NORTH SYDNEY BOYS HIGH SCHOOL

EXTENSION 1 MATHEMATICS

2015 Preliminary Course Assessment Task 3 (Yearly)

Friday August 28, 2015

General Instructions

- Working time – 2 hours (plus 5 minutes reading time).
- Use a blue or black pen. Diagrams may be sketched in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt all questions.

Section I

- Mark answers on the answer grid provided.

Section II

- Commence each question on a new page.
- Write on both sides of the paper.
- Show all necessary working for every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** ....

Teacher (please ✓) ☐ Mr Hwang ☐ Mr Weiss ☐ Mr Ireland

☐ Mr Lin ☐ Ms Ziaziaris ☐ Ms Lee

Question	1-5	6	7	8	9	10	11	12	Total	%
Marks	$\overline{5}$	$\overline{8}$	$\overline{12}$	$\overline{9}$	$\overline{13}$	$\overline{12}$	$\overline{12}$	$\overline{9}$	$\overline{80}$	$\overline{100}$
Working out grade:					Legibility grade:					

Section I

5 Marks

Attempt Questions 1 to 5

Mark your answers on the answer sheet provided on page 2.

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1. Which of the following is a factor of $8x^3 + 27$?

   A) $x - 3$

   B) $x + 3$

   C) $2x - 3$

   D) $2x + 3$

2. Let α and β be the roots of the equation $3x^2 - 7x + 12 = 0$. What is the value of $\alpha + \beta$?

   A) $-\frac{7}{3}$

   B) $\frac{7}{3}$

   C) 4

   D) 7

3. What is the value of :

   $$\lim_{x \to \infty} \frac{x^2 - 6}{x^2 - 3}$$

   A) -6

   B) 1

   C) 2

   D) No Limit

4. Which of the following is equivalent to $\cos(A - B)$

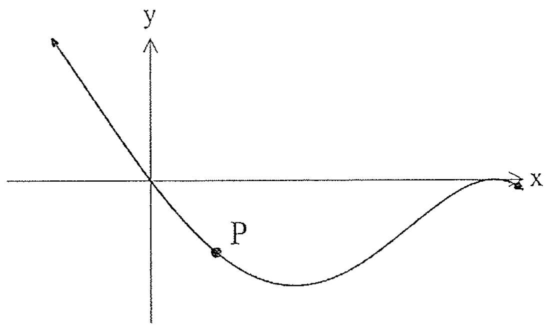
(A) $\cos A \cos B\,(\cos A - \cos B)$

(B) $\cos A \cos B\,(\cos A + \cos B)$

(C) $\cos A \cos B\,(1 - \tan A \tan B)$

(D) $\cos A \cos B\,(1 + \tan A \tan B)$

5. Which of the following is true for the point P:



(A) $f'(x) > 0,\ f''(x) > 0$

(B) $f'(x) > 0,\ f''(x) < 0$

(C) $f'(x) < 0,\ f''(x) > 0$

(D) $f'(x) < 0,\ f''(x) < 0$

END OF SECTION I

Section II

Total of 75 marks

Attempt Questions 6 to 12.

Write your answers in the writing book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (8 marks)

(a) Sketch the following graphs on the **same set of axes**: You should allow $\frac{1}{2}$ a page for your diagram.

(i) $y = 3^{x-1}$ **2**

(ii) $y = 4 - 3^{x-1}$ **2**

(b) Given the function **2**

$$f(x) = \frac{3x^2 - 2}{x^3},$$

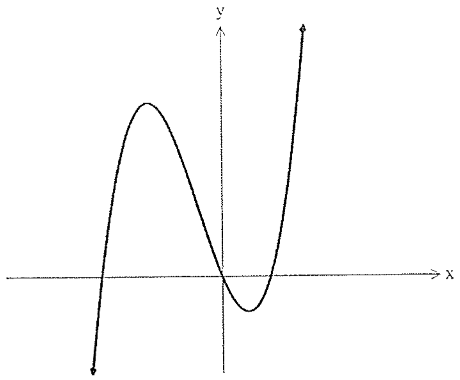
is $f(x)$ an odd function, an even function or neither? Show your working.

(c) Sketch the graph $y = |\, x^2 - 16 \,|$ **2**

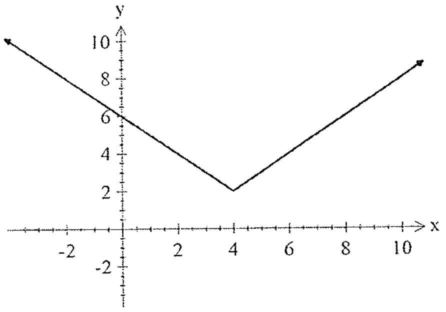
Question ~~5~~ 7 (12 marks)

(a) Copy the following graphs into your workbook, then sketch the gradient function on a set of axes directly below each graph:

(i) **2**



(ii) **2**



(b) Differentiate $y = 3x^2 - 2x$ from first principles. **3**

(c) Differentiate and fully simplify the following:

(i) $y = 3\sqrt{x}$ **1**

(ii) $y = (4 - 3x)^5 \cdot (2x + 1)$ **2**

(iii) $y = \dfrac{x^2 + 2}{x + 4}$ **2**

Question 8 (9 marks)

(a) Find the values of k for which $x^2 - kx + 4 = 0$ has:

(i) One root equal to -1. **1**

(ii) Real roots **2**

(iii) One root which is double the other root. **2**

(b) Consider the parabola $-12y = x^2 - 6x - 3$

(i) Find the equation of its directrix. **2**

(ii) Find the coordinates of its focus **2**

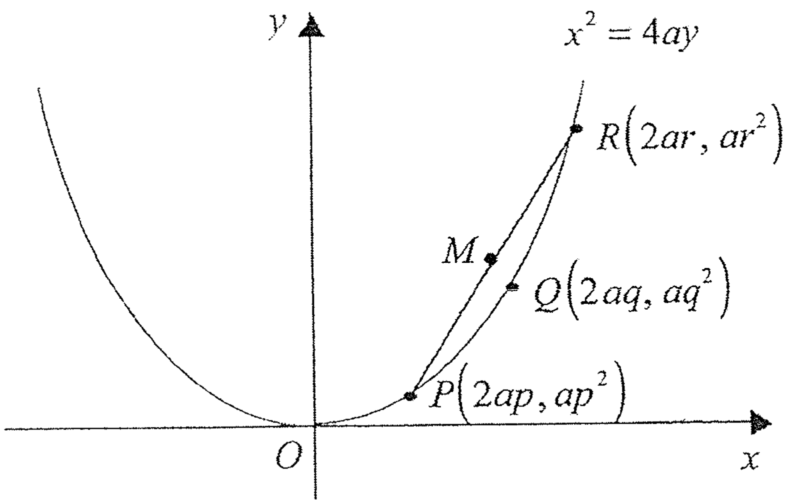
Question 9 (13 marks)

(a) Solve $|2x - 3| = 3x + 7$ **2**

(b) Calculate the acute angle between the lines $x + y = 4$ and $3x - 4y = 2$ **3**

(c) Solve the inequality : $\frac{2}{x} \geq x - 1$ **3**

(d)



$Q(2aq, aq^2)$ is a fixed point on the parabola $x^2 = 4ay$ where $a > 0$. $P(2ap, ap^2)$ and $R(2ar, ar^2)$ are variable points which move on the parabola such that the chord PR is parallel to the tangent at Q.

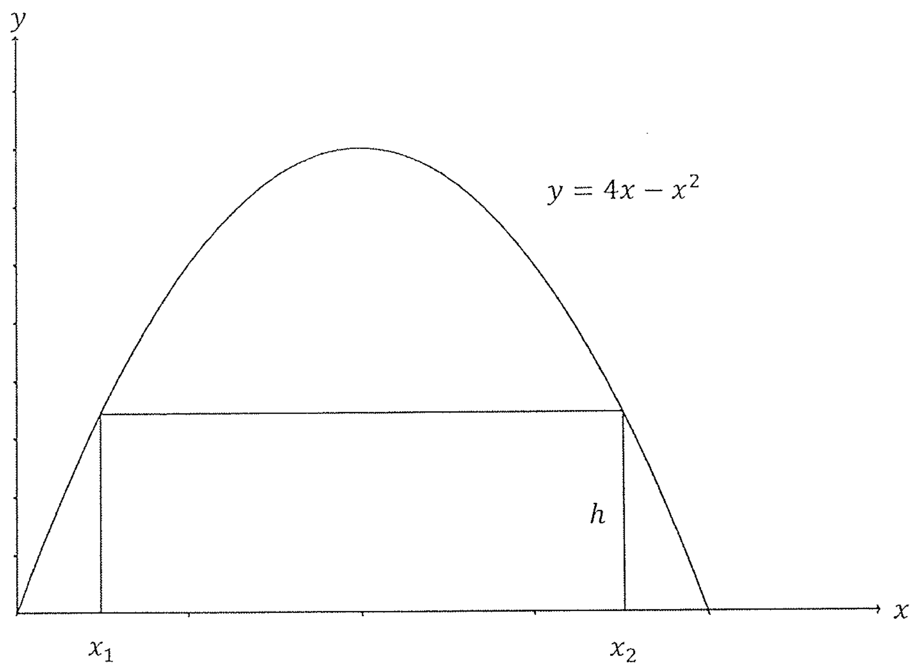
(i) Show that $p + r = 2q$ **3**

(ii) Hence show that the coordinate of the midpoint, M is given by $(2aq, \frac{a(r^2+p^2)}{2})$ **1**

(iii) Explain why $r^2 + p^2 \geq 2q^2$ **1**

Question 10 (12 marks)

(a) A rectangle has two of its vertices on the curve $y = 4x - x^2$ and the other two on the x axis in the interval $0 \leq x \leq 4$.



(i) If the height of the rectangle is $'h'$ units, show that its area is $2h\sqrt{4-h}$ units^2. **2**

(ii) Hence find the height of the rectangle when the area is a maximum. **3**

(b) Given the relation: $y = \pm\dfrac{x}{2}\sqrt{4-x^2}$

and also given that $\dfrac{dy}{dx} = \pm\dfrac{2-x^2}{\sqrt{4-x^2}}$

(i) Find the domain of this graph. **2**

(ii) Find the point(s) on the curve where the tangent(s) are horizontal **2**

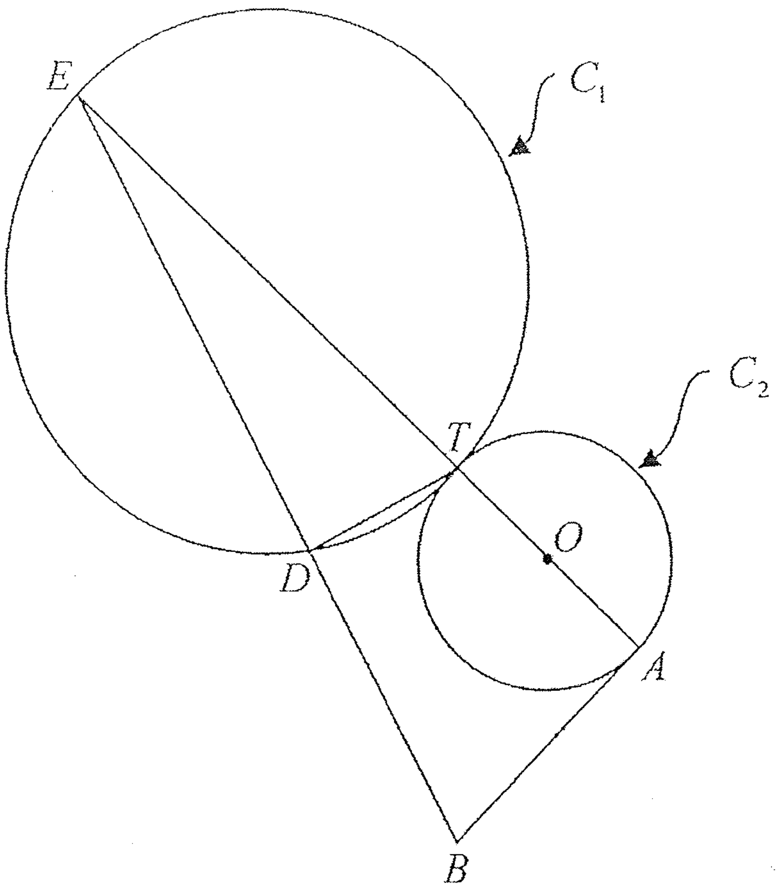
(iii) Where on the curve are the tangents vertical (i.e. when the gradient is undefined)? **1**

(iv) Find the gradient(s) at the point $(0, 0)$. **1**

(v) Hence draw a *rough* sketch of the curve. **1**

Question 11 (12 marks)

(a)



Two circles C_1 and C_2 touch at T.

The line AE passes through O, the centre of C_2, and through T.

The point A lies on C_2 and E lies on C_1.

The line AB is tangent to C_2 at A, D lies on C_1 and BE passes through D.

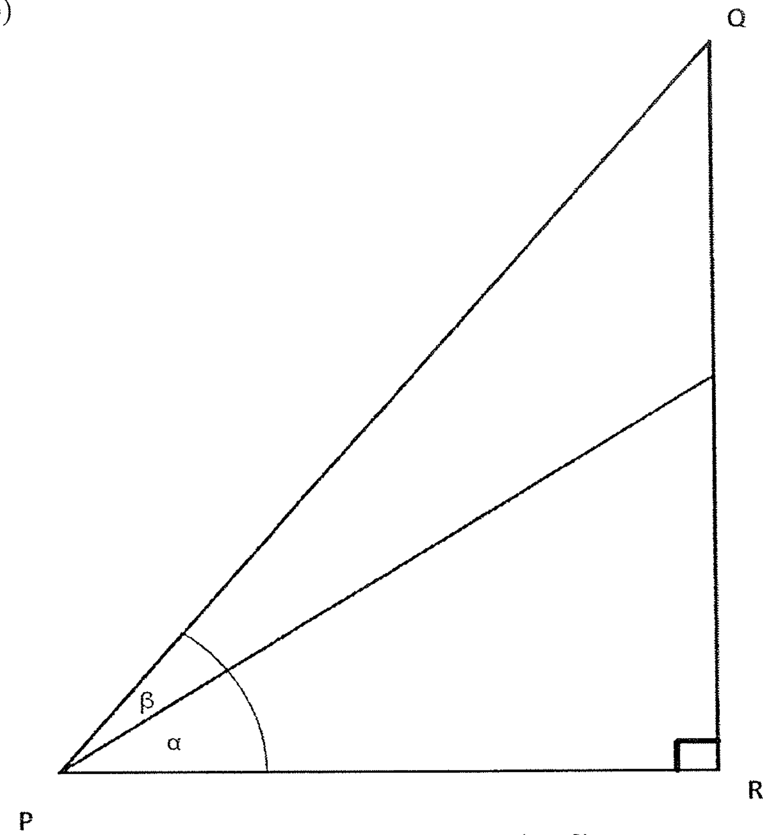
The radius of C_1 is R and the radius of C_2 is r.

(i) Explain why $\angle EDT = 90°$ **2**

(ii) If $DE = 2r$, show that $EB = \dfrac{2R(R+r)}{r}$ **2**

Question 11 continued overleaf

(b)



NOT TO SCALE

In the diagram $\alpha > \beta > 0$, and $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{4}{5}$

(i) Show that $\tan\alpha \tan\beta = \frac{1}{9}$ **2**

(ii) If $PR = QR$, Show that $9(\tan\alpha + \tan\beta) = 8$ **3**

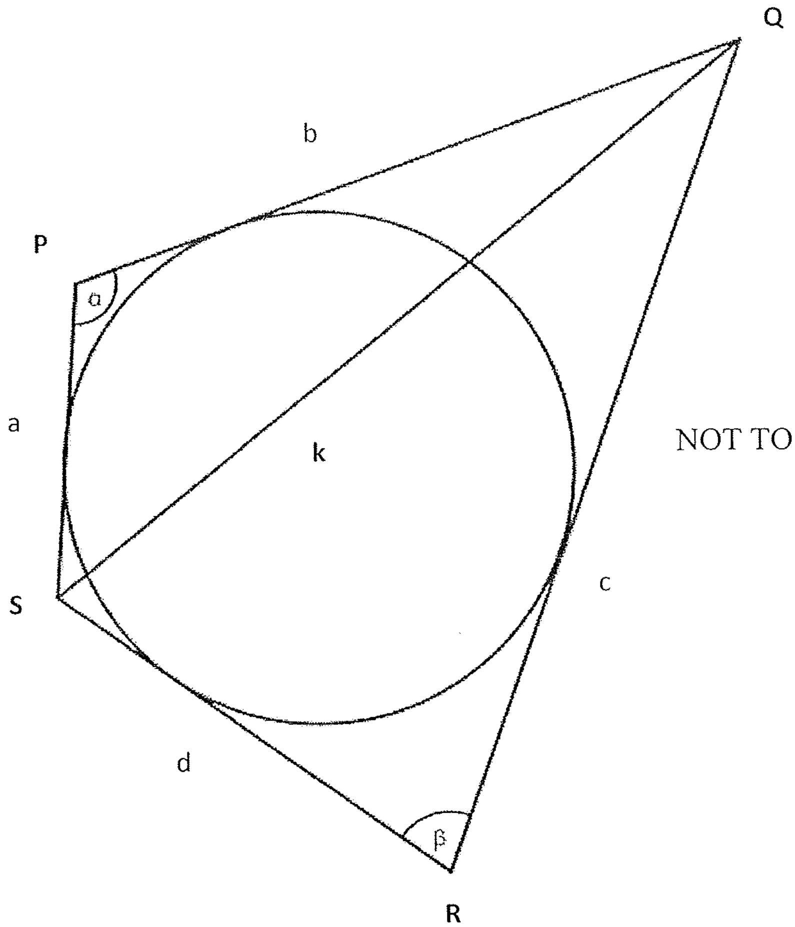
(iii) Hence, find the value of α and β to the nearest minute. **3**

Question 12 (9 marks)

A convex quadrilateral $PQRS$ has side lengths a, b, c, d and an area of $\sqrt{abcd}$ units^2.

The quadrilateral has an inscribed circle as shown in the diagram below.

Let k be the length of the diagonal SQ and $\angle SPQ = \alpha$ and $\angle SRQ = \beta$.



NOT TO SCALE

(i) By using the ***cosine rule***, find two different relationships for k^2. **1**

(ii) By using trigonometry find an expression for the area of the quadrilateral $PQRS$. **1**

(iii) Explain why $(a-b)^2 = (c-d)^2$ **2**

(iv) By using part (i) and (iii), show that **2**

$$ab(1-\cos\alpha) = cd(1-\cos\beta)$$

(v) Hence, show that the quadrilateral $PQRS$ is a cyclic quadrilateral **3**

END OF EXAMINATION

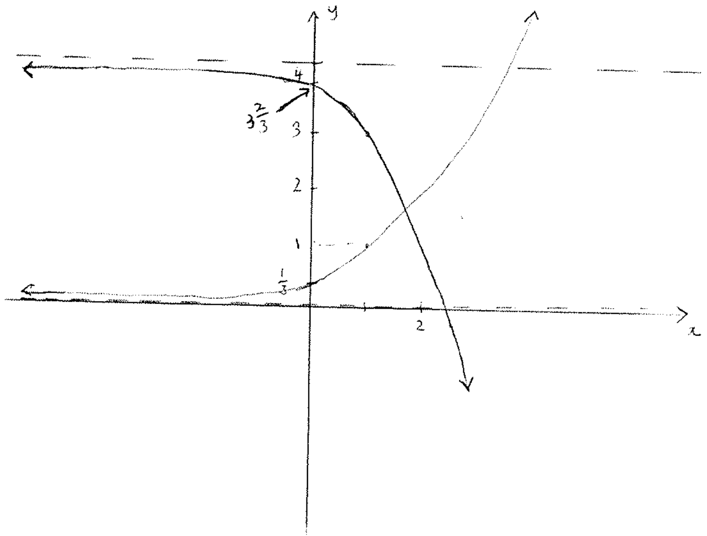
SECTION I

1. D
2. B
3. B
4. D
5. C

SECTION II

Question 6 (8 marks)

a)



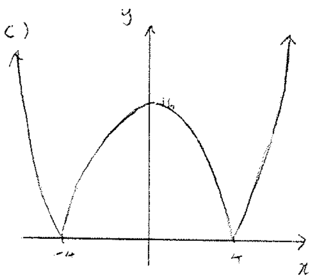
b) $f(x) = \dfrac{3x^2-2}{x^3}$, $f(-x) = \dfrac{3(-x)^2-2}{(-x)^3}$

$= -\dfrac{3x^2-2}{x^3}$

$= -f(x)$

$\therefore$ $f(x)$ is odd function

c)



Question 8

a) when $x = -1$

(i) $(-1)^2 - k(-1) + 4 = 0$

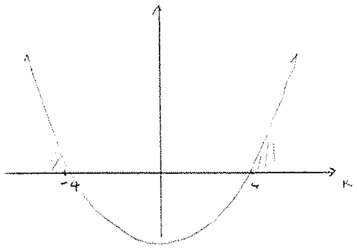
$k + 5 = 0$

$k = -5$

(ii) Real roots occur when $\Delta \geq 0$

$\Delta = (-k)^2 - 4 \times 1 \times 4$

$= k^2 - 16 \geq 0$



$\Delta \geq 0$ when

$k \leq -4$ or $k \geq 4$

(iii) let one root be α

$\therefore$ 2nd root is 2α

Product of roots: $2\alpha^2 = 4$

$\alpha^2 = 2$

$\alpha = \pm\sqrt{2}$

From sum of roots: $\alpha + 2\alpha = k$

$k = 3\alpha$

$= \pm 3\sqrt{2}$

b) $-12y = x^2 - 6x - 3$

$= x^2 - 6x + 9 - 9 - 3$

$= (x-3)^2 - 12$

$-12y + 12 = (x-3)^2$

$-12(y-1) = (x-3)^2$

(i) focal length = 3

vertex is at $(3, 1)$

parabola is concave down

$\therefore$ directrix is at $y = 4$

ii) vertex $(3, 1)$

focal length = 3

$\therefore$ focus: $(3, -2)$

Question 4

a) $|2x-3| = 3x+7$

This implies that $3x+7 \geq 0$

$x \geq -\frac{7}{3}$

Now we solve

$2x-3 = 3x+7$ & $2x-3 = -3x-7$

$x = -10$ or $5x = -4$

$x = -\frac{4}{5}$

Since $x \geq -\frac{7}{3}$

then only solution is

$x = -\frac{4}{5}$

b) Let l_1: $x+y=4$

l_2: $3x-4y=2$

$\therefore$ $m_1 = -1$ and $m_2 = \frac{3}{4}$

acute angle between two lines can be found by:

$$\tan\theta = \left|\frac{-1-\frac{3}{4}}{1+(-\frac{3}{4})}\right|$$

$$= \left|\frac{-\frac{7}{4}}{\frac{1}{4}}\right|$$

$$= 7$$

$\therefore$ $\theta = \tan^{-1} 7$

$\doteqdot 81^\circ$ (to nearest degree)

c) $\frac{2}{x} \geq x-1$

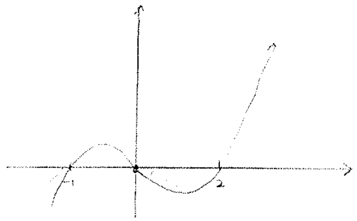
multiplying by the square of denominator to both sides

$2x \geq x^3 - x^2$ where $x \neq 0$

$x^3 - x^2 - 2x \leq 0$

$x(x^2-x-2) \leq 0$

$x(x-2)(x+1) \leq 0$



$x \leq -1$ or $0 < x \leq 2$

as $x \neq 0$

d) Gradient at Q: $\frac{dy}{dq} \times \frac{dq}{dx}$

$$= 2aq \times \frac{1}{2a}$$

$$= q$$

Now chord PQ has gradient of: $\frac{ar^2 - ap^2}{2ar - 2ap}$

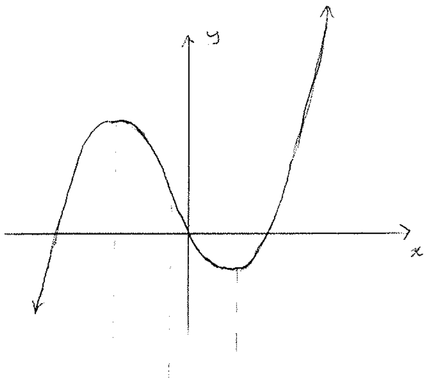
$$= \frac{a(r-p)(r+p)}{2a(r-p)}$$

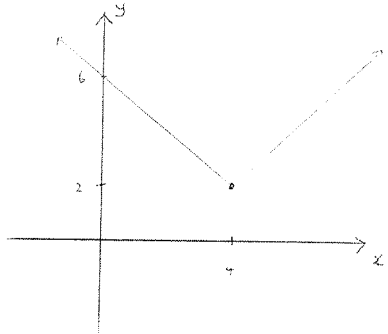
$$= \frac{(r+p)}{2}$$

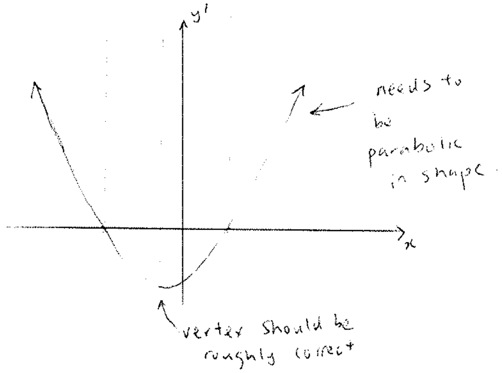
Since they have equal gradients

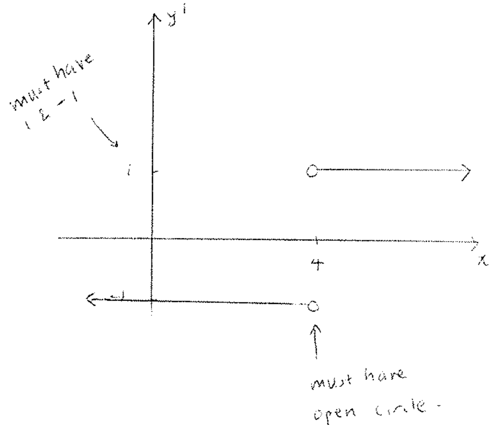
$\frac{r+p}{2} = q \Rightarrow p+r = 2q$

Question 7









b) $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x+h) = 3(x+h)^2 - 2(x+h)$

$= 3(x^2+2xh+h^2) - 2x - 2h$

$= 3x^2 + 6xh + 3h^2 - 2x - 2h$

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{(3x^2+6xh+3h^2-2x-2h) - (3x^2-2x)}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x+3h-2)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x+3h-2$$

$$= 6x-2$$

c) i) $y = 3x^{\frac{1}{2}}$

$= \frac{3}{2} x^{-\frac{1}{2}}$

$= \frac{3}{2\sqrt{x}}$

ii) $y = (4-3x)^5(2x+1)$

$\frac{dy}{dx} = (4-3x)^5 \cdot \frac{d}{dx}(2x+1) + (2x+1) \cdot \frac{d}{dx}(4-3x)^5$

$= 2(4-3x)^5 + (2x+1) \cdot 5 \cdot -3 \cdot (4-3x)^4$

$= 2(4-3x)^5 - 15(4-3x)^4(2x+1)$

$= (4-3x)^4[2(4-3x) - 15(2x+1)]$

$= (4-3x)^4 [8 - 6x - 30x - 15]$

$= (4-3x)^4 [-36x - 7]$

iii) $y = \frac{x^2+2}{x+4}$

$u = x^2+2$ $\quad$ $v = x+4$

$\frac{du}{dx} = 2x$ $\quad$ $\frac{dv}{dx} = 1$

$\therefore \frac{dy}{dx} = \frac{(x+4)\cdot 2x - (x^2+2)}{(x+4)^2}$

$= \frac{2x^2+8x-x^2-2}{(x+4)^2}$

$= \frac{x^2+8x-2}{(x+4)^2}$

Question 9 (con't)

ii) Midpoint: $M\left(\frac{2ar+2ap}{2}, \frac{ar^2+ap^2}{2}\right)$

$M\left(a(r+p), \frac{a(r^2+p^2)}{2}\right)$

Since $r+p = 2q$ (from (i))

$M\left(2aq, \frac{a(r^2+p^2)}{2}\right)$ [1]

iii) Since the x value of M is equal to the x value of Q. M lies directly above Q. Since P and R lie to the left and right of Q, M is always above the point Q. Although equality exists when P, Q, R are the same points. [1] for reasonable explaination.

$\therefore \frac{a(r^2+p^2)}{2} \geq aq^2$

$r^2+p^2 \geq 2q^2$ (as $a>0$)

Question 10

i) To find x_1 and x_2. Find the intersection of the parabola $y = -x^2 + 4x$ with $y = h$

hence x_1 and x_2 are the roots of $x^2 - 4x + h = 0$

where $x_1 + x_2 = 4$ and $x_1 x_2 = h$

$\therefore$ The length of the rectangle can be expressed as $x_2 - x_1$.

We know $(x_2 - x_1)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$= 16 - 4h$

$= 4(4-h)$

$\therefore x_2 - x_1 = 2\sqrt{4-h}$

$\therefore$ Area of rectangle $(A) = h(x_2 - x_1)$

$= 2h\sqrt{4-h}$

Note: Quadratic formula to find x_1 and x_2 can be used.

ii) $\dfrac{dA}{dh} = 2h \times \dfrac{d}{dh}(4-h)^{\frac{1}{2}} + (4-h)^{\frac{1}{2}} \times \dfrac{d}{dh}(2h)$

$= -\dfrac{h}{\sqrt{4-h}} + 2\sqrt{4-h}$

$= \dfrac{2(4-h) - h}{\sqrt{4-h}}$

$= \dfrac{8-3h}{\sqrt{4-h}}$

Stationary point occurs when $\dfrac{dA}{dh} = 0$

i.e. $8 - 3h = 0$

$3h = 8$

$h = \dfrac{8}{3}$

Students might find 2nd derivative. It should be

$\dfrac{d^2A}{dh^2} = \dfrac{3h - 16}{2(4-h)^{\frac{3}{2}}}$

$\therefore$ when $h = \dfrac{8}{3}$

$\dfrac{d^2A}{dh^2} = -2.598$

$\therefore$ concave up

Testing the nature of the stationary point

h	2	$\frac{8}{3}$	3
$\frac{dA}{dh}$	$\frac{2}{\sqrt{2}}$	0	-1

For testing nature students must give actual output! They can use first or 2nd derivative, but check for algebraic mistake for 2nd derivative.

$\therefore$ maximum stationary point when $h = \dfrac{8}{3}$

$\therefore h = 8$ is the height of rectangle when the area is a maximum.

b) i) Domain: $4 - x^2 \geq 0$

$-2 \leq x \leq 2$

ii) tangents horizontal when $\dfrac{dy}{dx} = 0$

i.e. $2 - x^2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$\therefore$ the points are

$(\sqrt{2}, 1), (\sqrt{2}, -1), (-\sqrt{2}, 1), (-\sqrt{2}, -1)$

iii) when gradient is undefined

$4 - x^2 = 0$

i.e. when $x = \pm 2$

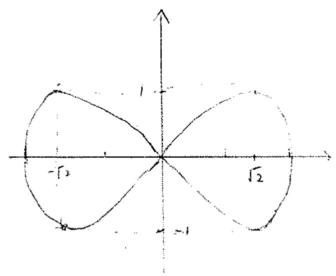
$(2, 0)$, $(-2, 0)$

iv) when $x = 0$

$\dfrac{dy}{dx} = \pm 1$

$\therefore$ gradient $= m = \pm 1$

v)



Question 11

(i) When two circles touch, the two centres and the point of contact are collinear, hence AE must pass through the centre of C_1.

$\therefore \angle EDT = 90^\circ$ (angle in a semicircle)

(ii) $\because \angle OAB = 90^\circ$ (tangent AB meet radius OA at right angles)

and $\angle EDT$ is common

$\triangle EDT ||| \triangle EAB$ (equiangular)

$\therefore \dfrac{x}{2R} = \dfrac{2R+2r}{2r}$ (corresponding sides in proportion)

$x = \dfrac{2R \cdot 2(R+r)}{2r}$

$= \dfrac{2R(R+r)}{r}$

) (i) $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \dfrac{4}{5}$

$5\cos(\alpha+\beta) = 4\cos(\alpha-\beta)$

$5\cos\alpha\cos\beta - 5\sin\alpha\sin\beta = 4\cos\alpha\cos\beta + 4\sin\alpha\sin\beta$

$\cos\alpha\cos\beta = 9\sin\alpha\sin\beta$

divide through by $9\cos\alpha\cos\beta$

$\tan\alpha\tan\beta = \dfrac{1}{9}$

(ii) If PR = QR then $\tan(\alpha+\beta) = 1$

$\therefore \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$

from part (i)

$\tan\alpha + \tan\beta = 1 - \tan\alpha\tan\beta$

$= 1 - \dfrac{1}{9}$

$= \dfrac{8}{9}$

$\therefore 9(\tan\alpha + \tan\beta) = 8$

(iii) From part (i) and (ii) we can let $\tan\alpha$ and $\tan\beta$ be the roots of

$x^2 - \dfrac{8}{9}x + \dfrac{1}{9} = 0$

$\therefore 9x^2 - 8x + 1 = 0$

$x = \dfrac{8 \pm \sqrt{64-36}}{18}$

$\tan\alpha = \dfrac{8+2\sqrt{7}}{18}$

$\alpha = 36^\circ 27'$

$\tan\beta = \dfrac{8-2\sqrt{7}}{18}$

$\beta = 8^\circ 33'$

Question 12

i) $k^2 = a^2 + b^2 - 2ab\cos\alpha$

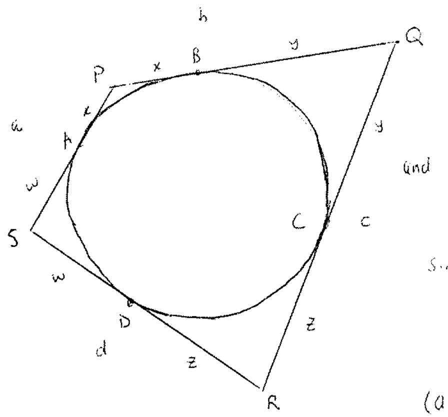
$k^2 = c^2 + d^2 - 2cd\cos\beta$

ii) Area PQRS = Area $\triangle PQS$ + Area $\triangle SRQ$

$= \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\beta$

$= \frac{1}{2}(ab\sin\alpha + cd\sin\beta) = \sqrt{abcd}$

iii)



Let the point of contact be A, B, C, D

and AP = PB = x (Two tangents from an external point are equal in length)

Similarly BQ = QC = y

RC = RD = z

SD = SA = w

$(a-b)^2 = [(w+x) - (x+y)]^2$

$= [w-y]^2$

$= [-(y-w)]^2$

$= (y-w)^2$

$= ((y+z) - (w+z))^2$

$= (c-d)^2$

iv) From (i)

$a^2+b^2-2ab\cos\alpha = c^2+d^2-2cd\cos\beta$

subtract $(a-b)^2 = (c-d)^2$

$2ab - 2ab\cos\alpha = 2cd - 2cd\cos\beta$

$ab(1-\cos\alpha) = cd(1-\cos\beta)$

v) From (ii)

$$\frac{1}{2}(ab\sin\alpha + cd\sin\beta) = \sqrt{abcd}$$

squaring both sides we get

$$a^2b^2\sin^2\alpha + c^2d^2\sin^2\beta + 2abcd\sin\alpha\sin\beta = 4abcd$$

$$a^2b^2(1-\cos^2\alpha) + c^2d^2(1-\cos^2\beta) + 2abcd\sin\alpha\sin\beta = 4abcd$$

$$ab\cdot ab(1-\cos\alpha)(1+\cos\alpha) + cd\cdot cd(1-\cos\beta)(1+\cos\beta) + 2abcd\sin\alpha\sin\beta = 4abcd$$

From (iii)

$$ab(1+\cos\alpha)cd(1-\cos\beta) + cd\cdot ab(1-\cos\alpha)cd(1+\cos\beta) + 2abcd\sin\alpha\sin\beta = 4abcd$$

dividing through by abcd

$$(1+\cos\alpha)(1-\cos\beta) + (1-\cos\alpha)(1+\cos\beta) + 2\sin\alpha\sin\beta = 4$$

$$1 - \cancel{\cos\beta} + \cancel{\cos\alpha} - \cos\alpha\cos\beta + 1 + \cancel{\cos\beta} - \cancel{\cos\alpha} - \cos\alpha\cos\beta + 2\sin\alpha\sin\beta = 4$$

$$2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta = 4$$

$$2 - 2\cos(\alpha+\beta) = 4$$

$$\cos(\alpha+\beta) = -1$$

$$\therefore \alpha+\beta = 180^\circ$$

$\therefore$ quadrilateral is cyclic as opposite angles are supplementary