North Sydney Boys High School

MATHEMATICS EXTENSION 1

2018 Preliminary Course Yearly Examination

12th of September, 2018

General instructions

- Working time – 2 hours 30 minutes (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.

QUESTIONS 1-9

- Commence each new question on a **new page**. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

- ◯ Mr Berry
- ◯ Mr Hwang
- ◯ Mr Ireland
- ◯ Dr Jomaa
- ◯ Ms Lee
- ◯ Mr Lin
- ◯ Ms Ziaziaris

Marker's use only.

QUESTION	1	2	3	4	5	6	7	8	9	Total
MARKS	$\overline{}$ 18	$\overline{}$ 24	$\overline{}$ 13	$\overline{}$ 11	$\overline{}$ 9	$\overline{}$ 9	$\overline{}$ 11	$\overline{}$ 8	$\overline{}$ 7	$\overline{}$ 110

Start EACH question in a SEPARATE answer booklet.

Start a NEW booklet

Question 1 (18 marks) **Marks**

a) Fully factorise $2x^3 - 128$ **2**

b) Solve $\frac{2x}{5-x} \geq 1$ **3**

c) Solve $|2x - 1| < 3x + 2$ **3**

d) Solve $x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6 = 0$ **3**

e) Given $\sin\theta = \frac{4}{5}$ and $\frac{\pi}{2} \leq \theta \leq \pi$ find $\sin 2\theta$ **2**

f) Simplify $\frac{\cos(270-\theta)}{\cos(180-\theta)}$ **2**

g) Solve $2tan\,^3\theta - 3\tan^2\theta - 2\tan\theta + 3 = 0$ for $0° \leq \theta \leq 360°$, giving your answers to the nearest minute, where necessary **3**

Start a NEW booklet

Question 2 (24 marks) **Marks**

a) Show that $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sin x + \cos x}{\sqrt{2}}$. **4**

Hence or otherwise solve $\frac{\sin x + \cos x}{\sqrt{2}} = \frac{\sqrt{3}}{2}$ for $0 \leq x \leq 360°$

b) Prove that $\frac{\cos 2\theta}{\sin\theta} + \frac{\sin 2\theta}{\cos\theta} = cosec\,\theta$ **3**

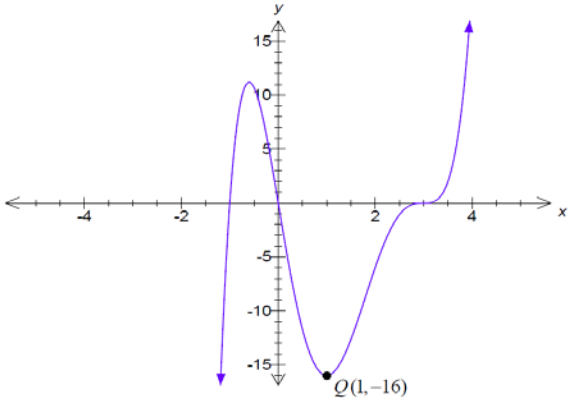
c) Find the size of the acute angle between the lines $y = 2x + 3$ and $4x - y + 1 = 0$ **2**

d) The point $(8, -5)$ divides the interval AB externally in the ratio $3:2$. If A is the point $(-1,4)$, find the coordinates of $B(x, y)$. **2**

e) Sketch the region of the number plane where $y \leq 1$ and $y < |x - 2|$ hold simultaneously **3**

f) Write $\frac{2^{1001}+2^{1000}}{3}$ as power of 2 **2**

g) **4**



i) The above graph has equation $f(x) = ax(x - b)(x + c)^d$. Find the values of $a, b, c\ d$ **3**

ii) Write down the domain of $y = \sqrt{f(x)}$ **1**

Start a NEW booklet

Question 3 (13 marks) **Marks**

a) Differentiate and fully simplify

(i) $y = x(1-x)^7$ **2**

(ii) $y = x\,.\sqrt[3]{x}$ **2**

(iii) $y = \frac{5-x^2}{2+x}$ **2**

b) Find the value of a given that the gradient of the tangent at $(2,4)$ on $y = 3x^2 - 2ax + 1$ is -1. **2**

c) For what values of x is the curve $y = 4x - x^3$ concave upwards? **2**

d) Differentiate from first principles $f(x) = 3x^2 - 2x$ **3**

Start a NEW booklet

Question 4 (11 marks) **Marks**

a) Consider the function $f(x) = \frac{x+1}{x^2+3}$ **6**

i) Find the x and y intercepts
ii) Find the coordinates of any stationary points on the curve $y = f(x)$ and determine their nature
iii) Find $\lim_{x\to\infty} \frac{x+1}{x^2+3}$
iv) Sketch $y = f(x)$

b) If α and β are the roots of the equation $4x^2 - 2x - 1 = 0$ find (without solving for x) **5**

i) $\frac{1}{2\alpha} + \frac{1}{2\beta}$
ii) $\alpha^2 + \beta^2$
iii) $|\alpha - \beta|$

Start a NEW booklet

Question 5 (9 marks) **Marks**

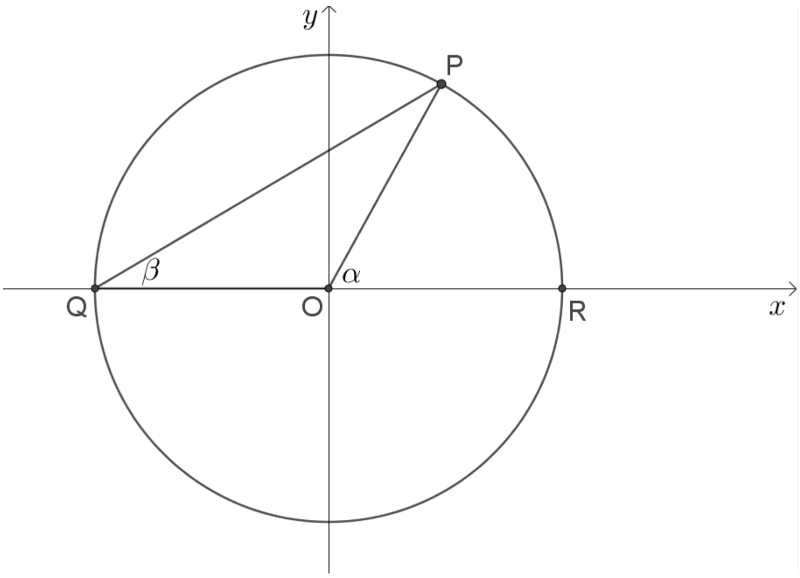
a) Find the values of k for which $(k+1)x^2 - 2(k-1)x + (2k-5)$ is a perfect square **3**

b) Find the coordinates of the vertex, focus and equation of the directrix of the parabola $x^2 + 6x + 8y - 7 = 0$ **4**

c) Write down the Cartesian equation of the locus of a point $P(x,y)$ where $x = 2\cos\theta$ and $y = \frac{1}{2}\sin\theta$ **2**

Start a NEW booklet

Question 6 (9 marks) **Marks**



In the diagram Q is the point $(-1,0)$, R is the point $(1,0)$, and P is another point on the circle with centre O and radius 1. Let $\angle POR = \alpha$ and $\angle PQR = \beta$, and let $\tan\beta = m$.

a) Explain why ΔOPQ is isosceles, and hence deduce that $\alpha = 2\beta$ **2**

b) Find the equation of the line PQ in terms of m **2**

c) Show that the x-coordinates of P and Q are solutions of the equation **2**

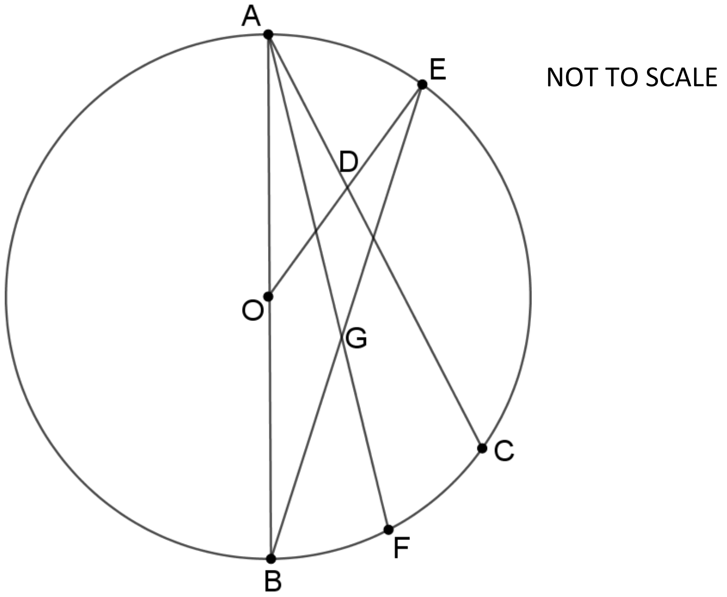
$$(1+m^2)x^2 + 2m^2x + m^2 - 1 = 0$$

d) Write an expression for the sum of the roots of this quadratic equation **1**

e) Hence or otherwise find the coordinates of P in terms of m **2**

Start a NEW booklet

Question 7 (11 marks) **Marks**



a) In the figure, AOB is the diameter of a circle centre O. D is a point on chord AC such that $DA = DO$ and OD is produced to E on the circumference. AF is the bisector of $\angle BAC$ and cuts BE at G.

Draw the diagram into your answer booklet

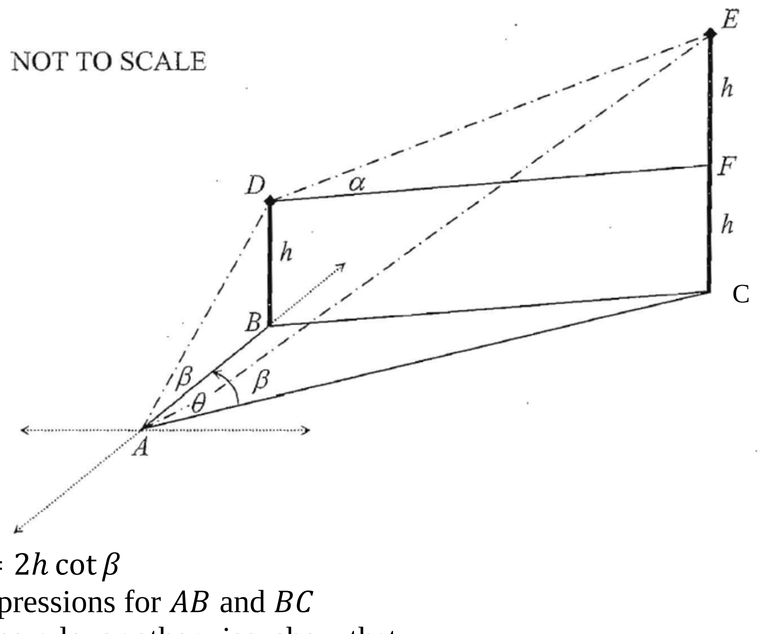
Prove that

i) $GA = GB$ **3**

ii) $AOGE$ is a cyclic quadrilateral **2**

iii) If CE is joined, then $CE \parallel AF$ **1**

c) A man standing on level ground at A notices two vertical towers BD and CE.
The foot of tower $BD, B,$ is due North of A and the foot of tower $CE, C,$ is on a bearing of θ from A.
The height of BD is h metres and the height of CE is twice the height of BD.
The angle of elevation from A to the top of both towers is β.
The angle of elevation to the top of CE from the top of BD is α



i) Show that $AC = 2h\cot\beta$ **1**

ii) Find similar expressions for AB and BC **2**

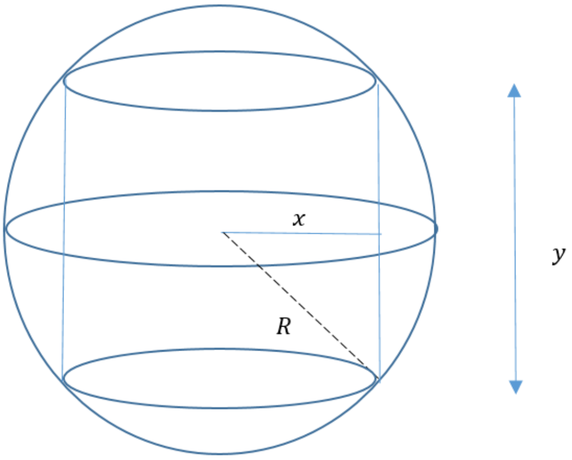
iii) Using the cosine rule, or otherwise, show that **2**

$$\cos\theta = \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}$$

Start a NEW booklet

Question 8 (8 marks) **Marks**

a) A right cylinder of radius x and height y is inscribed in a sphere of fixed radius R.



(The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the volume of a right cylinder is $V = \pi r^2 h$)

i) Show that the volume of the cylinder V, in the diagram above can be written as **2**

$$V = \pi R^2 y - \frac{\pi y^3}{4}$$

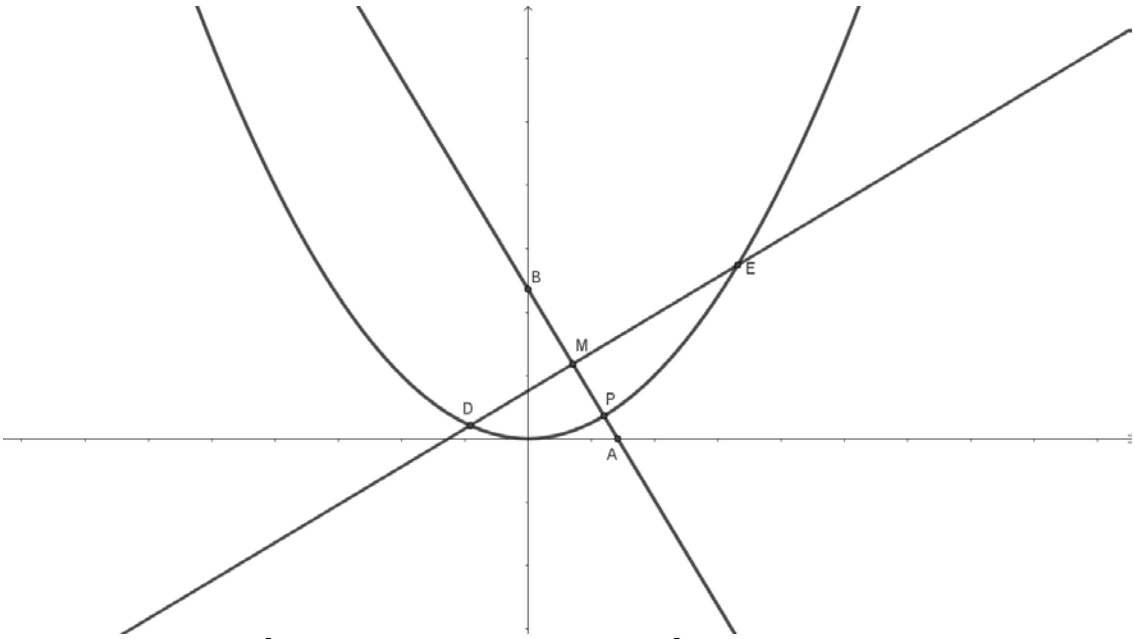
ii) Prove that the maximum volume of the cylinder occurs when $y = \frac{2R}{\sqrt{3}}$ **3**

iii) Find the maximum volume of the cylinder V in terms of R in simplest form **2**

iv) Show that when the cylinder has maximum volume, the ratio of the volume of the cylinder to the volume of the sphere is $1 : \sqrt{3}$. **1**

Start a NEW booklet

Question 9 (7 marks) **Marks**



The point $P(2at, at^2)$ lies on the parabola $4ay = x^2$.

The equation of the normal to the parabola at the point ***P*** is given by

$$y = -\frac{1}{t}x + 2a + at^2$$

a) Find the coordinates of the points **A** and **B**, the x and y intercepts of the normal and hence find the coordinates of their midpoint **M** **3**

b) Find the equation of the normal to the normal in part **a)** that passes through the midpoint **M** **2**

c) The normal in part **c)** cuts the parabola at **D** and **E**. Without finding their coordinates, find the locus of their midpoint, stating all restrictions. **2**

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Question 1 (18 marks)

2 a) $2x^3-128$

$=2(x^3-64)$

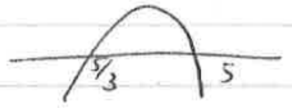
$=2(x-4)(x^2+4x+16)$

3 b) $\dfrac{2x}{5-x} \geq 1$

$2x(5-x)-(5-x)^2 \geqslant 0$

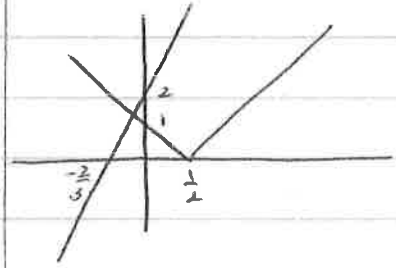
$(5-x)(2x-(5-x)) \geqslant 0$

$(5-x)(3x-5) \geqslant 0$



$\frac{5}{3} \leq x < 5$

3 c) $|2x-1| < 3x+2$



$-(2x-1) = 3x+2$

$-1 = 5x$

$x = -\frac{1}{5}$

$\therefore\ x > -\frac{1}{5}$

3 d) $x^{2/3} + x^{1/3} - 6 = 0$

Let $u = x^{1/3}$

$u^2 + u - 6 = 0$

$(u+3)(u-2) = 0$

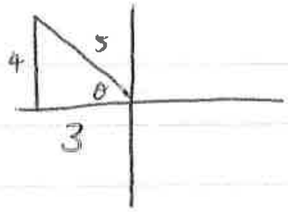
$u = -3 \qquad u = 2$

$x^{1/3} = -3 \qquad x^{1/3} = 2$

$x = (-3)^3 \qquad x = 2^3$

$x = -27$ or $+8$

2 e)



$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \frac{4}{5} \times -\frac{3}{5}$$
$$= \frac{-24}{25}$$

2 f)

$$\frac{\cos(270-\theta)}{\cos(180-\theta)}$$
$$= \frac{\cos(180+(90-\theta))}{\cos((180-\theta))}$$
$$= \frac{-\cos(90-\theta)}{-\cos\theta}$$
$$= \frac{-\sin\theta}{-\cos\theta}$$
$$= \tan\theta$$

3 g)

$$2\tan^3\theta - 2\tan^2\theta - 2\tan\theta + 3 = 0$$
$$\tan^2\theta(2\tan\theta - 3) - (2\tan\theta - 3) = 0$$
$$(2\tan\theta - 3)(\tan^2\theta - 1) = 0$$
$$\tan^2\theta = \pm 1 \qquad \tan\theta = 3/2$$
$$\theta = 45, 135, 225, 315, 56°19', 236°19'$$

Question 2 (20 marks)

4 a) $LHS = \sin(x + \frac{\pi}{4})$

$= \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$

$= \frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}$

$= \frac{\sin x + \cos x}{\sqrt{2}}$

$= RHS$

$\sin(x + \frac{\pi}{4}) = \frac{\sqrt{3}}{2}$

$x + \frac{\pi}{4} = 60°, 120°$

$x + 45 = 60°, 120°$

$x = 15°, 75°$

3 b) $LHS = \frac{\cos 2\theta}{\sin\theta} + \frac{\sin 2\theta}{\cos\theta}$

$= \frac{1 - 2\sin^2\theta}{\sin\theta} + \frac{2\sin\theta \cancel{\cos\theta}}{\cancel{\cos\theta}}$

$= \frac{1 - 2\sin^2\theta}{\sin\theta} + \frac{2\sin^2\theta}{\sin\theta}$

$= \frac{1}{\sin\theta}$

$= \operatorname{cosec}\theta$

$= RHS$

OR LHS

$= \frac{\cos 2\theta\cos\theta + \sin 2\theta\sin\theta}{\sin\theta\cos\theta}$

$= \frac{\cos(2\theta - \theta)}{\sin\theta\cos\theta}$

$= \frac{\cancel{\cos\theta}}{\sin\theta\cancel{\cos\theta}}$

$= \operatorname{cosec}\theta$

$= RHS$

2 c) $m_1 = 2,\ m_2 = 4$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2 - 4}{1 + 2 \times 4}\right|$

$= 2/9$

$\theta = 12°32'$ (or $12.53°$ or 0.219 radians)

2 d) $A(-1,4)$ $B(x,y)$

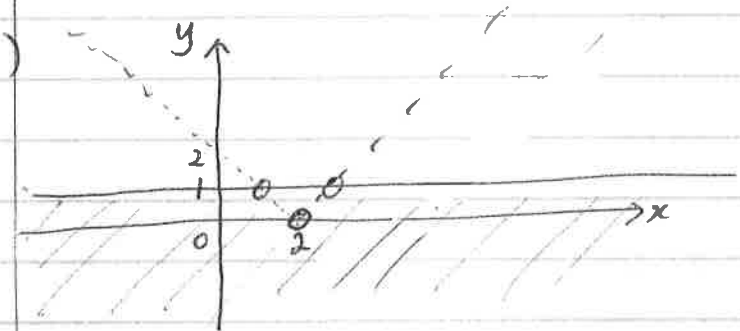
$-3 : 2$

$8 = \dfrac{-3x-2}{-3+2}$ $-5 = \dfrac{-3y+2(4)}{-3+2}$

$-8 = -3x-2$ $5 = -3y+8$

$3x = 6$ $3y = 3$

$x = 2$ $y = 1$

3 e) 

$y \leq 1$ and $y < |x-2|$

2 f) $\dfrac{2^{1001}+2^{1000}}{3}$

$= \dfrac{2^{1000}(2+1)}{3}$

$= 2^{1000}$

4 g) i) $f(x) = a(x-b)(x+c)^d$

$b = -1$

$c = -3$

$d = 3$

$f(x) = a(x+1)(x-3)^3$

$(1,-16)$

$-16 = a(2)(-2)^3$

$\therefore a = 1$

ii) $D: -1 \leq x \leq 0$ or $x \geq 3$

Question 3 (13 marks)

a)

(i) $(1-x)^7 \cdot 1 + x \cdot 7(1-x)^6 \cdot (-1)$

$= (1-x)^7 - 7x(1-x)^6$ 2

$= (1-x)^6(1-x-7x)$

$= (1-x)^6(1-8x)$ 1

(ii) $y = x \cdot \sqrt[3]{x}$

$y = x \times x^{1/3}$

$y = x^{4/3}$ 1

$\frac{dy}{dx} = \frac{4}{3}x^{1/3}$

$= \frac{4}{3} \cdot \sqrt[3]{x}$ 1

(iii) $\frac{dy}{dx} = \frac{(2+x) \cdot (-2x) - (5-x^2) \cdot 1}{(2+x)^2}$ 1

$= \frac{-4x - 2x^2 - 5 + x^2}{(2+x)^2}$

$= \frac{-4x-5-x^2}{(2+x)^2}$ 1

b) $y = 3x^2 - 2ax + 1$

$\frac{dy}{dx} = 6x - 2a$ 1

When $x = 2$, $\frac{dy}{dx} = -1$

$-1 = 12 - 2a$

$2a = 13$ 1

$a = \frac{13}{2}$

(c) $y = 4x - x^3$

$\frac{dy}{dx} = 4 - 3x^2$

$\frac{d^2y}{dx^2} = -6x$ 1

concave up when

$\frac{d^2y}{dx^2} > 0$

$-6x > 0$

$x < 0$ 1

(d) $y = 3x^2 - 2x$

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h}$

$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{2x} - 2h - \cancel{3x^2} + \cancel{2x}}{h}$

$= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h}$

$= \lim_{h \to 0} 6x + 3h - 2$

$= 6x - 2$ 3.

Question 4 (11 marks)

6 a) $f(x) = \dfrac{x+1}{x^2+3}$

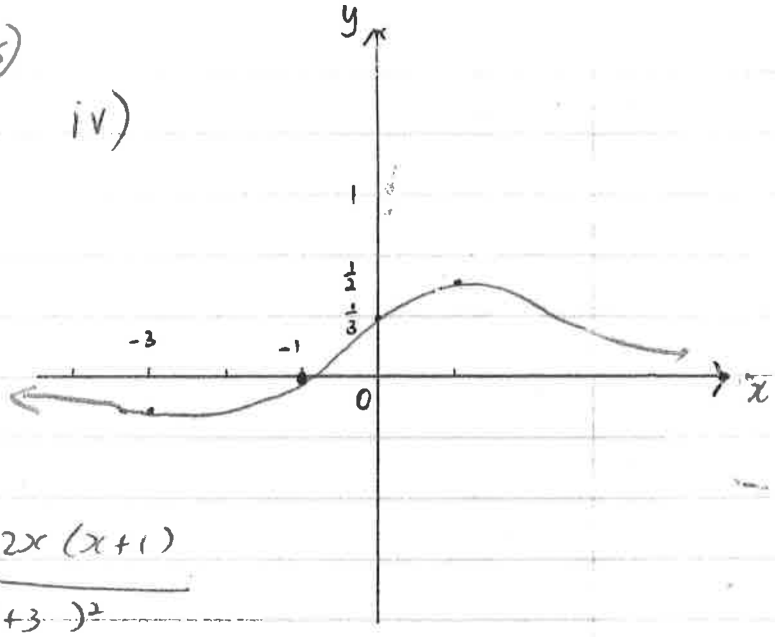
i) x interept : $y = 0$

$x = -1$

y-intercept : $x = 0$

$y = \frac{1}{3}$

iv)



ii) $f'(x) = \dfrac{x^2+3 - 2x(x+1)}{(x^2+3)^2}$

$= \dfrac{x^2+3-2x^2-2x}{(x^2+3)^2}$

$= \dfrac{-x^2-2x+3}{(x^2+3)^2}$

Stat pts: $f'(x) = 0$

$-x^2-2x+3 = 0$

$-(x^2+2x-3) = 0$

$-(x+3)(x-1) = 0$

$x = -3 \ , \ y = \frac{-2}{12} = -\frac{1}{6}$

$x = 1 \ , \ y = \frac{2}{4} = \frac{1}{2}$

x	-4	-3	0	1	2
$f'(x)$	$-5/192$	0	$\frac{3}{9}$	0	$-5/49$
	\	—	/	—	\

$\therefore$ min t.p. at $(-3, -\frac{1}{6})$ and max t.p. at $(1, \frac{1}{2})$

iii) $\displaystyle\lim_{x\to\infty} \frac{x+1}{x^2+3}$

$= \displaystyle\lim_{x\to 0} \frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}$

$= \displaystyle\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}$

$= 0$

5 b) $4x^2-2x-1=0$

i) $\frac{1}{2\alpha}+\frac{1}{2\beta}$

$=\frac{\beta+\alpha}{2\alpha\beta}$

$=\frac{\frac{2}{4}}{2(-\frac{1}{4})}$

$=-1$

ii) $\alpha^2+\beta^2$

$=(\alpha+\beta)^2-2\alpha\beta$

$=(\frac{2}{4})^2-2(-\frac{1}{4})$

$=\frac{1}{4}+\frac{1}{2}$

$=3/4$

iii) $|\alpha-\beta| = \sqrt{(\alpha-\beta)^2}$

$=\sqrt{\alpha^2+\beta^2-2\alpha\beta}$

$=\sqrt{3/4-2(-\frac{1}{4})}$

$=\frac{\sqrt{5}}{2}$

Question 5 (9 marks)

3 a) $(k+1)x^2 - 2(k-1)x + (2k-5)$

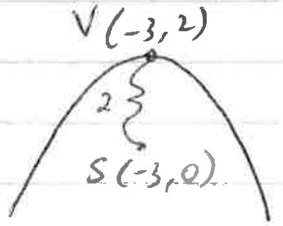
Perfect square : $\Delta = 0$

$$(-2(k-1))^2 - 4(k+1)(2k-5) = 0$$
$$4k^2 - 8k + 4 - 4(2k^2 - 3k - 5) = 0$$
$$4k^2 - 8k + 4 - 8k^2 + 12k + 20 = 0$$
$$-4k^2 + 4k + 24 = 0$$
$$k^2 - k - 6 = 0$$
$$(k-3)(k+2) = 0$$
$$k = 3 \text{ or } -2$$

4 b) $x^2 + 6x + 8y - 7 = 0$

$$(x+3)^2 = -8y + 16$$

$y = 4$ - - - - - - - - $= -8(y-2)$



$4a = 8$

$a = 2$

Vertex : $(-3, 2)$

Focus : $S(-3, 0)$

Directrix : $y = 4$

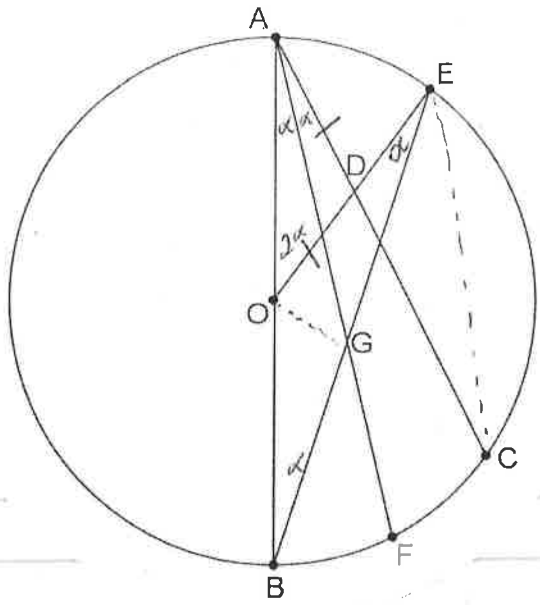
2 c) $x = 2\cos\theta \qquad y = \frac{1}{2}\sin\theta$

$\cos\theta = \frac{x}{2}$ ① $\qquad \sin\theta = 2y$ ②

$①^2 + ②^2 \qquad \cos^2\theta + \sin^2\theta = \left(\frac{x}{2}\right)^2 + (2y)^2$

$$\frac{x^2}{4} + 4y^2 = 1$$

Question 6 (9marks)



2 a) $OQ = OP$ (equal radii of circle)

$\therefore \triangle OQP$ is isosceles (2 equal sides)

$\therefore \angle OQP = \angle OPQ = \beta$ (base angles of isosceles triangle)

$\therefore \alpha = 2\beta$ (exterior angle of a triangle)

2 b) $\tan\beta = m \quad (-1, 0)$

$y - 0 = m(x+1)$

Equation of PQ: $y = mx + m$

2 c) $x^2 + y^2 = 1 \quad (1)$

$y = mx + m \quad (2)$

Solving simultaneously gives points of intersection P, Q.

Sub (1) into (2)

$x^2 + (mx+m)^2 = 1$

$x^2 + m^2x^2 + 2m^2x + m^2 - 1 = 0$

1 d) $\alpha + \beta = \dfrac{-2m^2}{1+m^2}$

2 e) One of the roots is -1

$\therefore \quad -1 + \beta = \dfrac{-2m^2}{1+m^2}$

$\beta = \dfrac{-2m^2}{1+m^2} + 1$

$= \dfrac{-2m^2 + 1 + m^2}{1+m^2}$

$= \dfrac{1-m^2}{1+m^2}$

$\therefore \quad P\left(\dfrac{1-m^2}{1+m^2}, \dfrac{2m}{1+m^2}\right)$

Question 7

3 a) i) Let $\angle BAF = \alpha$

$\angle BAF = \angle FAC = \alpha$ (AF is the bisector of $\angle BAC$ given)

$\angle BAD = 2\alpha$

$DA = DO$ (Given)

$\angle BAD = \angle AOD = 2\alpha$ (Equal angles opposite equal sides)

$2\angle ABE = \angle AOD$ (Angle at centre is twice angle at circumference)

$\therefore \angle ABE = \alpha$

Now $\angle ABE = \angle GAO = \alpha$

$\therefore GA = GB$ (Equal sides opposite equal angles)

2 ii) $OB = OE$ (Equal radii)

$\angle ABE = \angle OEB = \alpha$ (Equal angles opposite equal sides)

$\therefore \angle BAF = \angle OEB = \alpha$

$\therefore$ AOGE is a cyclic quadrilateral (Interval OG subtends equal angles at A and E on the same side of the interval).

1 iii) $\angle ABE = \angle ACE = \alpha$ (Angles in the same segment)

$\therefore \angle FAC = \angle ACE = \alpha$

$\therefore CE \parallel AF$ (Alternate angles are equal)

1 (b) i) In $\triangle ACE$

$$\tan(90-\beta) = \frac{AC}{2h}$$

$$AC = 2h\tan(90-\beta)$$
$$= 2h\cot\beta$$

2 ii) In $\triangle ABD$,

$$\tan(90-\beta) = \frac{AB}{h}$$

$$AB = h\tan(90-\beta)$$
$$= h\cot\beta$$

$$\tan(90-\alpha) = \frac{DF}{h}$$

$$DF = h\cot\alpha$$

But $DF = BC$

$\therefore BC = h\cot\alpha$

2 iii)

$$\cos\theta = \frac{AB^2 + AC^2 - BC^2}{2\times AB\times AC}$$

$$= \frac{h^2\cot^2\beta + 4h^2\cot^2\beta - h^2\cot^2\alpha}{2\times h\cot\beta \times 2h\cot\beta}$$

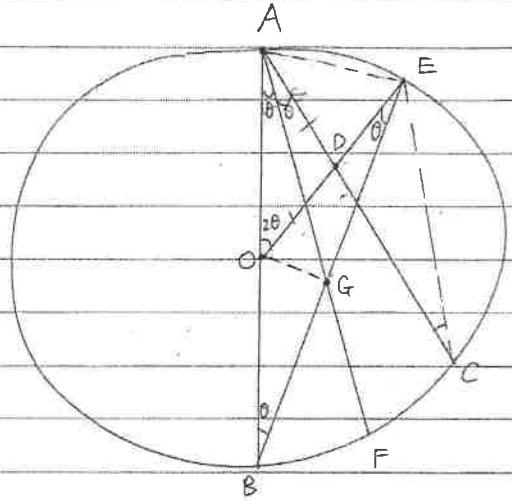
$$= \frac{\cancel{h^2}(\cot^2\beta + 4\cot^2\beta - \cot^2\alpha)}{4\cancel{h^2}\cot^2\beta}$$

$$= \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}$$

Model Response

Question 7 :

a)



i) RTP: $GA = GB$

$\because$ AF bisects $\angle BAC$

$\therefore \angle BAF = \angle FAC$

let $\angle BAF = \angle FAC = \theta$

$\therefore \boxed{\angle OAD = 2\theta}$

In triangle ODA

$\because AD = DO$ (Given)

$\therefore$ triangle ODA is isosceles (Two sides of an isosceles triangle are equal)

$\therefore \angle OAD = \angle DOA$ (Base angles of isosceles triangles are equal)

$\therefore \angle OAD = \angle DOA = 2\theta$

$\because \angle AOE = 2\theta$

$\therefore \angle ABE = \frac{2\theta}{2} = \theta$ (Angle at circumference standing on the same arc, is half the angle at centre on the same arc)

$\because \angle ABE = \theta$

$\therefore \angle ABE = \angle BAF = \angle FAC = \theta$

$\because \angle BAF = \theta = \angle ABE$

$\therefore$ Triangle AGB is isosceles (Two angle are equal in an isosceles triangle)

$\therefore AG = BG$ (Base sides of an isosceles triangle are equal)

ii). Construct line AE and OG

In triangle BOE,

$OB = OE$ (Radii in the same circle are equal)

$\therefore \triangle BOE$ is isosceles (Two sides of an isosceles triangle are equal)

$\therefore \angle OBE = \angle OEB$ (Base angles of isosceles triangles are equal)

$\because \angle OBE = \theta$

$\therefore \angle OEB = \theta$

In quadrilateral, AOGE

because OG subtends equal angles of θ at $\angle OAG$ and $\angle OEG$, therefore it is cyclic.

iii). $\because \angle AOE = 2\theta$

$\therefore \angle ACE = \theta$ (Angle at circumference standing on the same arc is half the angle at centre standing on the same arc)

$\because \angle ACE = \theta = \angle FAC$

$\therefore AF \parallel EC$ (Alternate angles $\angle FAC$ and $\angle ACE$ are only equal if $AF \parallel EC$)

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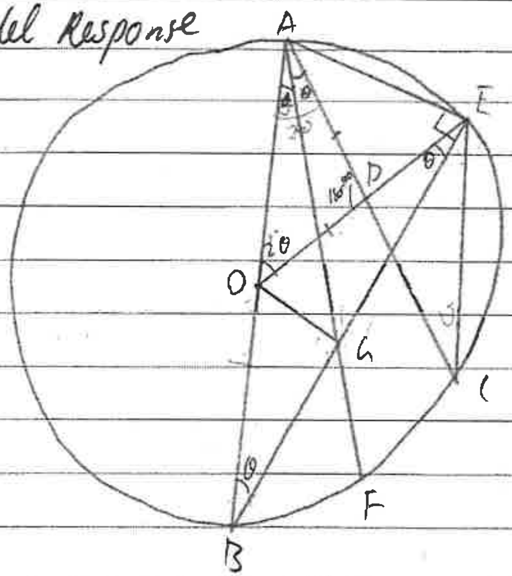
Model Response

A)



i) let $\angle OAG$ be θ

$\angle OAD = 2\theta$ (given)

$\angle ADE = 2\theta$ (base angles are equal on isosceles $\triangle ADO$)

$\therefore \angle ABE = \theta$ (angle at centre is twice angle at circumference)

$\therefore AG = BG$ (equal sides on isosceles $\triangle AGB$)

ii) $\angle OBG = \angle OEG$ (base angles are equal on isosceles $\triangle OBE$)

$\therefore$ ADGE is a cyclic quadrilateral as $\angle OAG = \angle OEG$ (converse of angles in the same segment are equal)

iii) $\angle ACE = \angle ABE = \theta$ (angles in the same segment are equal)

$\therefore \angle FAC = \angle ACE = \theta$

$\therefore AF \parallel CE$ (alternate angles are equal on parallel lines)

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Question 8

2 i) $V = \pi r^2 h$ $\qquad R^2 = x^2 + \left(\frac{y}{2}\right)^2$

$= \pi x^2 h$

$= \pi x^2 y$ $\qquad x^2 = R^2 - \frac{y^2}{4}$

$= \pi\left(R^2 - \frac{y^2}{4}\right) y$

$= \pi R^2 y - \frac{\pi y^3}{4}$ 2

3 ii) $\frac{dV}{dy} = \pi R^2 - \frac{3\pi y^2}{4}$

$\frac{d^2V}{dy^2} = -\frac{6\pi y}{4} = -\frac{3\pi y}{2}$

Let $\frac{dV}{dy} = 0$.

$\pi R^2 - \frac{3\pi y^2}{4} = 0$

$\frac{3\pi y^2}{4} = \pi R^2$

$y^2 = \frac{\cancel{\pi} R^2 \times 4}{3\cancel{\pi}}$

$y = \pm\sqrt{\frac{4R^2}{3}}$

$y = \frac{2R}{\sqrt{3}}$ $\quad y > 0$ as it is a length

3
1 – first der
1 – show find y
1 – Test

Test $y = \frac{2R}{\sqrt{3}}$

$\frac{d^2V}{dy^2} = -\frac{3\pi}{2}\left(\frac{2R}{\sqrt{3}}\right)$

< 0 for all $R > 0$. $\therefore$ max at $y = \frac{2R}{\sqrt{3}}$

2 iii) Max $V = \pi R^2\left(\frac{2R}{\sqrt{3}}\right) - \frac{\pi}{4}\left(\frac{2R}{\sqrt{3}}\right)^3$

$$= \frac{2\pi R^3}{\sqrt{3}} - \frac{\cancel{8}^{2} R^3 \pi}{\cancel{4} \times 3\sqrt{3}}$$

$$= \frac{6\pi R^3 - 2\pi R^3}{3\sqrt{3}}$$

$$= \frac{4\pi R^3}{3\sqrt{3}}$$

Sub in - 1mark
Correct answer - 1
2

1 iv) $V_{SPHERE} = \frac{4}{3}\pi R^3$

$$V_{CYL} = \frac{4\pi R^3}{3\sqrt{3}}$$

$$V_{SP} : V_{CYL}$$

$$\frac{\cancel{4}}{3}\cancel{\pi R^3} : \frac{\cancel{4\pi R^3}}{3\sqrt{3}}$$

$$\frac{1}{3} : \frac{1}{3\sqrt{3}}$$

$$\sqrt{3} : 1$$

$$\therefore V_{CYL} : V_{SP} = 1 : \sqrt{3}$$

1
Must show

Question 9

$4ay = x^2 \qquad P(2at, at^2)$

3 a) $y' = \frac{2x}{4a} = \frac{x}{2a} = \frac{2at}{2a} = t$

$m_{normal} = \frac{-1}{t}$

$y - at^2 = \frac{-1}{t}(x - 2at)$ Eq. of normal at P

$\boxed{y = \frac{-1}{t}x + 2a + at^2}$ (1)

x-intercept: $y = 0 \longrightarrow x = (2a + at^2)t$

y-intercept: $x = 0 \longrightarrow y = 2a + at^2$

$B(0, 2a + at^2)$, $A(2at + at^3, 0)$

M midpoint of AB $\left(at + \frac{1}{2}at^3, a + \frac{a}{2}t^2\right)$

2 b) Normal to (a) at M

$y - \left(a + \frac{a}{2}t^2\right) = t\left(x - at - \frac{1}{2}at^3\right)$

$y = tx - at^2 - \frac{1}{2}at^4 + a + \frac{at^2}{2}$

$\boxed{y = tx - \frac{1}{2}at^4 - \frac{a}{2}t^2 + a}$

2 c) $4ay = 4atx - 2a^2t^4 - 2a^2t^2 + 4a^2$

$x^2 - 4atx + 2a^2(t^4 + t^2 - 2) = 0$

midpoint $\left(2at, -\frac{1}{2}at^4 + \frac{3}{2}at^2 + a\right)$

$y = \frac{-1}{32a^3}x^4 + \frac{3}{8a}x^2 + a$

$$\Delta = 16a^2t^2 - 8a^2(t^4 + t^2 - 2)$$

$$= -8a^2t^4 + 8a^2t^2 + 16a^2$$

$$= 8a^2(-t^4 + t^2 + 2)$$

$$= 8a^2(-t^2+2)(t^2+1) \geq 0$$

$$-t^2 + 2 \geq 0$$

$$-\sqrt{2} \leq t \leq \sqrt{2}$$

$$-2a\sqrt{2} \leq x \leq 2a\sqrt{2}.$$