NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2017 Preliminary Course Yearly Assessment Task

Wednesday, 30 August 2017

General instructions

- Working time – **150 minutes**.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided

SECTION II

- Commence each new question on **a new page.** Write on both sides of the paper.
- **All necessary working should be shown** in every question. Marks may be deducted for **illegible or incomplete working**.

STUDENT NUMBER: ....   **# BOOKLETS USED:** ....

Class: (please ✓)

- ○ Mr Berry
- ○ Mr Hwang
- ○ Mr Ireland
- ○ Dr Jomaa
- ○ Ms Lee
- ○ Mr Lin
- ○ Ms Ziaziaris

Marker's use only

QUESTION	1-5	6	7	8	9	10	11	12	Total
MARKS	$\overline{\ \ }$ 5	$\overline{\ \ }$ 15	$\overline{\ \ }$ 11	$\overline{\ \ }$ 16	$\overline{\ \ }$ 12	$\overline{\ \ }$ 18	$\overline{\ \ }$ 9	$\overline{\ \ }$ 10	$\overline{\ \ }$ 96

Section I

5 marks
Attempt Question 1 to 5

Mark your answers on the answer grid provided

1. What is the number of asymptotes to the graph of

$$y = \frac{x^2}{x^2 - 4x - 12}$$

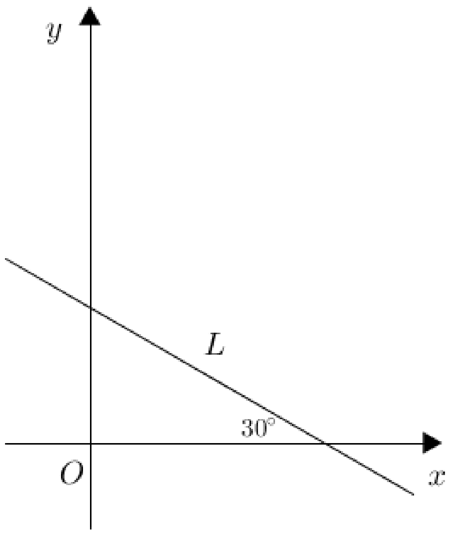
(A) 1 (B) 2 (C) 3 (D) 4

2. What is the focus of the parabola $(x+3)^2 = 12(y-3)$?

(A) $(-3, 3)$ (B) $(0,3)$

(C) $(-3,6)$ (D) $(-3,0)$

3. The diagram shows the line L.



What is the slope of the line L?

(A) $-\sqrt{3}$ (B) $-\frac{1}{\sqrt{3}}$

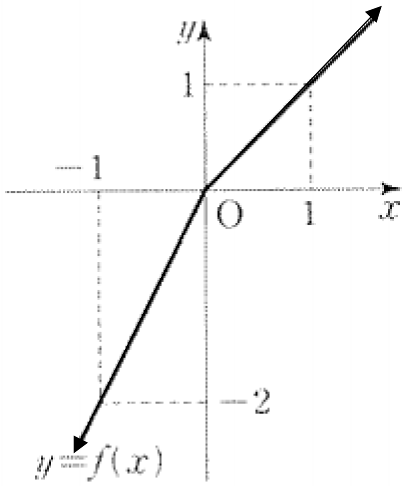
(C) $\frac{1}{\sqrt{3}}$ (D) $\sqrt{3}$

4. What is the value of $\tan(\alpha - \beta)$ if $\sin\alpha = -\frac{3}{5}$ for $\frac{3\pi}{2} < \alpha < 2\pi$ and $\sin\beta = \frac{4}{5}$ for $\frac{\pi}{2} < \beta < \pi$?

(A) $-\dfrac{4}{3}$ (B) $-\dfrac{5}{24}$

(C) 5 (D) $\dfrac{7}{24}$

5. The graph of the function $y = f(x)$ consists of two rays that pass through the origin and the two points (1, 1) and (-1,-2) as shown below. Choose the **TRUE** statements from the following.



Statements

$a)$ $f^{-1}(5) = f^{-1}(f^{-1}(5))$

$b)$ $f^{-1}(-2) = -1$

$c)$ There is only one solution for $f^{-1}(x) = f(x)$

(A) a, b (B) $a, b,$ c

(C) a, c (D) b, c

End of Section I.

Examination continues overleaf.

Section II

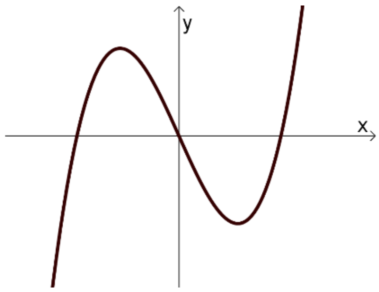
91 marks
Attempt Question 6 to 12

Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

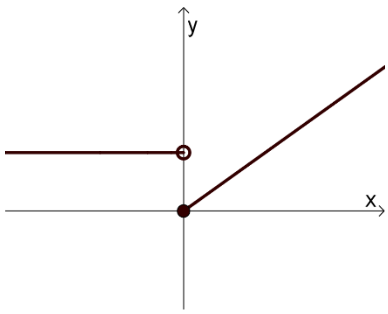
Question 6 (15 marks)

(a) Copy the following graphs, then sketch their gradient function on a set of axes directly below each graph:

(i) **2**



(ii) **2**



(b) Differentiate $f(x) = 3x^2 - 7x$ from first principles. **3**

(c) Differentiate and **FULLY** simplify the following

(i) $y = 4x^2 + \frac{3}{\sqrt{x}}$ **2**

(ii) $y = (1 - 2x)^5 \cdot (2x + 1)$ **3**

(iii) $y = \frac{5x^2+3}{1-x}$ **3**

Question 7 (11 marks)

(a) Find the equation of the normal to the curve $y = 2x^3 - 10x + 1$ at $x = 2$. **3**

(b) For the curve $y = -x^3 - x^2 + 16x - 20$,

(i) Find and classify any stationary points. **3**

(ii) Find any points of inflexion. **3**

(iii) Sketch the graph of $y = -x^3 - x^2 + 16x - 20$. (You are not required to find the *x*-intercepts) **2**

Question 8 (16 marks)

(a) If α and β are the roots of $3x^2 - 4x + 2 = 0$, without solving the equation find the values of:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\dfrac{2}{\alpha} + \dfrac{2}{\beta}$ **2**

(iv) $(\alpha - \beta)(\beta - \alpha)$ **2**

(b) If $3x^2 - 2x + 5 \equiv Ax(x-2) + B(x-2) + C$, find *A*, *B* and *C*. **3**

(c) Solve $3^{2x} - 12(3^x) + 27 = 0$, giving all answers in exact form. **2**

(d) For what values of *m* are the roots of $x^2 + 4x - 3 = m(x-2)$

(i) real? **2**

(ii) both negative? **3**

Question 9 (12 marks)

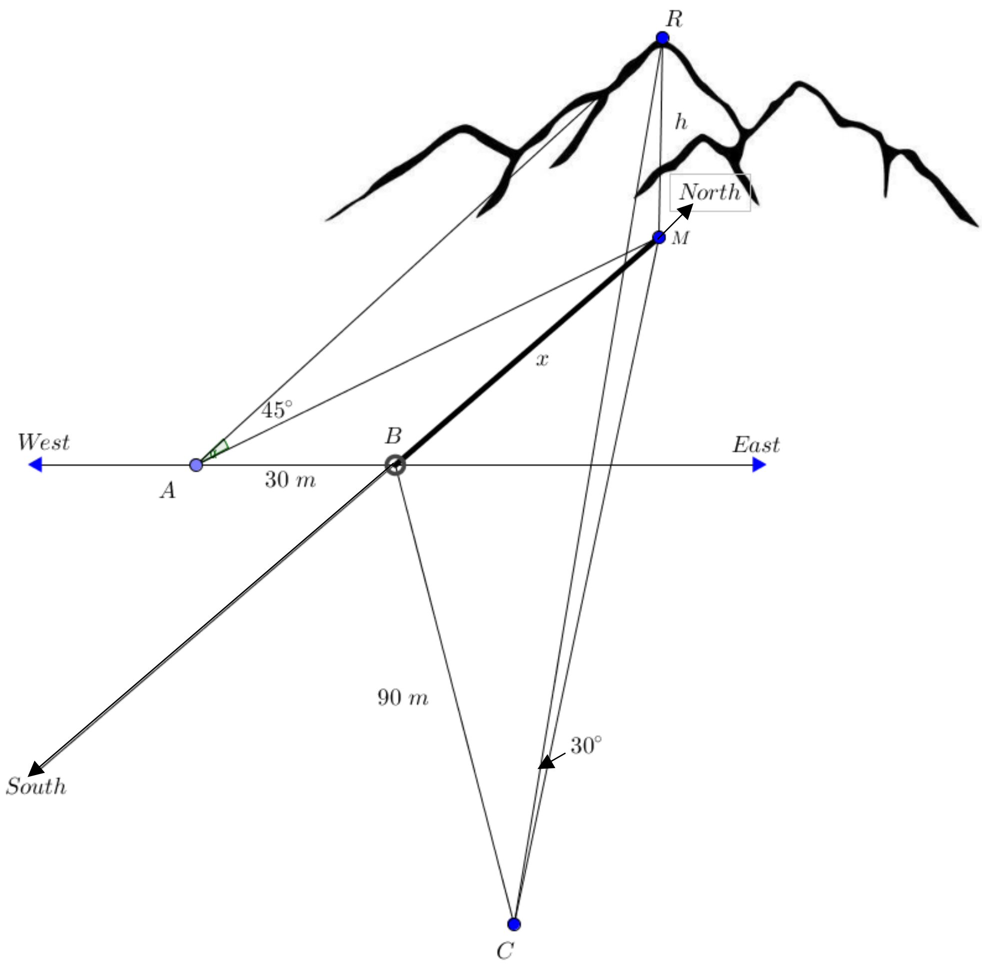
(a) Find the exact value of $\operatorname{cosec} 240°$ **1**

(b) Sketch the curve $y = 2\ \sin 3x$ for $-\pi \leq x \leq \pi$, using at least $\frac{1}{3}$ of the page. **2**

(c) Prove the identity: **3**

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} + \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\sec 2\theta$$

(d) The base camp (B) of an expedition is located *x* metres due south of the base of a mountain (M) with height *h* metres. Amelie (A) moved 30 metres due west of the camp and spots a ruin (R) at the top of the mountain peak with her scope showing an angle of elevation of 45°. At the same time, Clement (C) was 90 metres directly south-east of the camp and spots the same ruin with his scope showing an angle of elevation of 30°



(i) Show that $h^2 = x^2 + 900$. **1**

(ii) Show that $MC^2 = x^2 + 90\sqrt{2}x + 90^2$ **2**

(iii) Find the value of h to the nearest metre. **3**

Question 10 (18 marks)

(a) The line $3x - 4y = 4$ is more than 2 units from the point $(-2, \delta)$. Find the possible values of δ. **2**

(b) Give a geometric description of the locus of a point P which moves so that $PA=2PB$, where $P = (x, y)$, $A = (-2,4)$ and $B = (4,1)$. **3**

(c) $P(2ap, -ap^2)$ and $Q(2aq, -aq^2)$ are two points on the parabola $x^2 = -4ay$.

(i) Find the equation of the chord of PQ. **2**

(ii) What is the value of pq when PQ is a focal chord? **2**

(d) P and Q are two points on the parabola $x^2 = 4ay$ with coordinates $(2ap, ap^2)$ and $(2aq, aq^2)$ respectively. The tangents at P and Q meet at T, which is situated on the parabola $x^2 = -4ay$.

(i) Write down the equations of the tangents at P and Q. **2**

(ii) Show that T is the point $(a(p + q), apq)$. **2**

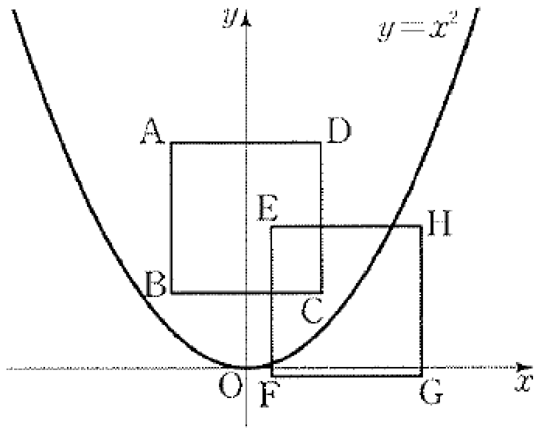
(iii) Prove that $p^2 + q^2 = -6pq$. **3**

(iv) Find the equation of the locus of the midpoint of PQ. **2**

Question 11 (9 marks)

(a) Find the function $f(x)$ that satisfies $2f(x) + 3f(1-x) = x^2$ for all real x. **3**

(b) From the following diagram, the square *ABCD* has a side length of 1 and its diagonals intersect at $(0,1)$. The square *EFGH* also has a side length of 1 and its diagonals intersect on the parabola $y = x^2$. (All sides of the squares are parallel to the *x* and *y* axes)



(i) Show that the coordinates of *C* are $(\frac{1}{2}, \frac{1}{2})$ **1**

(ii) Let the intersection of diagonals in *EFGH* be (p, p^2) and find the coordinates of *E* in terms of *p*. **1**

(iii) Show that the area equation of the overlapping region of square *ABCD* and *EFGH* is **2**

$$A(p) = -p^3 + p^2$$

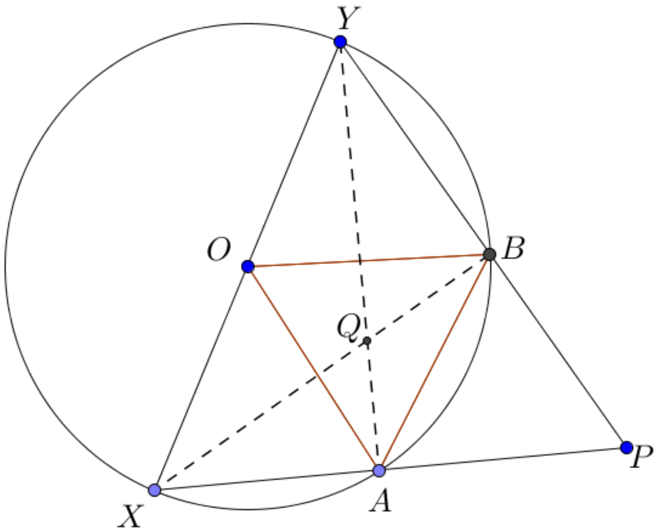
(iv) What is the maximum area of the overlapping region? **2**

Question 12 (10 marks)

(a) Suppose that $\sin\alpha + \sin\beta = \sqrt{\frac{5}{3}}$ and $\cos\alpha + \cos\beta = 1$.

Find the value of $\cos(\alpha - \beta)$. **3**

(b) In a circle centred at *O*, points *A* and *B* are chosen so that *OAB* forms an equilateral triangle. Let points *XY* be two points of a diameter, and the intersection between lines *XA*, *YB* produced be *P*. Connect points *AY* and *BX* and let the intersection of *AY* and *BX* be Q.



(i) Show that $\angle PAQ = 90°$. **1**

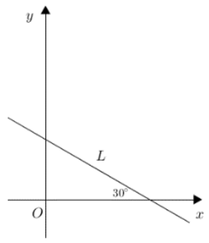
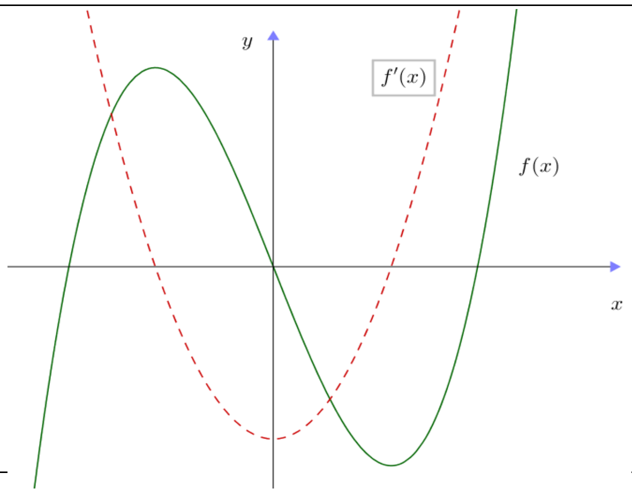
(ii) Show that *APBQ* is a cyclic quadrilateral. **2**

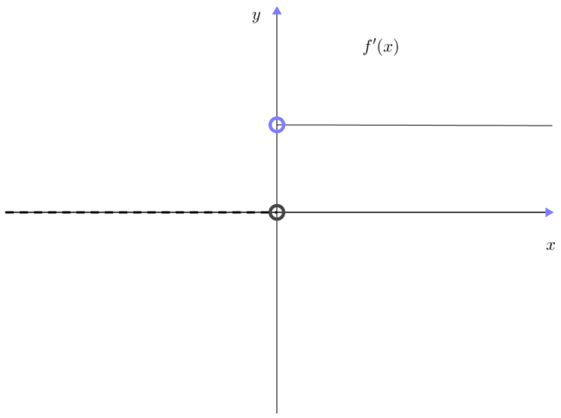
(iii) State why $\angle XQA = \angle XYA + \angle YXB$. **1**

(iv) Hence, find the size of $\angle APB$. **2**

(v) Describe the locus of *P* as *X* and *Y* move around the circle. **1**

End of paper

Section 1 - Multiple Choice	
1	$y = \dfrac{x^2}{x^2 - 4x - 12} = \dfrac{x^2}{(x-6)(x+2)}$ $\therefore$ 2 vertical asymptotes + 1 horizontal asymptote = 3 (C)
2	$(x+3)^2 = 12(y-3)$ $V = (-3,3), a = 3$ $\therefore S = (-3,6)$ (C)
3	$\dfrac{\text{rise}}{\text{run}} = -\dfrac{1}{\sqrt{3}}$ (B) 
4	$\tan\alpha = -\dfrac{3}{4}, \tan\beta = -\dfrac{4}{3}$ $\tan(\alpha - \beta) = \dfrac{-\frac{3}{4} + \frac{4}{3}}{1+1} = \dfrac{7}{24}$ (D)
5	$f(x)\begin{cases} x, & x > 0 \\ 0, & x = 0 \\ 2x, & x < 0 \end{cases}, \qquad f^{-1}(x)\begin{cases} x, x > 0 \\ 0, x = 0 \\ \frac{x}{2}, x < 0 \end{cases}$ a ✓, b ✓, c × (A)
Section 2	
6a) (i)	

(ii)	 $x = 0, when\, x < 0$
b)	$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ $\lim_{h\to 0} \frac{\left(3(x+h)^2 - 7(x+h)\right) - 3x^2 + 7x}{h} = \lim_{h\to 0} \frac{3(2x+h)h - 7h}{h}$ $= \lim_{h\to 0} 3(2x+h) - 7 = 6x - 7$
c) (i)	$y = 4x^2 + 3x^{-\frac{1}{2}}$ $y' = 8x - \frac{3}{2}x^{-\frac{3}{2}} = 8x - \frac{3}{2\sqrt{x^3}}$
(ii)	$y = (1-2x)^5(2x+1)$ $y' = -10(1-2x)^4(2x+1) + 2(1-2x)^5$ $= (1-2x)^4\{2(1-2x) - 10(2x+1)\}$ $= (1-2x)^4(-8-24x)$ $= -8(1-2x)^4(1+3x)$
(iii)	$y = \frac{5x^2+3}{1-x}$ $y' = \frac{10x(1-x) + 5x^2 + 3}{(1-x)^2} = \frac{3 + 10x - 5x^2}{(1-x)^2}$

7 a)

$$y = 2x^3 - 10x + 1, \qquad x = 2 \ \therefore y = 16 - 20 + 1 = -3$$
$$y' = 6x^2 - 10, \qquad at\ x = 2, \qquad y' = 24 - 10 = 14$$
$$m_{normal} = -\frac{1}{14}$$
$$y + 3 = -\frac{1}{14}(x - 2)$$
$$y = -\frac{x}{14} - \frac{20}{7}$$

b) (i)

$$y = -x^3 - x^2 + 16x - 20$$
$$y' = -3x^2 - 2x + 16 = -(3x + 8)(x - 2)$$
$$y' = 0\ at\ x = 2\ or\ x = -\frac{8}{3}$$
$$y'' = -6x - 2$$
$$at\ x = 2, y'' = -14 < 0 \quad \text{local max}$$
$$at\ x = -\frac{8}{3}, y'' = 14 > 0\ \text{local min}$$
$$(2,0) local\ max, \left(-\frac{8}{3}, -50.8\right) local\ min$$

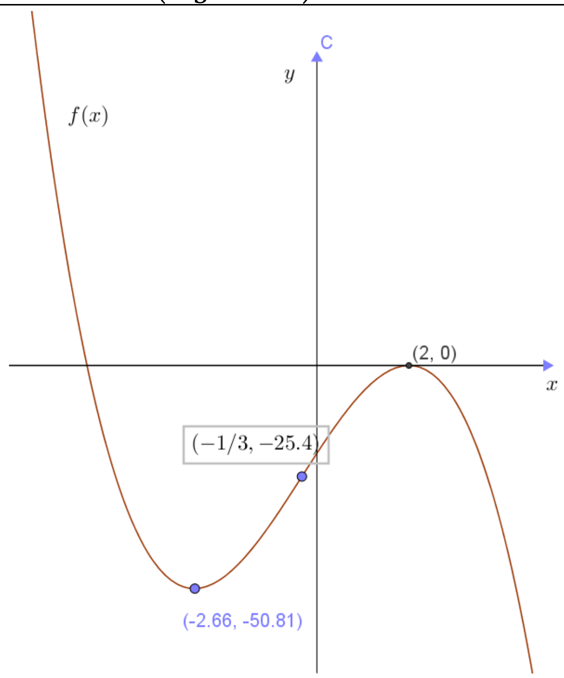
(ii)

$$y'' = -6x - 2 = 0$$
$$x = -\frac{1}{3}, y = -25.4$$

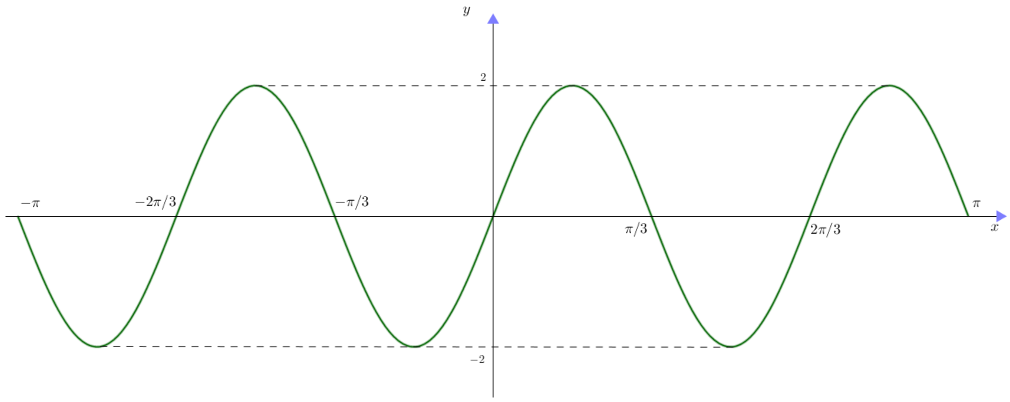
x	-1	$-\frac{1}{3}$	0
y''	4	0	-2

$$\therefore \left(-\frac{1}{3}, -25.4\right) is\ a\ point\ of\ inflexion$$

(iii)

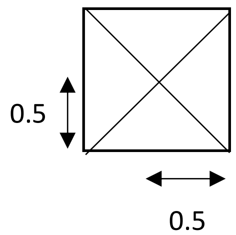


8 a)	$$\text{(i) } \alpha+\beta=-\frac{b}{a}=\frac{4}{3}$$ $$\text{(ii) } \alpha\beta=\frac{c}{a}=\frac{2}{3}$$ $$\text{(iii) } \frac{2}{\alpha}+\frac{2}{\beta}=\frac{2(\alpha+\beta)}{\alpha\beta}=\frac{2\left(\frac{4}{3}\right)}{\frac{2}{3}}=4$$ $$\text{(iv)} (\alpha-\beta)(\beta-\alpha)=-(\alpha-\beta)^2$$ $$=-\{(\alpha+\beta)^2-4\alpha\beta\}$$ $$=4\alpha\beta-(\alpha+\beta)^2$$ $$=4\left(\frac{2}{3}\right)-\left(\frac{4}{3}\right)^2=\frac{8}{3}-\frac{16}{9}$$ $$=\frac{8}{9}$$
b)	$$3x^2-2x+5\equiv Ax^2-2Ax+Bx-2B+C$$ $$A=3\ (only\ coefficient\ of\ x^2)$$ $$B-2A=-2\,,\therefore B=-2+2A=4$$ $$C-2B=5,\qquad \therefore C=5+2B=13$$
c)	$$\text{let } 3^x=t$$ $$t^2-12t+27=0$$ $$(t-3)(t-9)=0,\therefore t=3,9$$ $$3^x=3,\therefore x=1$$ $$3^x=9,\therefore x=2$$
d) (i)	$$\text{real,}\qquad \Delta\geq 0$$ $$\Delta=(4-\mathrm{m})^2-4(2m-3)$$ $$=16-8m+m^2-8m+12$$ $$=m^2-16m+28\geq 0$$ $$(m-14)(m-2)\geq 0$$ $$\therefore m\leq 2 \text{ or } m\geq 14$$
(ii)	$$\Delta\geq 0,$$ $$\text{and } m-4<0,\therefore m<4$$ $$\text{and } 2m-3>0,\therefore \mathrm{m}>\frac{3}{2}$$ $$\frac{3}{2}<m\leq 2$$ (Draw a number line)

9a)	$cosec\ 240° = \frac{1}{\sin 240\,°} = -\frac{1}{\sin 60} = -\frac{1}{\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}\ or\ -\frac{2\sqrt{3}}{3}$
b)	
c)	$LHS = \frac{\cos\theta + \sin\theta}{(\cos\theta - \sin\theta)} + \frac{\cos\theta - \sin\theta}{(\cos\theta + \sin\theta)}$ $= \frac{(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2}{(\cos\theta - \sin\theta)\,(\cos\theta + \sin\theta)}$ $= \frac{\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta}$ $= \frac{2(\cos^2\theta + \sin^2\theta)}{\cos^2\theta - \sin^2\theta}, (since\ \cos^2\theta + \sin^2\theta = 1, \cos^2\theta - \sin^2\theta = \cos 2\theta)$ $= \frac{2}{\cos 2\theta} = 2\sec 2\theta = RHS$
d) (i)	$\Delta RMA\ is\ a\ right\ angled\ at\ M, AM = h\,, \quad \left(\tan 45° = \frac{h}{AM} = 1\right)$ $AM^2 = 30^2 + x^2 \quad \therefore h^2 = 30^2 + x^2$
(ii)	From cosine rule $MC^2 = x^2 + 90^2 - 2\times 90\times x\times\cos\angle MBC,\ \ Where\ \angle MBC = 90 + 45 = 135°$ $= x^2 + 90^2 - 2\times 90\times x\times -\frac{1}{\sqrt{2}} = x^2 + 90^2 + 90\sqrt{2}\,x$ $Q.E.D.$
(iii)	From (i) and (ii) $From\ RMC, \frac{h^2}{MC^2} = \frac{x^2 + 900}{x^2 + 90\sqrt{2} + 90^2} = \tan^2 30° = \frac{1}{3}$ $3x^2 + 2700 = x^2 + 90\sqrt{2} + 90^2$ $2x^2 - 90\sqrt{2}x - 5400 = 0$ $x^2 - 45\sqrt{2} - 2700 = 0$ $\therefore x = \frac{45\sqrt{2} + \sqrt{4050 - 4(1)(-2700)}}{2}, \qquad as\ x > 0.$ $h = \sqrt{x^2 + 900} = 97m\ (nearest\ m)$

10 a)	$$perpendicular\ distance = \frac{\lvert 3\times(-2)-4\times\delta-4\rvert}{\sqrt{3^2+4^2}} = \frac{\lvert -10-4\delta\rvert}{5}$$ $$\frac{\lvert -10-4\delta\rvert}{5} > 2, \quad \therefore \lvert -10-4\delta\rvert > 10$$ $$(-10-4\delta)^2 > 100$$ $$4(5+2\delta)^2 > 100, \quad \therefore (5+2\delta)^2 > 25$$ $$\delta > 0\ or\ \delta < -5$$
b)	$$PA = 2PB, \therefore PA^2 = 4PB^2$$ $$(x+2)^2+(y-4)^2 = 4\{(x-4)^2+(y-1)^2\}$$ $$x^2+4x+4+y^2-8y+16 = 4(x^2-8x+16+y^2-2y+1)$$ $$4x^2-32x+72+4y^2-8y+4-x^2-4x-4-y^2+8y-16=0$$ $$3x^2-36x+3y^2+56=0$$ $$x^2-12x+y^2=-16$$ $$(x-6)^2+y^2=20$$ It is a circle locus with centre :(6,0) with radius of $2\sqrt{5}$
c) (i)	$$m_{PQ} = \frac{-aq^2+ap^2}{2aq-2ap} = \frac{a(p-q)(p+q)}{-2a(p-q)} = -\frac{p+q}{2}$$ $$y+ap^2 = -\frac{p+q}{2}(x-2ap)$$ $$y = -\frac{(p+q)}{2}x+ap(p+q)-ap^2$$ $$y = -\frac{p+q}{2}x+apq$$
(ii)	PQ as focal chord passes through (0,-a) Sub (0,-a) into the equation from (i) $$-a = apq$$ $$\therefore pq = -1$$
d) (i)	$$y-ap^2 = p(x-2ap)$$ $$y = px-ap^2$$ Similarly $$y = qx-aq^2$$
(ii)	$$px-ap^2 = qx-aq^2$$ $$(p-q)x = a(p-q)(p+q)$$ $$x_T = a(p+q)$$ $$y_T = p\{a(p+q)\}-ap^2 = apq, \quad \therefore T(a(p+q), apq)$$
(iii)	T lies on $x^2 = -4ay$ $$a^2(p+q)^2 = -4a(apq)$$ $$a^2[p^2+q^2+2pq] = -4a^2pq$$ $$p^2+q^2 = -6pq$$

(iv)	$$M\left(a(p+q),\frac{a(p^2+q^2)}{2}\right)$$ $$x=a(p+q),\ \ \frac{2y}{a}=p^2+q^2$$ $$x^2=a^2(p+q)^2=a^2(p^2+q^2+2pq)$$ $$=\frac{1}{3}a^2(3(p^2+q^2)+6pq)$$ $$=\frac{2}{3}a^2(p^2+q^2)$$ $$=\frac{2}{3}a^2\left(\frac{2y}{a}\right)=\frac{4}{3}ay$$ $$\therefore x^2=\frac{4}{3}ay$$

11 a)	$$2f(x) + 3f(1-x) = x^2 \qquad - \ (1)$$ Let $x = 1 - x$ $$2f(1-x) + 3f(x) = (1-x)^2 \qquad - \ (2)$$ $$3 \times (2) - 2 \times (1)$$ $$6f(1-x) + 9f(x) - 4f(x) - 6f(1-x) = 3x^2 - 6x + 3 - 2x^2$$ $$5f(x) = x^2 - 6x + 3$$ $$\therefore f(x) = \frac{1}{5}(x^2 - 6x + 3)$$
b) (i)	 $$C = \left(0 + \frac{1}{2}, 1 - \frac{1}{2}\right)$$ $$= (\frac{1}{2}, \frac{1}{2})$$
(ii)	Like part (i) $$E = (p - \frac{1}{2}, p^2 + \frac{1}{2})$$
(iii)	$$A(p) = b \times h = \left(\frac{1}{2} - \left(p - \frac{1}{2}\right)\right)\left(\left(p^2 + \frac{1}{2}\right) - \frac{1}{2}\right)$$ $$A(p) = (1-p)p^2 = -p^3 + p^2$$
(iv)	$$\frac{dA}{dp} = -3p^2 + 2p = p(-3p + 2)$$ $$Stationary\ point = p(-3p+2) = 0$$ $$p = 0\ or\ p = \frac{2}{3}$$ $$\frac{d^2A}{dp^2} = -6p + 2$$ $$f''(0) = 2, \qquad \text{Min}$$ $$f''\left(\frac{2}{3}\right) = -2, \qquad Max$$ $$\therefore Max\ A = -\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 = -\frac{8}{27} + \frac{12}{27} = \frac{4}{27}\ u^2$$

12 a)	$\sin\alpha + \sin\beta = \sqrt{\frac{5}{3}} \quad (1)$ $\cos\alpha + \cos\beta = 1 \quad (2)$ $(1)^2 + (2)^2 = \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta + \cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta$ $= \sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta + 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$ $= 2 + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 + 2\cos(\alpha-\beta)$ $\frac{5}{3} + 1 = 2 + 2\cos(\alpha-\beta)$ $\therefore \cos(\alpha-\beta) = \frac{1}{3}$
b) (i)	$\angle PAQ = 180 - \angle XAY\ (Supplementary\ angle)$ $\angle XAY = 90°\ (angle\ in\ a\ semicircle)$ $\therefore \angle PAQ = 90°$ as given
(ii)	Similarly $\angle PBQ = 90°$ $\angle PBQ + \angle PAQ = 90° + 90° = 180°$ opposite angles are supplementary $\therefore$ APBQ is a cyclic quadrilateral
(iii)	$\angle XQA = \angle XYA + \angle YXB,$ $(exterior\ angle\ of\ a\ triangle\ is\ equal\ to\ the\ sum\ of\ other\ two\ interior\ angles)$
(iv)	$\angle XYA = \frac{1}{2}\angle XOA, \angle YXB = \frac{1}{2}\angle YOB$ $\angle XOA + \angle YOB = 180° - 60°(equilateral\ \Delta) = 120°$ $\therefore \angle XYA + \angle YXB = \angle XQA = 60°$ $\angle APB = \angle XQA = 60°$ (exterior angle of cyclic quadrilateral is equal to the opposite interior angle)
(v)	Point P traces circumference of a circle subtended from arc AB with angle at the circumeference of 60° with diameter of PQ.