NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 1)

2014 Preliminary Course Final Examination

Friday August 29, 2014

General instructions

- Working time – 2 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 13)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER (ERN): .... **# BOOKLETS USED**: .....

Class (please ✔)

- ◯ 11M3A – Ms Ziaziaris
- ◯ 11M3B – Mr Ireland
- ◯ 11M3C – Mr Lin
- ◯ 11M3D – Mr Weiss
- ◯ 11M3E – Mr Berry
- ◯ 11M3F – Mr Lam

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	**Total**
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

Section I

10 marks
Attempt Question 1 to 10
Mark your answers on the answer sheet provided.

Questions **Marks**

1. Evaluate $\dfrac{2}{\sqrt{3.142^2 - 0.8^3}}$ correct to three significant figures. **1**

(A) 0.653 (B) 0.669 (C) 0.654 (D) 0.668

2. If $\sqrt{25t} - \sqrt{9t} = \sqrt{8}$, the value of t will be **1**

(A) 2 (B) $\frac{1}{2}$ (C) 16 (D) 4

3. Fully simplify: $\dfrac{x^2-1}{x^2-x} \times \dfrac{x}{2x+2}$. **1**

(A) 2 (B) $\dfrac{x-1}{2(x+1)}$ (C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2x+2}$

4. Fully simplify: $\dfrac{5^{2x}-5^x}{5^x}$. **1**

(A) $5^x - 1$ (B) 25 (C) 5^x (D) 5^{2x}

5. What is $\left(510 \times 10^6\right) \div \left(4.93 \times 10^{-15}\right)$ in scientific notation, correct to one decimal place? **1**

(A) 1.0×10^{21}

(B) 1.0×10^{-7}

(C) 103.4×10^{-4}

(D) 1.0×10^{23}

6. Melissa buys a saxophone for $\$30pq$. If the cost represents $w\%$ of her monthly income, how much does she earn in the month? **1**

(A) $\dfrac{w}{100} \times 30pq$

(B) $30pq - w$

(C) $\dfrac{30pq}{w} \times 100$

(D) $30pq - \dfrac{w}{100}$

7. Which of the following is the tangent to $y = x^2 - 4x + 1$ at the point $(4, 3)$? **1**

(A) $y = 2x - 4$

(B) $y = 2x - 5$

(C) $y = 4x - 13$

(D) $y = 4x - 8$

8. Which of the following is the domain of $y = \sqrt{4 - x}$? **1**

(A) $x \leq 4$

(B) $x \geq 4$

(C) $x \geq 0$

(D) $x \in \mathbb{R}$

9. Which of the values of x will result in the curve $y = 2x^3 - 3x^2$, find the values of x concaving up? **1**

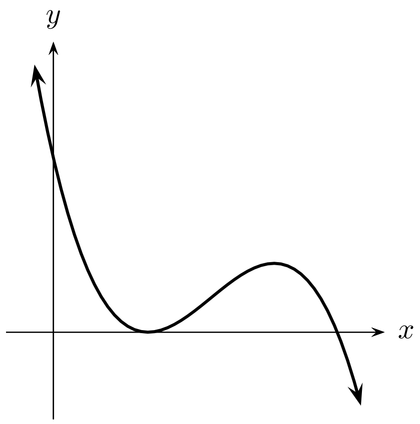
(A) $x > 2$

(B) $x > \frac{1}{2}$

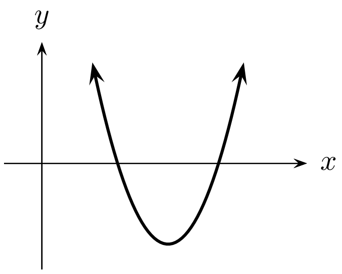
(C) $x > \frac{3}{8}$

(D) $x < 0$ or $x > 1$

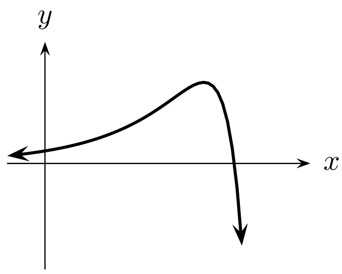
10. A sketch of $y = f(x)$ is given. Which of the following are the possible graphs for $y = f'(x)$? **1**



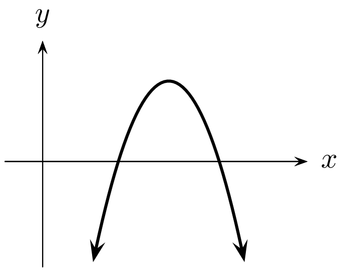
(A)



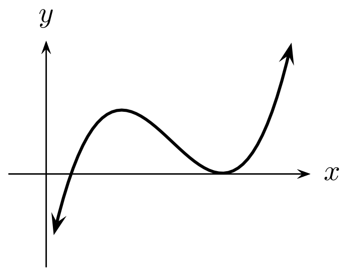
(C)



(B)



(D)



Examination continues overleaf...

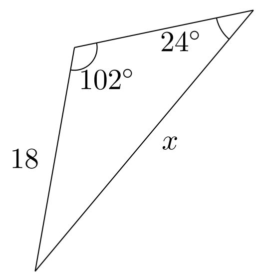
Section II

90 marks
Attempt Questions 11 to 16
Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Solve $\sin\beta - \dfrac{\sqrt{3}}{2} = 0$ for $0^\circ \leq \beta \leq 360^\circ$. **2**

(b) Find the value of x correct to the 2 decimal places. **2**



(c) Fully simplify:

i. $\sec^2\theta - \tan^2\theta$. **1**

ii. $\dfrac{\sin\theta}{\cos\theta\cot(90^\circ - \theta)}$. **1**

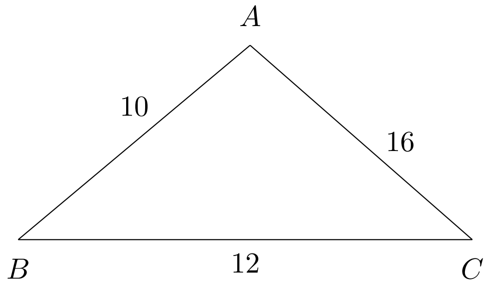
(d) If $\tan\alpha = \dfrac{8}{15}$ and $180^\circ \leq \alpha \leq 270^\circ$, find the exact value of:

i. $\sin\alpha$. **2**

ii. $\operatorname{cosec}\alpha - \sec\alpha$. **2**

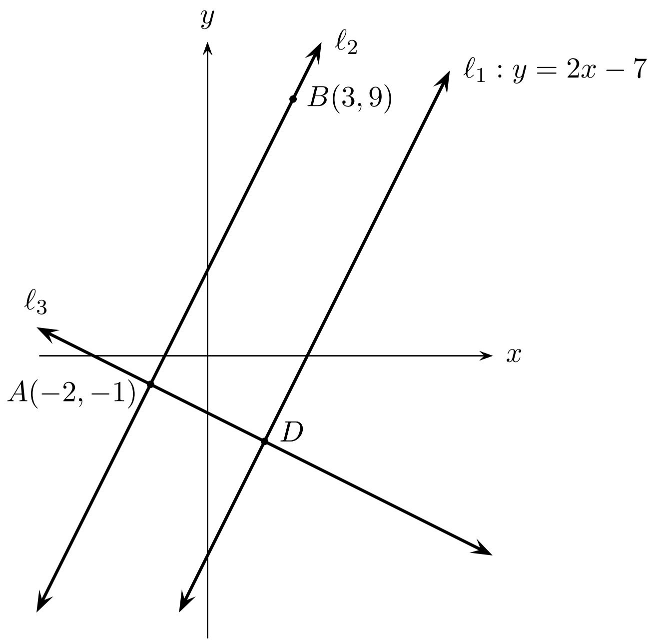
(e) Show that $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta}$. **2**

(f) Find the area of $\triangle ABC$, correct to 1 decimal place. **3**



Question 12 (15 Marks) Commence a NEW page. **Marks**

The lines ℓ_1 has equation $y = 2x - 7$. The line ℓ_2 passes through $A(-2,-1)$ and $B(3,9)$.



Copy or trace the diagram into your answer booklet. Allow **half a page** for the diagram.

(a) Find the gradient of ℓ_2. **1**

(b) Show that the line ℓ_3, which is perpendicular to AB and passes through the point $A(-2,-1)$ has equation $x + 2y + 4 = 0$. **2**

(c) Find the coordinates of D, which is the point of intersection of ℓ_1 $y = 2x - 7$ and the line $\ell_3 : x + 2y + 4 = 0$. **2**

(d) Find the length of the interval AB, leaving your answer in simplext exact form. **2**

(e) Find the exact perpendicular distance from $B(3,9)$ to the line $\ell_1 : y = 2x - 7$. **2**

(f) Find the exact area of $\triangle ABD$. **2**

(g) Does the point $(5,3)$ lie on the line $\ell_1 : y = 2x - 7$? **2**

(h) On the diagram, shade in the region satisfied simultaneously by the following inequalities: **2**

$$\begin{cases} y \geq 2x - 7 \\ x + 2y + 4 > 0 \\ y < 0 \end{cases}$$

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) Solve the inequality $\dfrac{2x-1}{x} \geq x$. **3**

(b) Find correct to the nearest degree, the acute angle between the lines $2x+y-3=0$ and $x-3y+2=0$. **2**

(c) $T(2at, at^2)$ is a point on the parabola $x^2 = 4ay$ with focus $S(0, a)$. The point M divides the interval ST externally in the ratio $2:1$.

i. Show that the point M has coordinates $(4at, 2at^2 - a)$. **2**

ii. Hence find the Cartesian equation of the locus of M as T moves on the parabola. **1**

(d) The equation of a parabola is given by

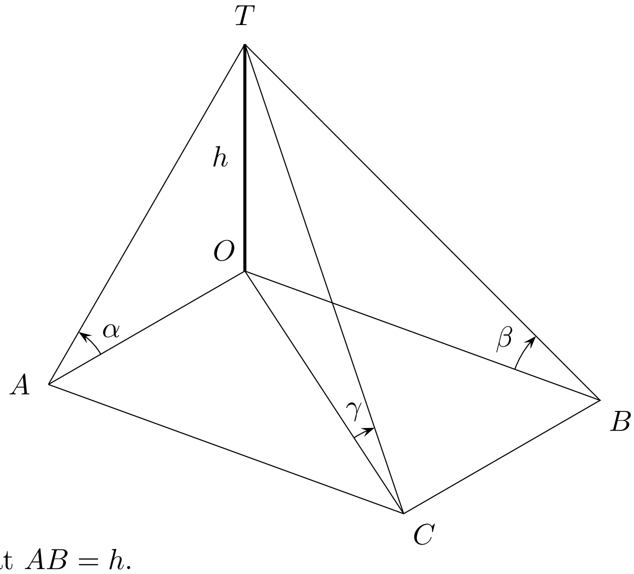
$$8y = x^2 + 4x + 12$$

i. Find the coordinates of the vertex. **2**

ii. Sketch the parabola, clearly showing the focus and the directrix. **2**

(e) A vertical flagpole OT of height h metres stands with its base O on horizontal ground. A is a point on the ground due south of O, B is a point on the ground due east of O and C is a point on the ground such that $OACB$ is a rectangle.

From A, B and C, the angles of elevation to the top T of the flagpole are α, β and γ respectively, where $\tan\alpha = \dfrac{5}{3}$ and $\tan\beta = \dfrac{5}{4}$.

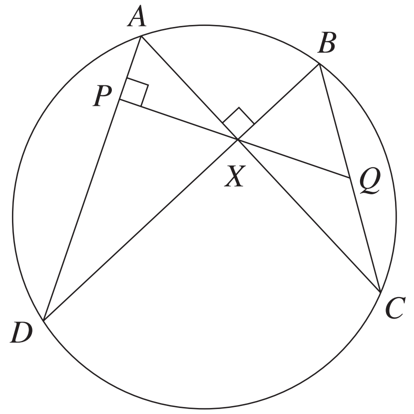


i. Show that $AB = h$. **2**

ii. Show that $\gamma = 45°$. **1**

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) The diagram shows points A, B, C and D on a circle. The lines AC and BD are perpendicular and intersect at X. The perpendicular to AD through X meets AD at P and BC at Q.

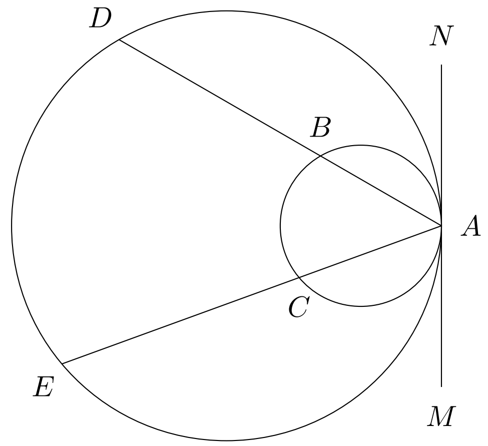


Copy or trace this diagram into your writing booklet.

i. Prove that $\angle QXB = \angle QBX$. **3**

ii. Prove that Q bisects BC. **2**

(b) Two circles touch each other internally at A. MAN is a common tangent to the circles at A. ABD and ACE are two straight lines which cut the smaller circle at B and C, and the larger circle at D and E.



i. Show that $\triangle ABC \parallel\!| \triangle ADE$. **3**

ii. Hence show that $\dfrac{AB \times AC}{AD \times AE} = \dfrac{BC^2}{DE^2}$. **2**

(c) The roots of the quadratic equation $x^2 + 4x + 2 = 0$ are α and β.

i. Find the value of $\alpha + \beta$ and $\alpha\beta$. **2**

ii. Show that $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} = -5$. **2**

iii. Hence or otherwise, find a quadratic equation with integer coefficients which has roots $\dfrac{1}{\alpha^3}$ and $\dfrac{1}{\beta^3}$. **1**

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) Consider the function $y = x^3 - 6x^2$ for $-1 \le x \le 6$.

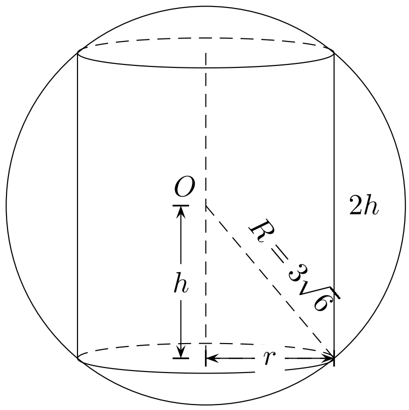
i. Find the x intercepts. **1**

ii. Find the stationary points and determine their nature. **4**

iii. Find the point of inflexion. **2**

iv. On half a page, sketch the graph over the given domain. **3**

(b) The diagram shows a cylinder inscribed inside a sphere. The radius of the sphere is $R = 3\sqrt{6}$ cm, and the height of the cylinder is $2h$ cm.



i. Show that the volume of the cylinder is **2**

$$V = 2\pi\left(54h - h^3\right)$$

ii. Find the maximum volume of the inscribed cylinder. **3**

Question 16 (15 Marks) Commence a NEW page. **Marks**

(a) The graphs of $y = x^2 - 3x - 5$ and $y = x + k$ have only one point of intersection at P.

i. Find the value of k. **2**

ii. Hence find the coordinates of P. **1**

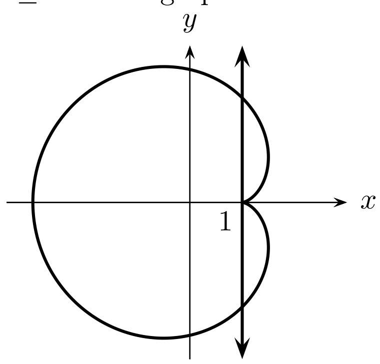
(b) i. Prove $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **2**

ii. Hence solve the equation $\sin 3\theta = 2\sin\theta$ for $0 \le \theta \le 2\pi$. **3**

(c) A *cardioid* curve is defined by the following pair of parametric equations **3**

$$\begin{cases} x = 2\cos\theta - \cos 2\theta \\ y = 2\sin\theta - \sin 2\theta \end{cases}$$

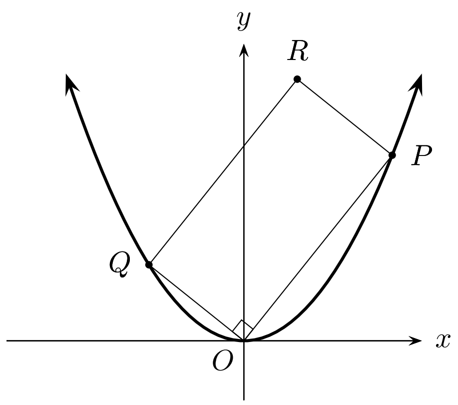
for $0 \le \theta \le 2\pi$. The graph is drawn below.



The cardioid curve and the vertical line $x = 1$ intersect in the 1st quadrant at a point P.

Find the coordinates of the point P.

(d) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ such that OP is perpendicular to OQ.



i. Prove that $pq = -4$. **2**

ii. Let R be the point such that $OPQR$ is a rectangle. **2**

R has coordinates

$$\left(2a(p+q), a\left(p^2+q^2\right)\right)$$

(Do NOT prove this).

Show that the locus of R is a parabola.

End of paper.

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Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: . . . .

Class (please ✔)

○ 11M3A – Ms Ziaziaris

○ 11M3B – Mr Ireland

○ 11M3C – Mr Lin

○ 11M3D – Mr Weiss

○ 11M3E – Mr Berry

○ 11M3F – Mr Lam

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
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8 – Ⓐ Ⓑ Ⓒ Ⓓ
9 – Ⓐ Ⓑ Ⓒ Ⓓ
10 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

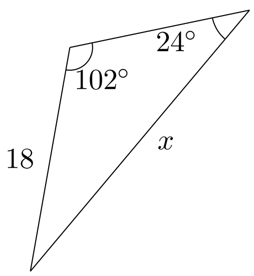
1. (B) **2.** (D) **3.** (C) **4.** (B) **5.** (D)
6. (B) **7.** (D) **8.** (C) **9.** (B) **10.** (B)

Question 11 (Lam)

(a) (2 marks)

$$\begin{aligned} \sin\beta - \frac{\sqrt{3}}{2} &= 0 \\ \sin\beta &= \frac{\sqrt{3}}{2} \\ \beta &= 60^\circ, 120^\circ \end{aligned}$$

(b) (2 marks)



Apply sine rule,

$$\begin{aligned} \frac{x}{\sin 102^\circ} &= \frac{18}{\sin 24^\circ} \\ x &= \frac{18\sin 102^\circ}{\sin 24^\circ} = 43.29 \end{aligned}$$

(c) i. (1 mark)

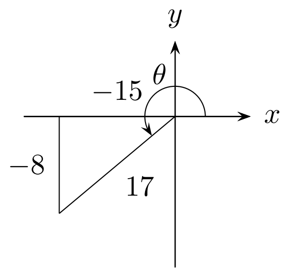
$$\begin{aligned} \sec^2\theta - \tan^2\theta &= \left(1+\tan^2\theta\right) - \tan^2\theta \\ &= 1 \end{aligned}$$

ii. (1 mark)

$$\begin{aligned} \frac{\sin\theta}{\cos\theta\cot\left(90^\circ-\theta\right)} &= \frac{\sin\theta}{\cos\theta\tan\theta} \\ &= \frac{\sin\theta}{\cos\theta\times\frac{\sin\theta}{\cos\theta}} \\ &= 1 \end{aligned}$$

(d) i. (2 marks)

$$\tan\theta = \frac{8}{15} \quad 180^\circ \le \theta \le 270^\circ$$



$$\sin\alpha = \frac{-8}{17}$$

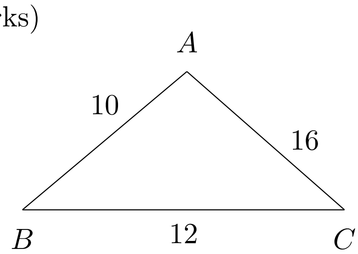
ii. (2 marks)

$$\begin{aligned} \operatorname{cosec}\alpha - \sec\alpha &= \frac{1}{\sin\alpha} - \frac{1}{\cos\alpha} \\ &= -\frac{17}{8} - \left(-\frac{17}{15}\right) \\ &= -\frac{119}{120} \end{aligned}$$

(e) (2 marks)

$$\begin{aligned} \tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\ &= \frac{1}{\sin\theta\cos\theta} \end{aligned}$$

(f) (3 marks)



Applying the cosine rule in $\triangle ABC$ to find $\angle BAC$ (students can find one of two other angles)

$$\begin{aligned} \cos\angle ABC &= \frac{10^2+16^2-12^2}{2\times 10\times 16} \\ &= \frac{53}{80} \\ \therefore \angle ABC &= 48^\circ 31' \text{ (approx)} \end{aligned}$$

Apply area of triangle via sine rule:

$$\begin{aligned} A &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}\times 10\times 16\times \sin 48^\circ 31' \\ &= 59.92\,\text{cm} \end{aligned}$$

Question 12 (Lin)

(a) (1 mark)

$$B(3,9) \quad A(-2,-1)$$
$$m_{\ell_1} = \frac{9-(-1)}{3-(-2)} = \frac{10}{5} = 2$$

(b) (2 marks)

$$m_{AB} = -\frac{1}{2}$$

Apply point gradient formula,

$$\begin{aligned} \ell_3 : \frac{y-(-1)}{x-(-2)} &= -\frac{1}{2} \\ 2y+2 &= -x-2 \\ x+2y+4 &= 0 \end{aligned}$$

(c) (2 marks)

$$\begin{cases} y = 2x-7 & (1) \\ x+2y+4=0 & (2) \end{cases}$$

Substitute (1) into (2):

$$\begin{aligned} x+2(2x-7)+4 &= 0 \\ x+4x-14+4 &= 0 \\ 5x-10 &= 0 \\ \therefore x &= 2 \\ y = 2(2)-7 &= -3 \\ \therefore D(2,&-3) \end{aligned}$$

(d) (2 marks)

$$\begin{aligned} d_{AB} &= \sqrt{(3-(-2))^2+(9-(-1))^2} \\ &= \sqrt{5^2+10^2} \\ &= \sqrt{125} = 5\sqrt{5} \end{aligned}$$

(e) (2 marks)

$$\begin{aligned} (3,9) &\longrightarrow 2x-y-7=0 \\ d_\perp &= \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} \\ &= \frac{|2(3)-9-7|}{\sqrt{2^2+1^2}} \\ &= \frac{|6-9-7|}{\sqrt{5}} \\ &= \frac{10}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5} \end{aligned}$$

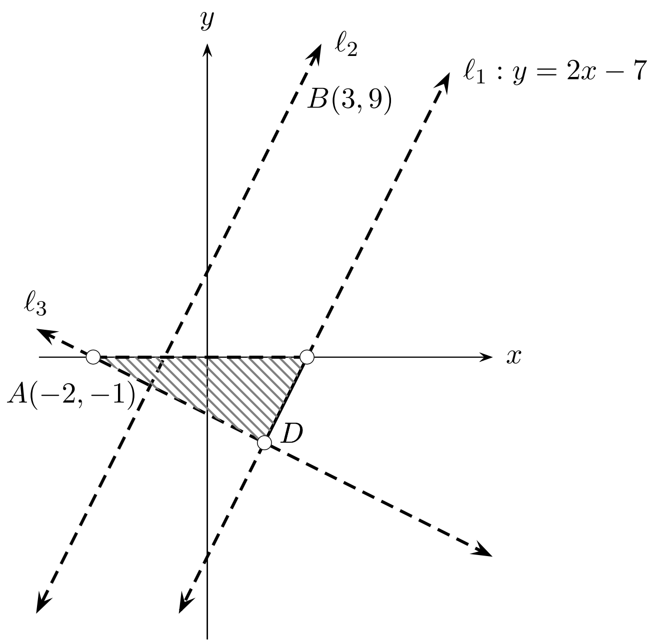
(f) (2 marks)

$$\begin{aligned} A_{\triangle ABD} &= \frac{1}{2} \times AB \times AD \\ &= \frac{1}{2} \times 5\sqrt{5} \times 2\sqrt{5} \\ &= 25 \end{aligned}$$

(g) (2 marks)
- ✓ [1] for reason
- ✓ [1] for "yes"

$$\begin{aligned} y = 2(5)-7 = 3 \\ \therefore x = 5, y = 3 \\ \therefore P(5,3) \text{ lies on the line } \ell_1 \end{aligned}$$

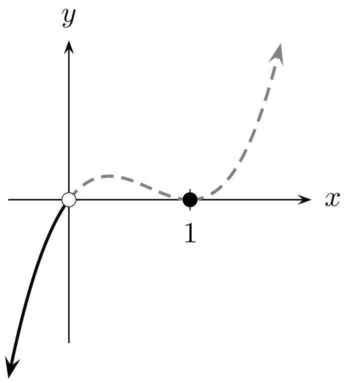
(h) (2 marks)



Question 13 (Berry)

(a) (3 marks)

$$\begin{aligned}
\underset{\times x^2}{\frac{2x-1}{x}} &\geq \underset{\times x^2}{x} \qquad (x \neq 0) \\
x(2x-1) &\geq x^3 \\
x^3 - x(2x-1) &\leq 0 \\
x\left(x^2 - 2x + 1\right) &\leq 0 \\
x(x-1)^2 &\leq 0
\end{aligned}$$



$$\therefore x < 0 \text{ or } x = 1$$

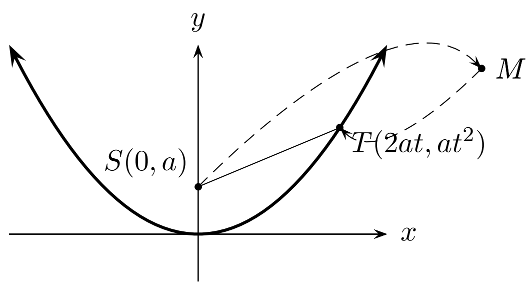
(b) (2 marks)

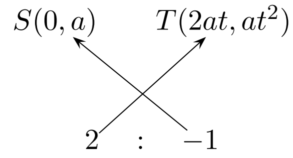
$$\begin{aligned}
2x + y - 3 = 0 \quad &\Rightarrow \quad y = -2x + 3 \\
x - 3y + 2 = 0 \quad &\Rightarrow \quad y = \frac{1}{3}x + \frac{2}{3}
\end{aligned}$$

$$m_1 = -2 \qquad m_2 = \frac{1}{3}$$

$$\begin{aligned}
\tan\theta &= \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \\
&= \left|\frac{-2 - \frac{1}{3}}{1 + (-2)\left(\frac{1}{3}\right)}\right| \\
&= 7 \\
\theta &= \tan^{-1} 7 \approx 82^\circ
\end{aligned}$$

(c) i. (2 marks)





$$M\left(\frac{(-1)(0) + 2(2at)}{2-1}, \frac{(-1)(a) + 2\left(at^2\right)(-a)}{2-1}\right)$$

$$= \left(4at, 2at^2\right)$$

ii. (1 mark)

$$\begin{cases} x = 4at & (1) \\ y = 2at^2 - a & (2) \end{cases}$$

Rearrange (1):

$$t = \frac{x}{4a}$$

Subtitute to (2):

$$\begin{aligned}
y &= 2a\left(\frac{x}{4a}\right)^2 - a \\
&= 2a\frac{x^2}{16a^2} - a \\
&= \frac{x^2}{8a} - a \\
y + a &= \frac{x^2}{8a} \\
x^2 &= 8a(y+a) = 4 \times (2a)(y+a)
\end{aligned}$$

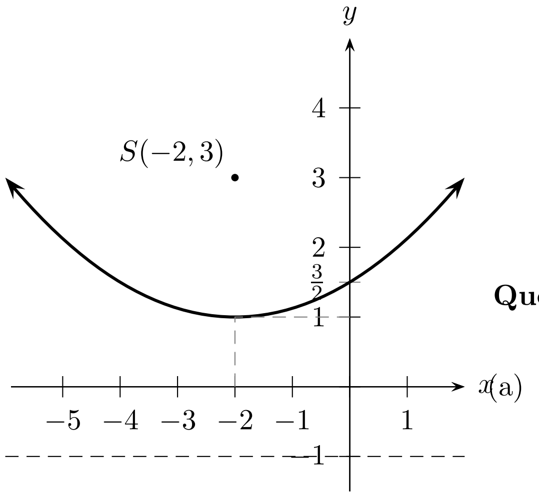
Locus of M is a parabola with vertex at $(0, -a)$ and focal length $2a$.

(d) i. (2 marks)

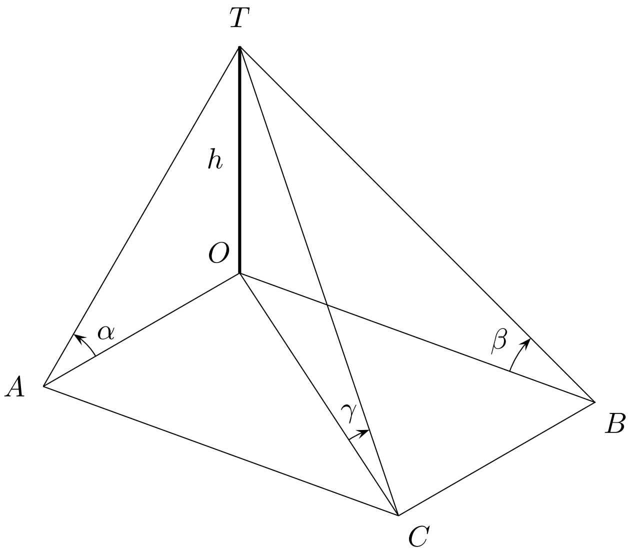
$$\begin{aligned}
8y &= x^2 + 4x + 12 \\
&= x^2 + 4x + 4 + 8 \\
8y &= (x+2)^2 + 8 \\
\therefore 8(y-1) &= (x+2)^2 \\
(x+2)^2 &= 4 \times 2(y-1)
\end{aligned}$$

Parabola with vertex $(-2, 1)$ and focal length 2.

ii. (2 marks)



(e) i. (2 marks)



$$\frac{h}{OA} = \tan\alpha = \frac{5}{3}$$

$$\frac{h}{OB} = \tan\beta = \frac{5}{4}$$

$$\therefore \frac{4}{5}h = OB \qquad \frac{3}{5}h = OA$$

In $\triangle OAB$,

$$OA^2 + OB^2 = AB^2$$

$$\therefore \left(\frac{4}{5}h\right)^2 + \left(\frac{3}{5}h\right)^2 = AB^2$$

$$\frac{\cancel{16+9}}{\cancel{25}}h^2 = AB^2$$

$$h^2 = AB^2$$

$$\therefore AB = h$$

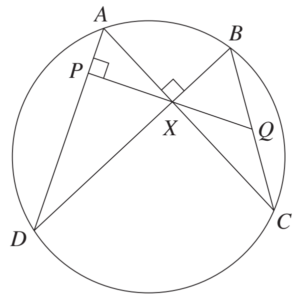
ii. (1 mark)
As $AB = OC$ (diagonal of rectangle)

$$\tan\gamma = \frac{h}{OC} = \frac{h}{h} = 1$$

$$\therefore \gamma = 45^\circ$$

Question 14 (Ziaziaris)

(a) i. (3 marks)



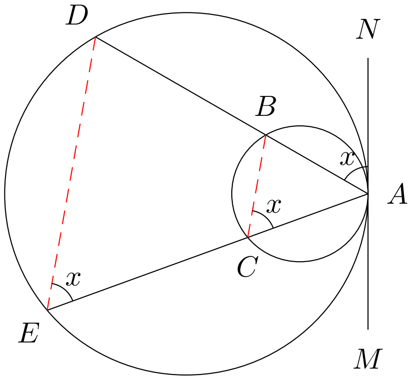
- Let $\angle DAC = x$
- $\angle DAC = \angle DBC = x$ (angles at the circumference, standing on the same arc)
- In $\triangle BAX$, $\angle AXB = 90^\circ - x$ ($\angle$ sum of $\triangle$)
- Also, as $\angle BXQ$ is a straight angle, then $\angle AXB + \angle QXB = 90^\circ$ (supplementary).
- Hence $90^\circ - x + \angle QXB = 90^\circ$, and

$$\angle QXB = x = \angle QBX$$

ii. (2 marks)

- As A, X and C are collinear, $\angle BXC = 90^\circ$.
- $\therefore$ $\angle QXC = 90^\circ - x$ (complementary)
- Also in $\triangle BXC$, $\angle XCQ = 90^\circ - x$ ($\angle$ sum of $\triangle$)
- $\therefore \triangle QXC$ is isosceles with $QX = QC$ as $\angle QXC = \angle QCX$
- But $\triangle QXB$ is also isosceles with $QX = QB$.
- $\therefore QX = QB = QC$
- Q bisects BC.

(b) i. (3 marks)



- Let $\angle DAN = x$.
- $\therefore \angle BCA = x$ ($\angle$ in the alternate segment)
- Likewise for $\angle DEC$.
- In $\triangle ABC$ and $\triangle DEA$,
  - $\angle DAE$ (common)
  - $\angle DEA = \angle BCA$ (previously proven)
  - $\angle EDA = \angle CBA$ (remaining $\angle$)

  $\therefore \triangle ABC ||| \triangle DEA$ (equiangular)

ii. (2 marks)

- As $\triangle ABC ||| \triangle DEA$, then all corresponding sides of the two triangles are in the same proportion.

$$\frac{AB}{AD} = \frac{BC}{DE} \qquad (14.1)$$
$$\frac{AC}{AE} = \frac{BC}{DE} \qquad (14.2)$$

(14.1) $\times$ (14.2)

$$\frac{AB}{AD} \times \frac{AC}{AE} = \left(\frac{BC}{DE}\right)^2$$

(c) i. (2 marks)

$$x^2 + 4x + 2 = 0$$
$$\alpha + \beta = -\frac{b}{a} = -4$$
$$\alpha\beta = \frac{c}{a} = 2$$

ii. (2 marks)

$$\begin{aligned}\frac{1}{\alpha^3} + \frac{1}{\beta^3} &= \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3} \\ &= \frac{(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^{)}}{(\alpha\beta)^3} \\ &= \frac{(\alpha+\beta)\left((\alpha+\beta)^2 - 2\alpha\beta - \alpha\beta\right)}{(\alpha\beta)^3} \\ &= \frac{(-4)\left((-4)^2 - 3(2)\right)}{2^3} \\ &= -5\end{aligned}$$

iii. (1 mark)

- Sum of roots of new quadratic: $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = 5$
- Product of roots of new quadratic:

$$\frac{1}{\alpha^3} \times \frac{1}{\beta^3} = \frac{1}{(\alpha\beta)^3} = \frac{1}{8}$$

- Hence new quadratic is

$$x^2 + 5x + \frac{1}{8} = 0$$
$$\text{or } 8x^2 + 40x + 1 = 0$$

Question 15 (Ireland)

(a) i. (1 mark)

$$f(x) = x^3 - 6x^2 = x^2(x-6)$$
$$\therefore f(x) = 0 \text{ when } x = 0 \text{ or } 6$$

ii. (4 marks)

$$f'(x) = 3x^2 - 12x$$
$$f''(x) = 6x - 12$$

Stationary pts when $f'(x) = 0$:

$$3x(x-4) = 0$$
$$\therefore x = 0 \text{ or } 4$$

- When $x = 0$,

$$f''(0) = -12 < 0$$
$$f(0) = 0$$

Hence $(0, 0)$ is a local maximum.

- When $x = 0$,

$$\begin{aligned} f''(4) &= 12 > 0 \\ f(4) &= 4^3 - 6(4^2) \\ &= -32 \end{aligned}$$

Hence $(4, -32)$ is a local minimum.

iii. (2 marks)

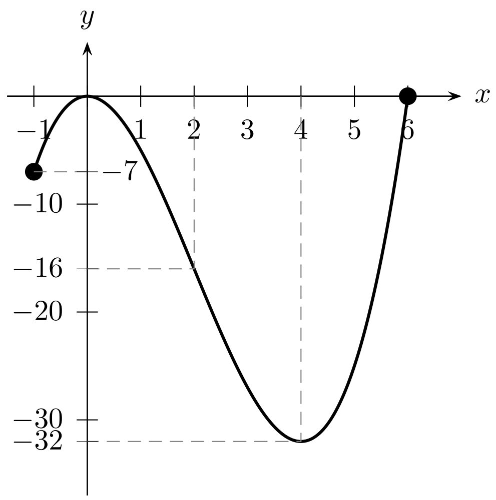
$$\begin{aligned} f''(x) &= 6x - 12 = 0 \\ \therefore x &= 2 \end{aligned}$$

x	0	2	4
$\frac{d^2y}{dx^2}$	$\overset{-}{-12}$	0	$\overset{+}{12}$
	$\frown$	0	$\smile$

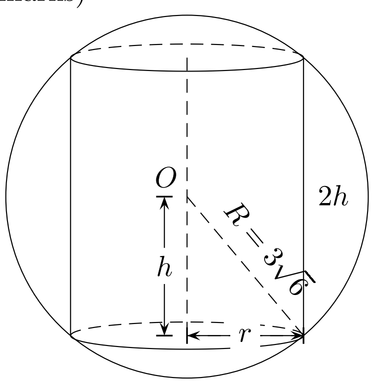
$$f(2) = 2^3 - 6(2^2) = -16$$

Hence $(2, -16)$ is a point of inflexion.

iv. (3 marks)



(b) i. (2 marks)



Volume of cylinder:

$$V = \pi r^2 (2h)$$

In the right triangle shown,

$$\begin{aligned} h^2 + r^2 &= \left(3\sqrt{6}\right)^2 \\ r^2 &= 54 - h^2 \\ \therefore V &= \pi\left(54 - h^2\right) \times (2h) \\ &= 2\pi\left(54h - h^3\right) \end{aligned}$$

ii. (3 marks)

- Differentiate to find $\dfrac{dV}{dh}$ and $\dfrac{d^2V}{dh^2}$:

$$\begin{aligned} \frac{dV}{dh} &= 2\pi\left(54 - 3h^2\right) \\ \frac{d^2V}{dh^2} &= 2\pi(-6h) = -12\pi h \end{aligned}$$

- Max volume requires $\dfrac{dV}{dh} = 0$ and $\dfrac{d^2V}{dh^2} < 0$:

$$\begin{aligned} \frac{dV}{dh} &= 2\pi\left(54 - 3h^2\right) = 0 \\ \therefore 3h^2 &= 54 \\ h^2 &= 18 \\ h &= \sqrt{18} = 3\sqrt{2} \end{aligned}$$

When $h = \sqrt{18}$,

$$\frac{d^2V}{dh^2} = -12\pi\sqrt{18} < 0$$

$\therefore\ h = 3\sqrt{2}$ produces a local maximum.

- Find maximal volume:

$$\begin{aligned} V &= 2\pi\left(54 \times 3\sqrt{2} - \left(3\sqrt{2}\right)^3\right) \\ &= 2\pi\left(54 \times 3\sqrt{2} - 27 \times 2 \times \sqrt{2}\right) \\ &= 2\pi\left(2 \times 54\sqrt{2}\right) \\ &= 216\pi\sqrt{2} \end{aligned}$$

Question 16 (Weiss)

(a) i. (2 marks)
Equating,

$$\begin{cases} y = x^2 - 3x - 5 \\ y = x + k \end{cases}$$

$$\begin{aligned} x^2 - 3x - 5 &= x + k \\ x^2 - 3x - x - 5 - k &= 0 \\ x^2 - 4x - (k+5) &= 0 \end{aligned}$$

Given only one point of intersection,

$$\begin{aligned} \Delta = b^2 - 4ac &= 0 \\ 4^2 - 4(k-5) &= 0 \\ 16 - 4k + 20 &= 0 \\ 4k &= 36 \\ k &= -9 \\ \therefore x^2 - 3x - 5 &= x - 9 \end{aligned}$$

ii. (1 mark)

$$\begin{aligned} x^2 - 3x - 5 &= x - 9 \\ x^2 - 4x + (9+5) &= 0 \\ (x-2)^2 &= 0 \\ \therefore x &= 2 \\ y = 2 - 9 &= -7 \\ \therefore P(2, &-7) \end{aligned}$$

(b) i. (2 marks)

$$\begin{aligned} \sin 3\theta &= \sin(2\theta + \theta) \\ &= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\ &= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta \\ &= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\ &= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta \end{aligned}$$

ii. (3 marks)

$$\begin{aligned} \sin 3\theta &= 2\sin\theta \\ 3\sin\theta - 4\sin^3\theta &= 2\sin\theta \\ 4\sin^3\theta - \sin\theta &= 0 \\ \sin\theta\left(4\sin^2\theta - 1\right) &= 0 \\ \sin\theta(2\sin\theta - 1)(2\sin\theta + 1) &= 0 \\ \therefore \sin\theta = 0, \pm\frac{1}{2} & \quad (0 \le \theta \le 2\pi) \\ \therefore \theta = 0, \pi, 2\pi, \frac{\pi}{6}, &\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

(c) (3 marks)

$$\begin{cases} x = 2\cos\theta - \cos 2\theta & (1) \\ y = 2\sin\theta - \sin 2\theta & (2) \end{cases}$$

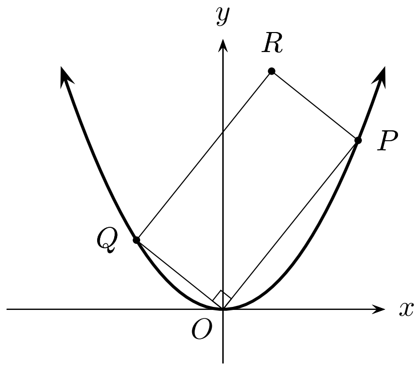
When $x = 1$,

$$\begin{aligned} 1 &= 2\cos\theta - \left(2\cos^2\theta - 1\right) \\ 1 &= 2\cos\theta - 2\cos^2\theta + 1 \\ \therefore 2\cos^2\theta - 2\cos\theta &= 0 \\ 2\cos\theta(\cos\theta - 1) &= 0 \\ \cos\theta &= 0, 1 \\ \theta &= 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi \end{aligned}$$

Substituting $\theta = 0$ into (1) results in $y = 0$, which is not the point P. Substitute $\theta = \frac{\pi}{2}$,

$$\begin{aligned} y = 2\sin\frac{\pi}{2} - \sin\pi &= 2 \\ \therefore x = 1, y &= 2 \end{aligned}$$

(d) i. (2 marks)



$$m_{OP} = \frac{ap^2 - 0}{2ap - 0} = \frac{p}{2}$$

Similarly, $m_{OQ} = \frac{q}{2}$. Given $OP \perp OQ$,

$$\begin{aligned} m_{OP} \times m_{OQ} &= -1 \\ \therefore \frac{p}{2} \times \frac{q}{2} &= -1 \\ \therefore pq &= -4 \end{aligned}$$

ii. (2 marks)

$$R\left(2a(p+q), a\left(p^2+q^2\right)\right)$$
$$\begin{cases} x = 2a(p+q) & (3) \\ y = a\left(p^2+q^2\right) & (4) \end{cases}$$

Note that

$$(p+q)^2 = p^2+q^2+2pq$$
$$\therefore p^2+q^2 = (p+q)^2 - 2pq$$

Square (3) and substitute into (4)

$$x^2 = 4a^2(p+q)^2$$
$$(p+q)^2 = \frac{x^2}{4a^2}$$
$$y = a\left(\frac{x^2}{4a^2} - 2pq\right)$$

Given $pq = -4$,

$$= a\left(\frac{x^2}{4a^2} + 8\right)$$
$$= \frac{x^2}{4a} + 8a$$
$$\therefore y - 8a = \frac{x^2}{4a}$$
$$x^2 = 4a(y-8a)$$

Locus is a parabola with vertex $(0, 8a)$ and focal length a.