* find the values of the intercepts or the y values of the stationary points.

(d) The normal to the curve $y = \sqrt{x}$ at the point P on the curve has an angle of inclination of $135°$ with the positive x axis. **3**

Find the coordinates of the point P.

Question 15 (15 Marks) Commence a NEW page. **Marks**

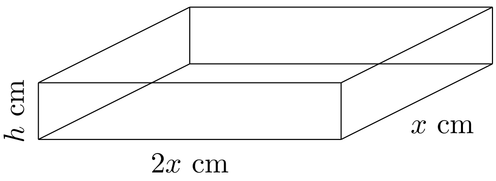
(a) Consider the function $f(x) = x^4 - 4x^3 + 1$.

i. Find the coordinates of the stationary points of $f(x)$, and determine their nature. **4**

ii. Find the coordinates of the points of inflexion. **2**

iii. Draw a neat sketch of the curve $y = f(x)$, showing the above features. **2**

(b) Joe is building a small *open topped* toy box, indicated by the diagram below. The box is twice as long as it is wide. The box has a total external surface area of $3\,750\,\mathrm{cm}^2$. **Note:** the box does not have a lid.



i. Show that the height of the box is given by **2**

$$h = \frac{625}{x} - \frac{x}{3}$$

ii. Find the dimensions of the box which would give the maximum volume. **3**

iii. Joe decides the height of the box cannot exceed $10\frac{5}{6}$ cm. **2**

Find the new dimensions of the box if the surface area was to remain at $3\,750\,\mathrm{cm}^2$.

Question 16 (15 Marks) Commence a NEW page. **Marks**

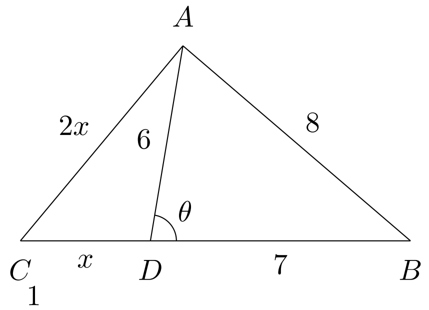
(a) The position x metres at time t seconds of a particle moving in a straight line is given by

$$x = 2t^3 - 9t^2 + 12t + 6$$

i. Find the times when the velocity is zero. **2**

ii. Find the average velocity between $t = 0$ and $t = 1$. **1**

(b) The diagram shows $\triangle ABC$ with D on CB such that $AD = 6$, $DB = 7$, $AB = 8$, $CD = x$, $AC = 2x$ and $\angle ADB = \theta$.



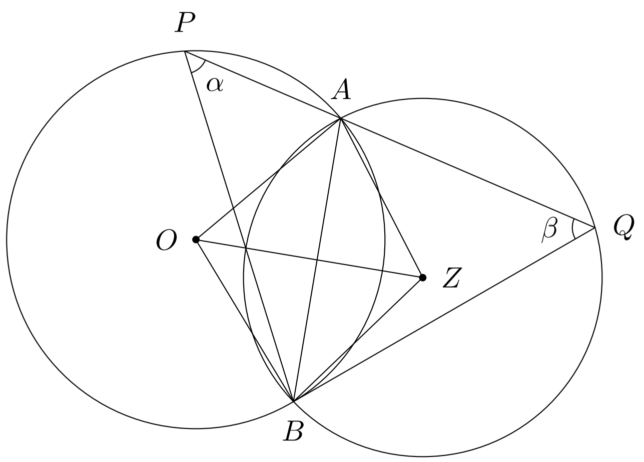
i. Show that $\cos\theta = \dfrac{1}{4}$. **1**

ii. Hence or otherwise, find the value of x. **3**

Question 16 continued on the next page...

Question 16 continued from the previous page...

(c) A line through A meets the two circles (with centres O and Z) again at P and Q. Let $\angle BPQ = \alpha$ and $\angle AQB = \beta$.



Copy or trace the diagram into your writing booklet.

Hint: all parts can be completed without additional constructions.

i. Prove that $\triangle AOZ \equiv \triangle BOZ$. **3**

ii. Prove that OZ bisects $\angle AOB$ and $\angle AZB$. **2**

iii. Hence prove that $\angle BOZ = \alpha$ and $\angle BZO = \beta$. **2**

iv. Prove that $\angle PBQ = \angle OBZ$. **1**

End of paper.

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Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

○ 11M3A – Mr Ireland

○ 11M3B – Mr Berry

○ 11M3C – Ms Ziaziaris

○ 11M3D – Mr Lam

○ 11M3E – Mr Lowe

○ 11M3F – Mr Choy

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
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8 – Ⓐ Ⓑ Ⓒ Ⓓ
9 – Ⓐ Ⓑ Ⓒ Ⓓ
10 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

1. (B) **2.** (D) **3.** (C) **4.** (B) **5.** (C)
6. (B) **7.** (D) **8.** (C) **9.** (B) **10.** (B)

Question 11 (Choy)

(a) (2 marks)

✓ [1] for significant progress towards answer.

✓ [1] for final answer.

$$\begin{aligned} & x^2 - 2x + 1 - 4y^2 \\ &= (x-1)^2 - 4y^2 \\ &= (x-1-2y)(x-1+2y) \end{aligned}$$

(b) (2 marks)

✓ [1] for significant progress towards answer.

✓ [1] for final answer.

$$\begin{aligned} \sqrt{\frac{3^{5k+2}}{27^k}} &= \sqrt{\frac{3^{5k+2}}{3^{3k}}} \\ &= \left(3^{2k+2}\right)^{\frac{1}{2}} \\ &= 3^{k+1} \end{aligned}$$

(c) (2 marks)

$$\begin{gathered} x^2 + 2x - 8 > 0 \\ (x+4)(x-2) > 0 \\ x < -4 \text{ or } x > 2 \end{gathered}$$

(d) (2 marks)

$$\begin{gathered} |x+3| < 2 \\ -2 < x + 3 < 2 \\ -5 < x < -1 \end{gathered}$$

(e) (2 marks)

$$\begin{aligned} & \lim_{x \to -1} \frac{x^2 - 2x - 3}{2x^2 - x - 3} \\ &= \lim_{x \to -1} \frac{(x-3)\cancel{(x+1)}}{(2x-3)\cancel{(x+1)}} \\ &= \frac{-4}{-5} = \frac{4}{5} \end{aligned}$$

(f) (2 marks)

$$\begin{aligned} & \frac{\log\left(a^2b^2\right) - \log\left(ab^2\right)}{\log\sqrt{a}} \\ &= \frac{\log\left(\frac{a^{\cancel{2}}\cancel{b^2}}{\cancel{a}\cancel{b^2}}\right)}{\frac{1}{2}\log a} \\ &= \frac{2\log a}{\log a} = 2 \end{aligned}$$

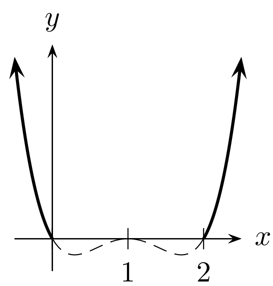
(g) (3 marks)

✓ [1] for eliminating denominator with unknown.

✓ [1] for further significant progress towards answer.

✓ [1] for final answer.

$$\begin{gathered} \underset{\times x^2(x-2)^2}{\frac{-1}{x(x-2)}} < \underset{\times x^2(x-2)^2}{1} \\ -x(x-2) < x^2(x-2)^2 \\ x^2(x-2)^2 + x(x-2) > 0 \\ x(x-2)\left(x(x-2)+1\right) > 0 \\ x(x-2)\left(x^2-2x+1\right) > 0 \\ x(x-2)(x-1)^2 > 0 \end{gathered}$$



$$\therefore x < 0 \text{ or } x > 2$$

Question 12 (Ziaziaris)

(a) (3 marks)

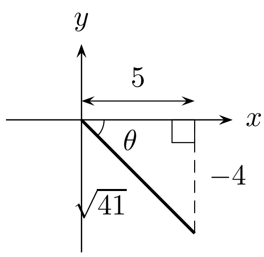
✓ [1] for significant progress towards answer.

✓ [1] for each pair of answers.

$$\begin{aligned} \tan^2\theta &= \frac{1}{3} \\ \tan\theta &= \pm\frac{1}{\sqrt{3}} \\ \therefore \theta &= 30^\circ, 150^\circ, 210^\circ, 330^\circ \end{aligned}$$

(b) (2 marks)

$$\tan\theta = -\frac{4}{5} \qquad \cos\theta > 0$$



$$\therefore \sin\theta = -\frac{4}{\sqrt{41}}$$

(c) (2 marks)

$$\begin{aligned} \frac{\sin\theta}{1-\cos\theta}\,\frac{\times(1+\cos\theta)}{\times(1+\cos\theta)} &= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} \\ &= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} \\ &= \frac{1+\cos\theta}{\sin\theta} \end{aligned}$$

(d) i. (2 marks)

✓ [1] for significant progress towards answer.

✓ [1] for final answer.

- $\angle DAC = 90^\circ - \beta$ (complementary $\angle$)
- $\therefore \angle ADC = 180^\circ - \theta - (90^\circ - \beta)$ ($\angle$ sum of $\triangle ADC$).
  $= 90^\circ - (\theta - \beta)$.

ii. (3 marks)

✓ [1] for correct application of sine rule

✓ [1] for further significant progress towards answer.

✓ [1] for final answer.

- In $\triangle ABC$,

$$\begin{aligned} \frac{q}{\sin\beta} &= \frac{AC}{\sin\alpha} \\ AC &= \frac{q\sin\alpha}{\sin\beta} \qquad (1) \end{aligned}$$

- In $\triangle ADC$,

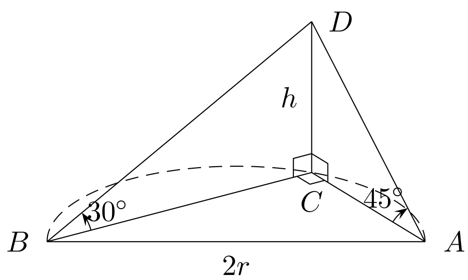
$$\begin{aligned} \frac{p}{\sin\theta} &= \frac{AC}{\sin(90^\circ - (\theta-\beta))} \\ \frac{p\cos(\theta-\beta)}{\sin\theta} &= AC \qquad (2) \end{aligned}$$

- Equate (1) and (2)

$$\begin{aligned} \frac{q\sin\alpha}{\sin\beta} &= \frac{p\cos(\theta-\beta)}{\sin\theta} \\ \therefore q &= \frac{p\sin\beta\cos(\theta-\beta)}{\sin\theta\sin\alpha} \end{aligned}$$

(e) (3 marks)

✓ [1] for each correct dot point.



- $\angle BCA = 90^\circ$ ($\angle$ in a semicircle)
- In $\triangle BCD$,

$$\begin{aligned} \tan 30^\circ &= \frac{h}{BC} \\ BC &= \frac{h}{\tan 30^\circ} = h\sqrt{3} \end{aligned}$$

Similarly,

$$AC = \frac{h}{\tan 45^\circ} = h$$

- In $\triangle ABC$,

$$\begin{aligned} AC^2 + BC^2 &= AB^2 \\ (2r)^2 &= h^2 + 3h^3 \\ 4h^2 &= 4r^2 \\ \therefore r &= h \end{aligned}$$

Question 13 (Berry)

(a) i. (1 mark)

✓ [1] All working must be shown to obtain full mark.

$$m_{PQ} = \frac{2-(-2)}{-2-(-4)} = \frac{4}{2} = 2$$

Equation of line ℓ: $m_{\perp} = -\frac{1}{2}$

$$y = -\frac{1}{2}x + b$$

As ℓ passes through $(-2, 2)$,

$$\begin{aligned} 2 &= -\frac{1}{2}(-2) + b \\ b &= 1 \\ \therefore \underset{\times -2}{y} &= \underbrace{-\frac{1}{2}x + 1}_{\times -2} \\ -2y &= x - 2 \\ x + 2y - 2 &= 0 \end{aligned}$$

ii. (1 mark) When $y = 0$,

$$\begin{aligned} x + 2(0) - 2 &= 0 \\ \therefore x &= 2 \\ \therefore R&(2,0) \end{aligned}$$

iii. (1 mark)

$$R(2,0) \qquad Q(-4,-2)$$

$$\begin{aligned} T &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{2+(-4)}{2}, \frac{0+(-2)}{2}\right) = (-1,-1) \end{aligned}$$

iv. (2 marks)

$$PR: x + 2y - 2 = 0 \qquad T(-1,-1)$$

$$\begin{aligned} d_{\perp} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \\ &= \frac{|1(-1) + 2(-1) - 2|}{\sqrt{1^2+2^2}} \\ &= \frac{|-1-2-2|}{\sqrt{1+4}} \\ &= \frac{5}{\sqrt{5}} = \sqrt{5} \end{aligned}$$

(b) i. (1 mark) If series is arithmetic,

$$\begin{aligned} T_2 - T_1 &= 708 - 721 = -13 \\ T_3 - T_2 &= 695 - 708 = -13 \end{aligned}$$

Hence series is arithmetic.

ii. (2 marks)

$$\begin{aligned} T_n &= 721 - 13(n-1) \\ &= 721 - 13n + 13 = 734 - 13n \end{aligned}$$

iii. (2 marks)
The first negative term occurs when $T_n < 0$.

$$\begin{aligned} \therefore 734 - 13n &< 0 \\ \underset{\div 13}{13n} &> \underset{\div 13}{734} \\ n &> \frac{734}{13} = 56.46\cdots \\ \therefore n &= 57 \\ T_{57} = 734 - 13(57) &= -7 \end{aligned}$$

(c) i. (1 mark) For a GP, $\frac{T_2}{T_1} = \frac{T_3}{T_2}$.

$$\begin{aligned} \frac{-2}{x} &= \frac{y}{-2} \\ \therefore xy &= 4 \end{aligned}$$

ii. (2 marks)
For the AP,

$$\begin{aligned} T_2 - T_1 &= T_3 - T_2 \\ y - (-2) &= x - y \\ 2y &= x - 2 \end{aligned}$$

As $xy = 4$, then $y = \frac{4}{x}$

$$\begin{aligned} 2\left(\frac{4}{x}\right) &= x - 2 \\ 8 &= x^2 - 2x \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ \therefore x &= 4, -2 \\ y &= 1, -2 \end{aligned}$$

Since x and $y > 0$, then $x = 4$ and $y = 1$ only.

iii. (2 marks)
Series: 4, -2, 1, with $r = -\frac{1}{2} < 1$.

$$S = \frac{a}{1-r} = \frac{4}{1-\left(-\frac{1}{2}\right)} = \frac{8}{3}$$

Question 14 (Ireland)

(a) i. (1 mark)

$$\frac{dy}{dx} = 5x^4 + 10x - 6$$

ii. (1 mark)

$$\frac{dy}{dx} = 5(3x-1)^4 \times 3 = 15(3x-1)^4$$

iii. (2 marks)

$$\begin{aligned} y &= \frac{x-1}{x^2+1} \\ u = x - 1 &\quad v = x^2 + 1 \\ u' = 1 &\quad v' = 2x \\ \frac{dy}{dx} &= \frac{(x^2+1) - 2x(x-1)}{(x^2+1)^2} \end{aligned}$$

iv. (2 marks)

$$\begin{aligned} y &= \frac{1}{2}x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= -\frac{1}{4}x^{-\frac{3}{2}} \end{aligned}$$

(b) i. (1 mark)

$$A\left(a, a^2\right) \quad B\left(b, b^2\right)$$

$$\begin{aligned} m_{AB} &= \frac{b^2 - a^2}{b - a} = \frac{\cancel{(b-a)}(b+a)}{\cancel{b-a}} \\ &= a + b \end{aligned}$$

ii. (1 mark)
As $b \to a$, $m_{AB} \to 2a$.

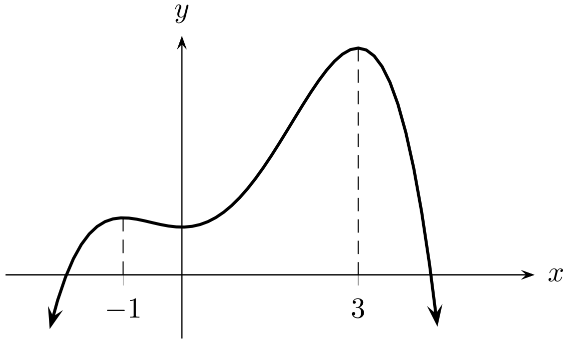
iii. (1 mark)
Value of the limit which leads to gradient of tangent to $y = x^2$ at $x = a$.

(c) (3 marks)

- ✓ [3] local max/min in correct locations, two zeros indicated, correct x values, no 'kinks', graph neat and clear.
- ✓ [2] quartic, and one of the criterion above is absent.
- ✓ [1] shows $y' = 0$ (max/min).

x		-1		0		3	
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$	0	$-$
y	$\nearrow$	—	$\searrow$	—	$\nearrow$	—	$\searrow$

Example:



(d) (3 marks)

- ✓ [1] for correct differentiation.
- ✓ [1] for $m_\perp$.
- ✓ [1] for point.

$$\begin{aligned} y &= x^{\frac{1}{2}} \\ m_{\tan} &= \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \end{aligned}$$

As angle of inclination is 135°,

$$\begin{aligned} m_\perp &= \tan 135^\circ = -1 \\ \therefore m_{\tan} &= 1 \\ \therefore \frac{1}{2}x^{-\frac{1}{2}} &= 1 \\ \frac{1}{2\sqrt{x}} &= 1 \\ 2\sqrt{x} &= 1 \\ \sqrt{x} &= \frac{1}{2} \\ x &= \frac{1}{4} \text{ (note domain: } x \ge 0) \\ \therefore y &= \sqrt{\frac{1}{4}} = \frac{1}{2} \\ \therefore P &\left(\frac{1}{4}, \frac{1}{2}\right) \end{aligned}$$

Question 15 (Lowe)

(a) i. (4 marks)

✓ [1] for each correct coordinate.

✓ [1] for each correct classification.

$$f(x) = x^4 - 4x^3 + 1$$
$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Stationary points occur when $f'(x) = 0$:

$$f'(x) = 0$$
$$4x^2(x-3) = 0$$
$$x = 0, 3$$
$$y = 1, -26$$

x	-1	0	1	3	4
$\frac{dy}{dx}$	$-$ -16	0	$-$ -8	0	$+$ 64
y	↘	1	↘	-26	↗

Hence $(-1, 1)$ is a horizontal point of inflexion, and $(3, -26)$ is a local minimum. (Alternatively, check stationary points via second derivative)

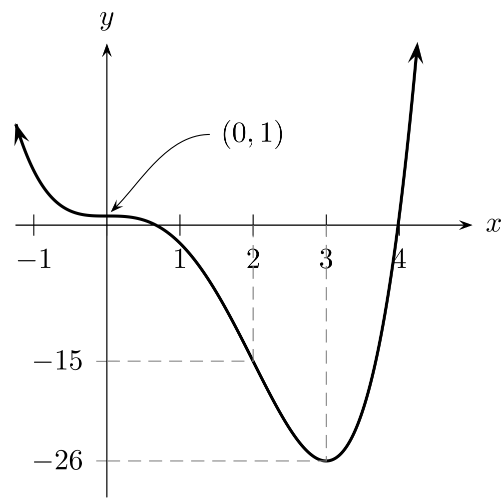
ii. (2 marks)
Points of inflexion occur when $f''(x) = 0$:

$$12x(x-2) = 0$$
$$\therefore x = 0, 2$$

x	-1	0	1	2	3
$\frac{d^2y}{dx^2}$	$+$ 12 $\smile$	0	$-$ -12 $\frown$	0	$+$ 36 $\smile$

As there are changes in concavity at $x = 0$ and $x = 2$, therefore $(0, 1)$ is a horizontal point of inflexion and $(2, -15)$ is a point of inflexion.

iii. (2 marks)



(b) i. (2 marks)

$$\begin{aligned} SA = 3\,750 &= 2x^2 + 2(2xh) + 2(xh) \\ 2x^2 + 6xh &= 3\,750 \\ x^2 + 3xh &= 1\,875 \\ 3xh &= 1\,875 - x^2 \\ h &= \frac{1875 - x^2}{3x} = \frac{625}{x} - \frac{x}{3} \end{aligned}$$

ii. (3 marks)

- ✓ [1] for finding $x = 25$.
- ✓ [1] for checking for local max.
- ✓ [1] for dimensions.

$$\begin{aligned} V &= 2x^2 h \\ &= 2x^2 \left(\frac{625}{x} - \frac{x}{3}\right) \\ &= 1\,250x - \frac{2}{3}x^3 \end{aligned}$$

Differentiating w.r.t x,

$$\frac{dV}{dx} = 1\,250 - 2x^2$$

Stationary points occur when $\frac{dV}{dx} = 0$:

$$\begin{aligned} 2x^2 &= 1\,250 \\ x^2 &= 625 \quad \Rightarrow \quad x = 25 \end{aligned}$$

Test for local maximum at $x = 25$:

$$\frac{d^2V}{dx^2} = -4x$$

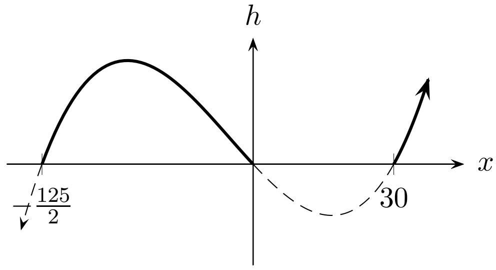
When $x > 0$, $\frac{d^2V}{dx^2} < 0$. Hence $x = 25$ is a local maximum. Finding dimensions:

$$h = \frac{625}{25} - \frac{25}{3} = \frac{50}{3}$$

Hence dimensions are 50 cm, $\frac{50}{3}$ cm, 25 cm

iii. (2 marks)
If $h < 10\frac{5}{6}$ $\left(= \frac{65}{6}\right)$,

$$\begin{aligned} \underbrace{\frac{625}{x} - \frac{x}{3}}_{\times 6x^2} &< \underset{\times 6x^2}{\frac{65}{6}} \\ 3\,750x - 2x^3 &< 65x^2 \\ 2x^3 + 65x^2 - 3\,750x &> 0 \\ x\left(2x^2 + 65x - 3\,750\right) &> 0 \\ x(2x + 125)(x - 30) &> 0 \end{aligned}$$



From the inequality, for $h < \frac{65}{6}$, $x > 30$ (and boundary condition of $x > 0$ applies as x is a length). Hence $x = 30$, $2x = 60$ and $h = \frac{65}{6}$.

Check:

$$\frac{625}{30} - \frac{30}{3} = \frac{125}{6} - 10 = 10\frac{5}{6}$$

Question 16 (Lam)

(a) i. (2 marks)

$$\begin{aligned}x &= 2t^3 - 9t^2 + 12t + 6\\ \frac{dx}{dt} &= 6t^2 - 18t + 12t\\ &= 6(t^2 - 3t + 2)\\ &= 6(t-2)(t-1)\\ \therefore \frac{dx}{dt} &= 0 \text{ when } t = 1, 2\end{aligned}$$

ii. (1 mark)

$$\begin{aligned}v_{\text{avg}} &= \frac{x_2 - x_1}{t_2 - t_1}\\ x(0) &= 6\\ x(1) &= 2 - 9 + 12 + 6 = 11\\ \therefore v_{\text{avg}} &= \frac{11-6}{1-0} = 5\,\text{ms}^{-1}\end{aligned}$$

(b) i. (1 mark)
In $\triangle ADB$,

$$\begin{aligned}\cos\theta &= \frac{6^2 + 7^2 - 8^2}{2 \times 6 \times 7}\\ &= \frac{21}{84} = \frac{1}{4}\end{aligned}$$

ii. (3 marks)
✓ [1] for $\cos(180^\circ - \theta) = -\cos\theta$
✓ [1] for correct application of cosine rule.
✓ [1] for final answer.

In $\triangle ADC$,

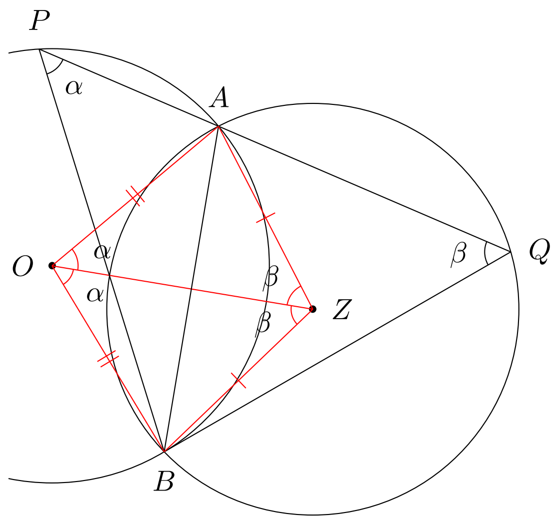
$$\begin{aligned}\cos(180^\circ - \theta) &= \frac{x^2 + 6^2 - (2x)^2}{2 \times x \times 6}\\ &= \frac{-3x^2 + 36}{12x}\\ &= \frac{12 - x^2}{4x}\end{aligned}$$

As $\cos(180^\circ - \theta) = -\cos\theta$,

$$\begin{aligned}-\frac{1}{4} &= \frac{12 - x^2}{4x}\\ -x &= 12 - x^2\\ x^2 - x - 12 &= 0\\ (x+3)(x-4) &= 0\\ \therefore x &= 4\end{aligned}$$

(as x is a length)

(c) i. (3 marks)
✓ [2] for body of proof.
✓ [1] for final answer.



In $\triangle AOZ$ and $\triangle BOZ$,

- $AO = BO$ (equal radii)
- $AZ = BZ$ (equal radii)
- OZ is common

$\therefore \triangle AOZ \equiv \triangle BOZ$ (SSS)

ii. (2 marks) As

$$\angle AOZ = \angle BOZ$$

(corresponding $\angle$ in congruent triangles), and similarly

$$\angle AZO = \angle BZO$$

then OZ must bisect $\angle AOB$ and $\angle AZB$.

iii. (2 marks)

- $\angle AOB = 2\alpha$ ($\angle$ at the centre is twice the $\angle$ at the circumference, subtended by the same arc)
- Since OZ bisects $\angle AOB$, then $\angle BOZ = \alpha$; this applies to $\angle BZO$ but with β as the angle.

iv. (1 mark)

- In $\triangle PBQ$, $\angle PBQ = 180^\circ - \alpha - \beta$ ($\angle$ sum of $\triangle$)
- In $\triangle OBZ$, $\angle OBZ = 180^\circ - \alpha - \beta$ ($\angle$ sum of $\triangle$)
- Hence $\angle PBQ = \angle OBZ$.