NORTH SYDNEY BOYS HIGH SCHOOL

2023 YEAR 11 ASSESSMENT TASK 3

YEARLY EXAMINATION

1st of September

Mathematics Extension 1

General Instructions

- Working time – 70 minutes **+** Reading time – 5 minutes
- Write in this booklet
- Write using blue or black pen
- NESA approved calculators may be used
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

Class Teacher: Please shade the circle.

- ○ Mr Berry
- ○ Ms Fu
- ○ Mr Ireland
- ○ Ms Lee
- ○ Ms Moss
- ○ Dr Vranešević

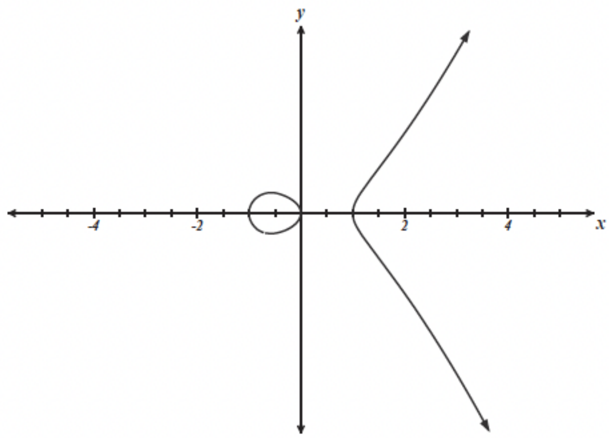
Student Number:____________________________

(To be used by the exam markers only.)

Section	A	B	C	D	E	Total	Total
Mark	$\overline{15}$	$\overline{14}$	$\overline{6}$	$\overline{20}$	$\overline{9}$	$\overline{64}$	$\overline{100}$

Section A (15 Marks)

1. The diagram shows the graph of $y^2 = f(x)$ 1



Which expression best represents the function $f(x)$

A) $x^2(1-x)$

B) $x^2(x-1)$

C) $x(1-x^2)$

D) $x(x^2-1)$

2. What is the natural domain of the function $f(x) = \frac{1}{2}\left(x\sqrt{x^2-1} - \log_e\left(x+\sqrt{x^2-1}\right)\right)$? 1

A) $x \geq 1$

B) $x \leq -1$

C) $-1 \leq x \leq 1$

D) $x \leq -1$ or $x \geq 1$

3. Solve for k in the following: $\frac{2k}{1-k} \geq k$ 3

....

....

....

....

....

....

....

....

....

4. Find the Cartesian equation which is defined parametrically as 3

$$\begin{cases} x = \sqrt{t} \\ y = 2\sqrt{1-t} \end{cases}$$

....

....

....

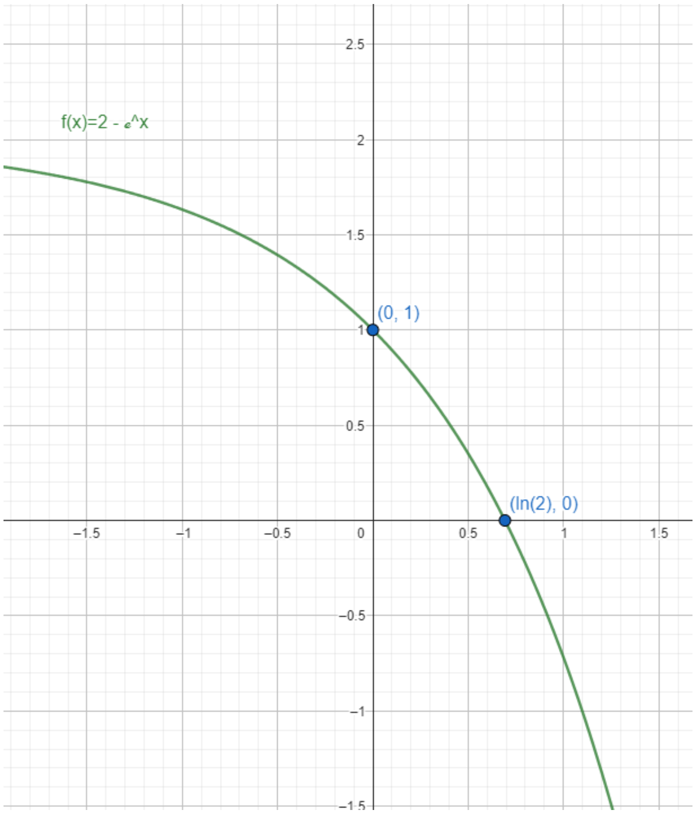
....

....

....

....

5. The diagram shows the graph of a function $f(x) = 2 - e^x$.



Sketch the following functions **in the provided grid**, indicate all important features.

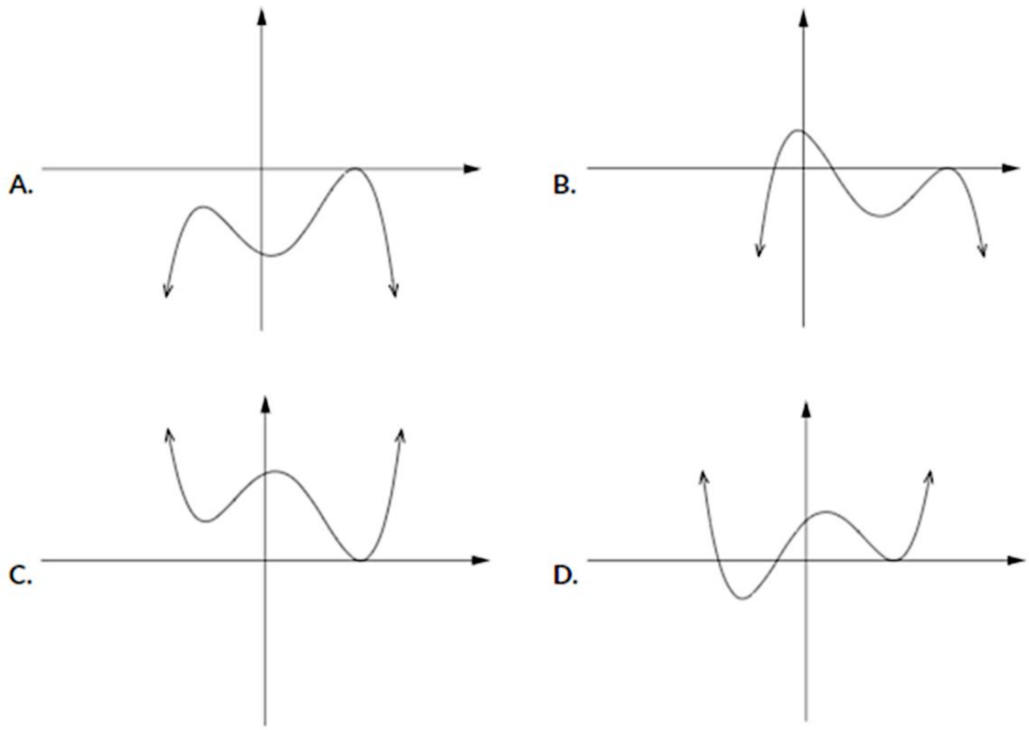
i) $y = \left(f(x)\right)^2$ 2

ii) $y = \frac{1}{f(x)}$ 2

iii) $y^2 = |f(x)|$ 3

Section B (14 Marks)

1. A monic polynomial $p(x)$ of degree 4 has one repeated zero of multiplicity 2 and is divisible by $x^2 + x + 1$. Which of the following could be the graph of $p(x)$? 1



2. Show that $(x-1)(x-2)$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$ where m and n are positive integers. 2

3. α, β and γ are solutions of $2x^3 - 4x^2 - 3x - 1$,

i) find the value of $\alpha\beta + \beta\gamma + \alpha\gamma$ 1

....

ii) find the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$ 2

....

....

....

....

....

....

iii) Hence or otherwise, find the value of $(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma)$ 2

....

....

....

....

....

....

....

....

....

4. The polynomial $P(x) = x^3 + ax^2 + b$ has a zero at r and a double zero at 4. 3

Find the values of a, b and r.

5. When a polynomial is divided by x, $(x-2)$ and $(x+3)$, the remainders are 1, 3 and 13 respectively. If the same polynomial is divided by $x(x-2)(x+3)$, the remainder is $g(x)$, then find the value of $g(1)$. 3

Section C (6 marks)

1. Four female and four male athletes are arranged in a row for the presentation of prizes. In how many ways can this be done if the males and females must alternate? 1

   A) $4! \times 4!$

   B) $4! \times 5!$

   C) $2 \times 4! \times 4!$

   D) $2 \times 4! \times 5!$

2. Four digit numbers are formed from the digits 1, 2, 3 and 4. 1

   Each digit is used once only. The sum of all the numbers that can be formed is?

   A) 6666

   B) 44440

   C) 66660

   D) 266640

3. How many distinct arrangements are there of the letters of the word ERATOSTHENES? 2

   ……………………………………………………………………………………………………

   ……………………………………………………………………………………………………

   ……………………………………………………………………………………………………

4. A child has the following six plastic magnetic numbers: 6,2,6,7,6,2 2

   How many different values can be created by using only five of the magnetic numbers?

   ……………………………………………………………………………………………………

   ……………………………………………………………………………………………………

   ……………………………………………………………………………………………………

   ……………………………………………………………………………………………………

   ……………………………………………………………………………………………………

Section D (20 marks)

1. Find the exact value of $\tan\left[\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right]$, show working out. 2

....

....

....

....

....

....

....

....

2. Show that if $\tan X = \frac{7m+6}{7-6m}$ and $\tan Y = \frac{4m+3}{4-3m}$, the value of $\tan(X - Y)$ is independent of m. 3

....

....

....

....

....

....

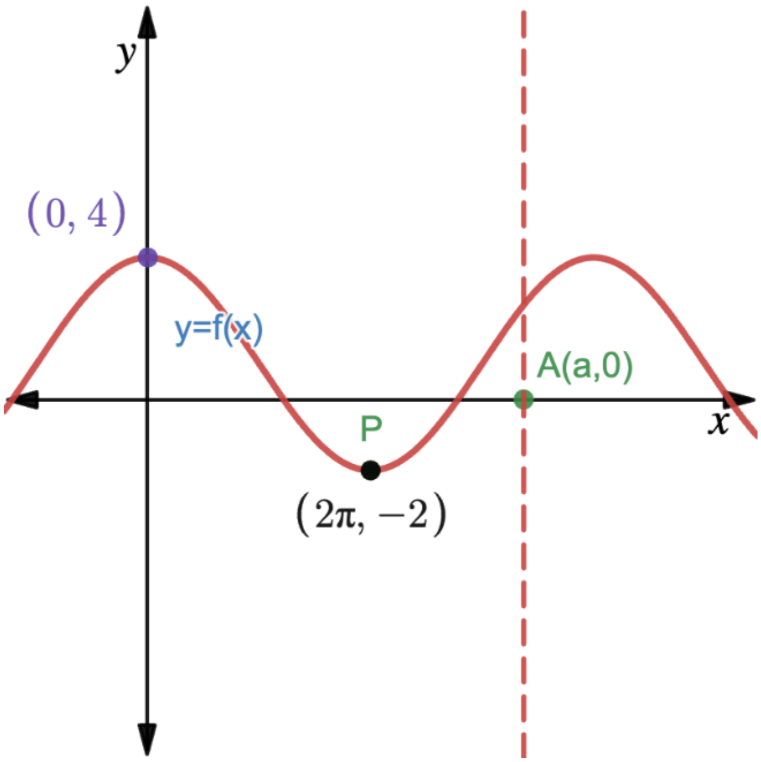
....

....

....

....

3. Let $f(x) = 1 + 3\cos\frac{x}{2}$. The diagram shows the graph of $y = f(x)$.



i) What is the largest positive domain, containing $x = 0$, for which $f(x)$ has an inverse function? 1

....

....

....

ii) Find the equation of $f^{-1}(x)$. 1

....

....

....

iii) Sketch the curve $y = f^{-1}(x)$, showing all important features. 2

iv) A(a,0) lies to the right of P$(2\pi, -2)$ as indicated in the diagram. Find a simplified expression for the exact value of $f^{-1}(f(a))$. 2

....

....

....

....

....

....

....

....

....

4. i) Use the substitution $t = \tan\frac{x}{2}$ to show that: 2

$$2\sin x - 3\cos x = \frac{3t^2 + 4t - 3}{1 + t^2}$$

....

....

....

....

....

....

....

....

ii) Hence solve the equation $2\sin x - 3\cos x = 2$ for $0° \leq x \leq 360°$. 3

Answer correct to the nearest degree.

....

....

....

....

....

....

....

....

5. i) Use the fact that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, solve $8x^3 - 6x - 1 = 0$ 2

....

....

....

....

....

....

....

....

....

....

ii) **Hence**, show that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$ 2

....

....

....

....

....

....

....

....

Section E (9 marks)

1. Find the term in x^5 in the expansion of $\left(2x - \frac{1}{x^2}\right)^8$. 2

....

....

....

....

....

....

....

....

....

....

2. i) Show that $(1+x)^m\left(1-\frac{1}{x}\right)^m = \left(x-\frac{1}{x}\right)^m$ 1

....

....

....

....

....

....

ii) By considering the term(s) independent of x in the expansion of the results from part i, show that: 3

$$\binom{2002}{0}^2 - \binom{2002}{1}^2 + \binom{2002}{2}^2 - \cdots + \binom{2002}{2002}^2 = -1\binom{2002}{1001}$$

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

iii) Hence, or otherwise, show that 3

$$\sum_{k=0}^{1001} (-1)^k \binom{2002}{k}^2 = -\frac{1}{2}\binom{2002}{1001}\left[1 + \binom{2002}{1001}\right]$$

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

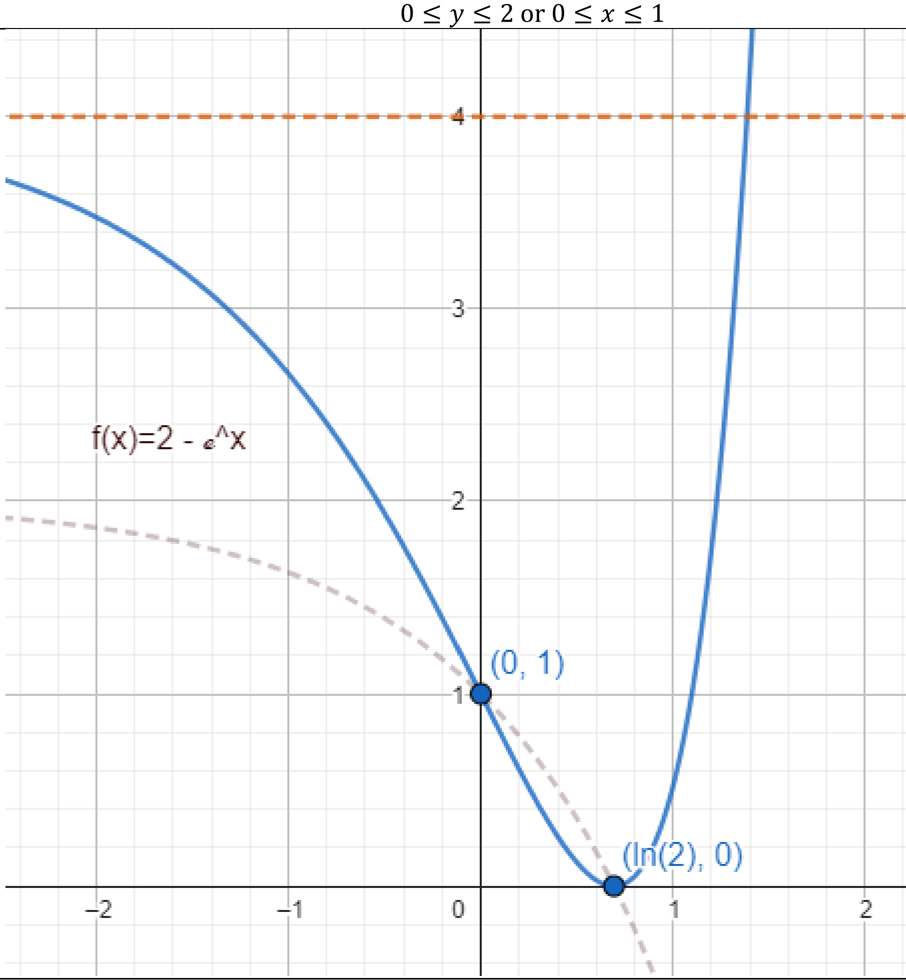
....

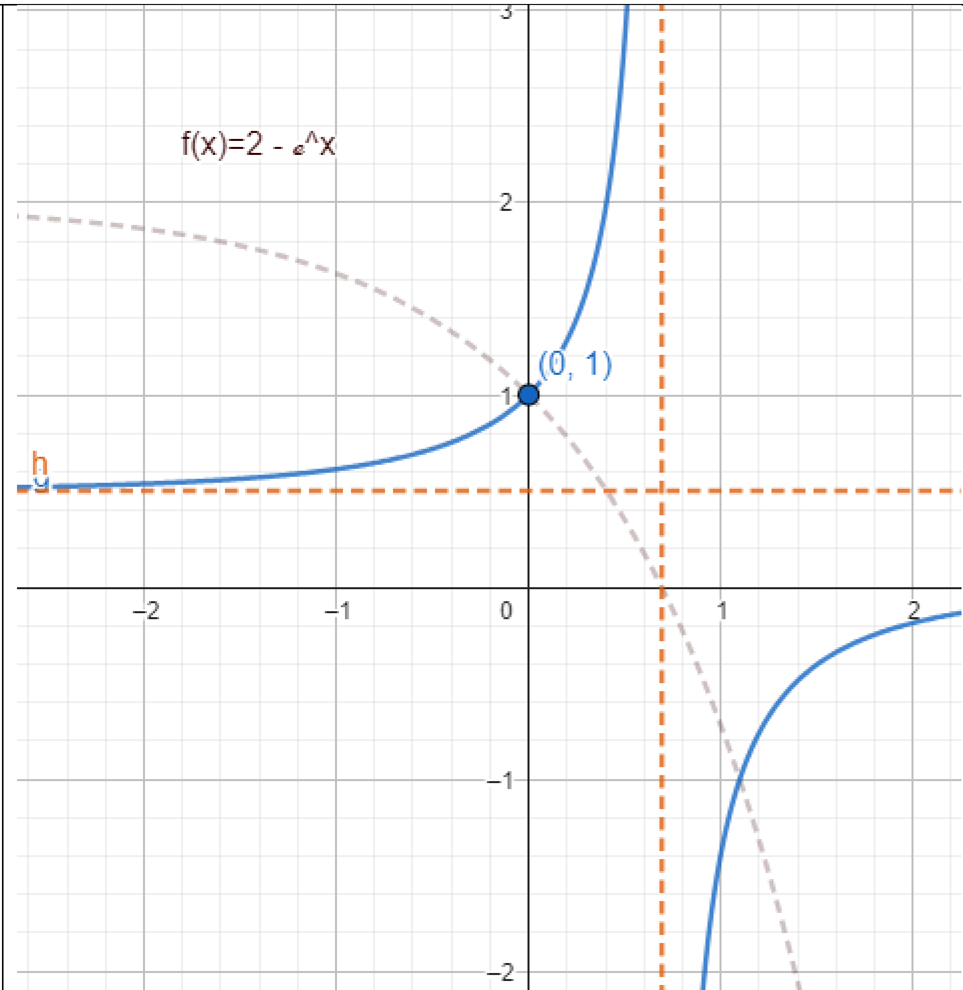
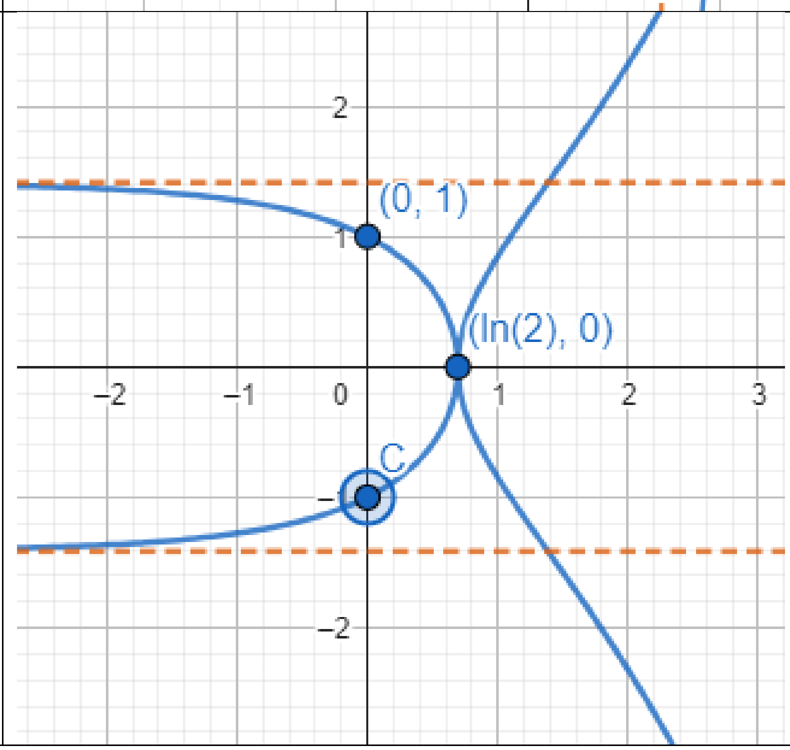
End of Examination

Extra Writing Space

Extra Writing Space

Section A

	Suggested Solutions	Marks Awarded & comments
1.	D Zeros at $x = -1, 0$ *and* 1, as $x \to \infty, y \to \infty$	1
2.	A $x^2 - 1 > 0$ and $x + \sqrt{x^2 - 1} > 0$, i.e. $x \leq -1, x \geq 1$; Thus, $x \geq 1$	1
3.	$k \neq 1$ $2k(1-k) \geq k(1-k)^2$ $2k(1-k) - k(1-k)^2 \geq 0$ $k(1-k)(2-1+k) \geq 0$ $k(1-k)(1+k) \geq 0$ $k \leq -1$ or $0 \leq k < 1$	1 1 1
4.	$x^2 = t$ $y^2 = 4(1-t)$ $y^2 = 4 - 4x^2$ Since $t \geq 0$ and $1 \geq 1 - t \geq 0$, $0 \leq y \leq 2$ or $0 \leq x \leq 1$	1 1 1 for $x \geq 0$
5 i	 	1 - $x \to -\infty, y \to 4$ asymptote 1 - two intercepts

5ii		1 – both Asymptotes Note: $x \to \infty, y \to 0$ $x \to -\infty, y \to \frac{1}{2}$ 1 – shape and intercept
5iii		1– all 3 intercepts 1- Shape of both graphs 1- Horizontal asymptotes

Section B

	Suggested Solutions	Marks Awarded & comments
1.	C Only two real roots, which means one x-intercept, and $x \to \pm\infty, y \to \infty$	1
2.	$P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$ $P(2) = 2^n(2^m - 1) + 2^m(1 - 2^n) + (2^n - 2^m)$ $= 2^n 2^m - 2^n + 2^m - 2^m 2^n + 2^n - 2^m$ $= 0$ Similar for $P(1) = 0$, thus $(x-1)(x-2)$ is a factor of $P(x) =$ $x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$	1 1
3i.	$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} = -\frac{3}{2}$	1
3ii.	$(\alpha - 1)(\beta - 1)(\gamma - 1)$ $= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \alpha\gamma) + (\alpha + \beta + \gamma) - 1$ $= \frac{1}{2} - \left(-\frac{3}{2}\right) + 2 - 1$ $= 3$	1 1
3iii.	Since $\alpha + \beta + \gamma = 2$, then $\alpha + \beta - \gamma = 2 - 2\gamma$ $\alpha - \beta + \gamma = 2 - 2\beta$ $-\alpha + \beta + \gamma = 2 - 2\alpha$ $(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma)$ $= (2 - 2\gamma)(2 - 2\beta)(2 - 2\alpha)$ $= -8(\gamma - 1)(\beta - 1)(\alpha - 1)$ $= -8 \times 3$ $= -24$	 1 1

4.	$P'(x) = 3x^2 + 2ax$ $P'(4) = 3 \times 4^2 + 2 \times 4a = 0$ $48 + 8a = 0$ $a = -6$ $P(4) = 4^3 + (-6) \times 4^2 + b = 0$ $b = 32$ $(x-4)^2 = x^2 - 8x + 16$ $$\begin{array}{r} x + 2 \\ x^2 - 8x + 16 \overline{)x^3 - 6x^2 \quad + 32} \\ \underline{x^3 - 8x^2 + 16x} \\ 2x^2 - 16x + 32 \\ \underline{2x^2 - 16x + 32} \\ 0 \end{array}$$ Thus, $r = -2$	1 1 1
5.	$P(x) = x(x-2)(x+3)Q(x) + R(x)$ Let $R(x) = ax^2 + bx + c$ $R(0) = 1, c = 1$ $R(2) = 3$, $4a + 2b + 1 = 3$ $4a + 2b = 2$ $2a + b = 1$ ------------- (1) $R(-3) = 13$, $9a - 3b + 1 = 13$ $9a - 3b = 12$ $3a - b = 4$ ------------- (2) (1) + (2):	1 1

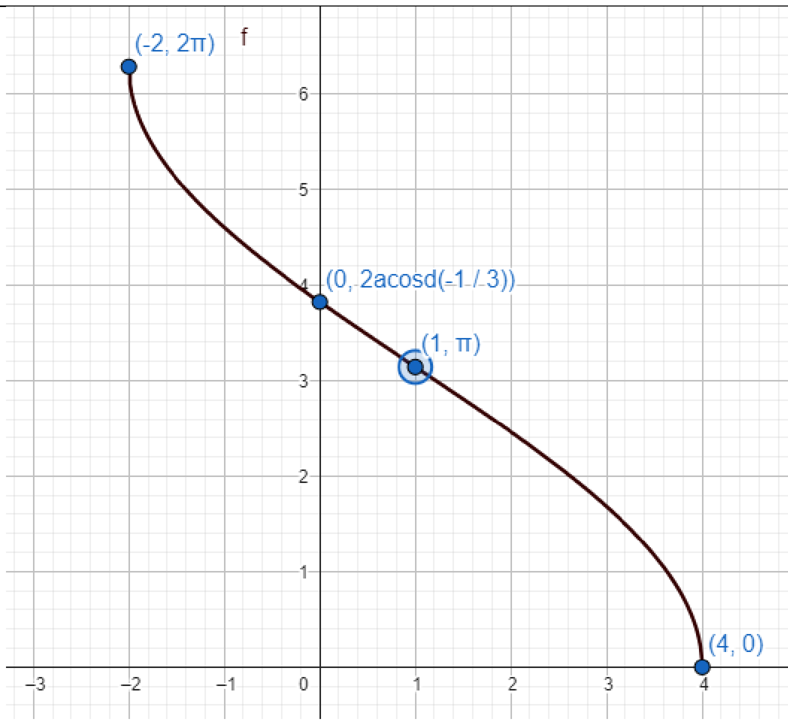
	$5a = 5$ $a = 1$ $b = -1$ Thus $R(x) = x^2 - x + 1$ $g(1) = R(1) = 1^2 - 1 + 1 = 1$	1

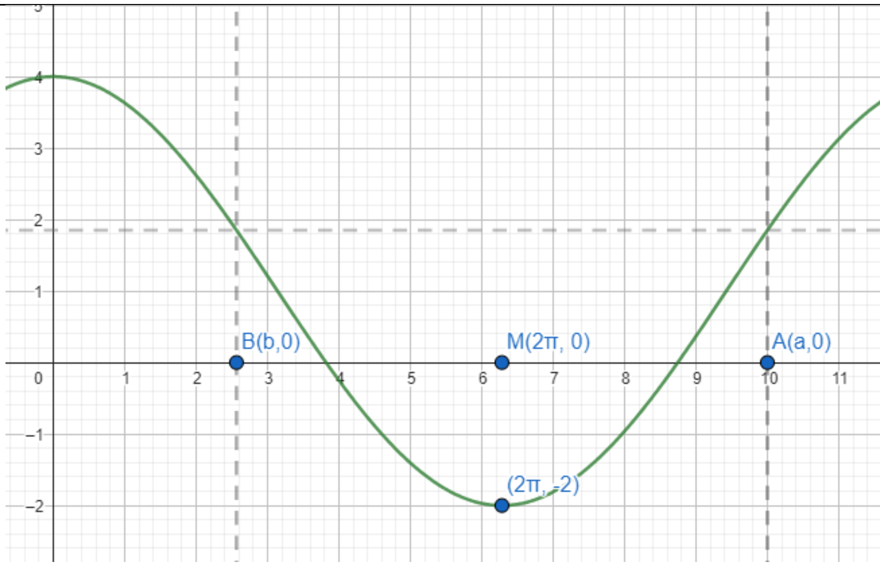
Section C

	Suggested Solutions	Marks Awarded & comments
1.	C 4! Arrangements for females, 4! Arrangements for males; number of arrangements for two groups: $2 \times 4! \times 4!$	1
2.	C The digit one can have a value of $(1000 + 100 + 10 + 1) = 1111$, with the other numbers arranged 3! Ways. The value of all numbers is $3! \times 1111 \times (1 + 2 + 3 + 4) = 66660$	1
3.	ERATOSTHENES has 12 letters, with 3 E's, 2 T's, 2 S's $\frac{12!}{3!2!2!} = 19{,}958{,}400$	2
4.	A group of five numbers out of 6, must contain either three 6's or two 6's. When three 6's are used, there's two ways to have 2's, either one 2 and one 7 or two 2s, 66622 or 66627: $\frac{5!}{3!2!} + \frac{5!}{3!} = 30$ 66227: $\frac{5!}{2!2!} = 30$ Total different numbers = 30+30=60	1 1

Section D

	Suggested Solutions	Marks Awarded & comments
1.	$\tan[\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)]$ $= \tan\left(\pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$ $= \tan\left(\pi - \frac{\pi}{4} + \frac{\pi}{4}\right)$ $= \tan(\pi)$ $= 0$	1 - correct use of identities or of exact value results 1

2.	$\tan(X-Y) = \dfrac{\tan X - \tan Y}{1+\tan X \tan Y}$ $= \dfrac{\dfrac{7m+6}{7-6m} - \dfrac{4m+3}{4-3m}}{1+\dfrac{7m+6}{7-6m}\times\dfrac{4m+3}{4-3m}}$ $= \dfrac{(7m+6)(4-3m)-(4m+3)(7-6m)}{(7-6m)(4-3m)+(7m+6)(4m+3)}$ $= \dfrac{3m^2+3}{46m^2+46}$ $= \frac{3}{46}$	1 1 1
3i.	$0 \le x \le 2\pi$	1
3ii.	$x = 1 + 3\cos\frac{y}{2}$ $\cos\frac{y}{2} = \frac{1}{3}(x-1)$ $y = 2\cos^{-1}(\frac{1}{3}(x-1))$	1
3iii.		1 - Two end points 1 – shape and 1 other point (y-intercept or $(1,\pi)$) OR 1- Correct domain & Range 1 – some correct points

3iv	 Let a, b have the same y-value, then The $AM = BM = a - 2\pi$ Thus $b = 2\pi - (a - 2\pi)$ by symmetry about P $f^{-1}(f(a)) = f^{-1}(f(b)) = b$ since b is in the domain of $f^{-1}(x)$ and $f(a) = f(b)$ $= 4\pi - a$	1 1
4i.	LHS $= 2 \times \frac{2t}{1+t^2} - 3 \times \frac{1-t^2}{1+t^2}$ $= \frac{4t-(3-3t^2)}{1+t^2}$ $= \frac{3t^2+4t-3}{1+t^2}$ = RHS	1 1
4ii.	$2\sin x - 3\cos x = 2$ From part I, $$\frac{3t^2+4t-3}{1+t^2} = 2$$ $$3t^2 + 4t - 3 = 2 + 2t^2$$ $$t^2 + 4t - 5 = 0$$ $$(t+5)(t-1) = 0$$	1

	Thus, $\tan\frac{x}{2} = 1$ $\frac{x}{2} = \tan^{-1}(1)$ $\frac{x}{2} = \frac{\pi}{4}$ $x = \frac{\pi}{2} = 90°$ or $\tan\frac{x}{2} = -5$ related angle $\frac{x}{2} = \tan^{-1}5$ $\frac{x}{2} \approx 101.3099 \ldots$ $x \approx 202.62$ Thus, $x = 90°, 203°$	2
5i	Let $x = \cos\theta$, then $8(\cos^3\theta) - 6(\cos\theta) - 1 = 0$ $$2(\cos 3\theta) = 1$$ $$\cos 3\theta = \frac{1}{2}$$ $$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$ Only taken the first three solutions as it's a cubic function. Thus, $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$	1 1

5ii	As x is the roots of the cubic function, then $$\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}+\cos\frac{7\pi}{9}=0$$ $$\cos\frac{\pi}{9}=-\cos\frac{5\pi}{9}-\cos\frac{7\pi}{9}$$ $$\cos\frac{\pi}{9}=\cos\left(\pi-\frac{5\pi}{9}\right)+\cos\left(\pi-\frac{7\pi}{9}\right)$$ $$\cos\frac{\pi}{9}=\cos\frac{4\pi}{9}+\cos\frac{2\pi}{9}$$ $\cos\frac{\pi}{9}=\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}$ as required	1 1

Section E

	Suggested Solutions	Marks Awarded & comments
1.	General term of $\left(2x-\frac{1}{x^2}\right)^8$ is $\binom{8}{i}(2)^{8-i}(x)^{8-i}(-1)^i(x)^{-2i}$, $$T_k=\binom{8}{i}(-1)^i(2)^{8-i}(x)^{8-3i}$$ For x^5, $$5=8-3i$$ $$i=1$$ Thus, the term in x^5 is $\binom{8}{1}(-1)^1(2)^{8-1}x^5=-1024x^5$	1 1
2i.	LHS = $(1+x)^m\left(1-\frac{1}{x}\right)^m$ $=\left((1+x)\left(1-\frac{1}{x}\right)\right)^m$ $=\left(1-\frac{1}{x}+x-1\right)^m$ $=\left(x-\frac{1}{x}\right)^m$ = RHS	1

2ii.	Let $m = 2002$, LHS = $(1+x)^{2002}\left(1-\frac{1}{x}\right)^{2002}$ $= \left[\binom{2002}{0} + \binom{2002}{1}x + \cdots + \binom{2002}{r}x^r + \cdots + \binom{2002}{2002}x^{2002}\right]\left[\binom{2002}{0} - \binom{2002}{1}x^{-1} + \cdots + (-1)^r\binom{2002}{r}x^{-r} + \cdots + \binom{2002}{2002}x^{-2002}\right]$ Then the coefficient of x^0 on LHS is $\left[\binom{2002}{0}\binom{2002}{0} + \binom{2002}{1}(-\binom{2002}{1}) + \cdots + \binom{2002}{r}((-1)^r\binom{2002}{r}) + \cdots + \binom{2002}{2002}\binom{2002}{2002}\right]$ $=\left[\binom{2002}{0}^2 - \binom{2002}{1}^2x + \cdots + (-1)^r\binom{2002}{r}^2 + \cdots + \binom{2002}{2002}^2\right]$ General term of RHS : $T_k = \binom{2002}{k}x^{2002-k}(-x)^{-k}$ $= (-1)^k\binom{2002}{k}x^{2002-2k}$ Then the coefficient of x^0 on RHS is when $2002 - 2k = 0, k = 1001$, which is $-1\binom{2002}{1001}$ Equating coefficients of LHS and RHS from (i), then LHS=RHS in part (ii)	1 1 1
3iii	LHS $= \left[\binom{2002}{0}^2 - \binom{2002}{1}^2x + \cdots + (-1)^r\binom{2002}{r}^2 + \cdots + \binom{2002}{2002}^2\right]$ $= 2\left[\binom{2002}{0}^2 - \binom{2002}{1}^2x + \cdots + \binom{2002}{1000}^2\right] - \binom{2002}{1001}^2$ as $\binom{n}{r} = \binom{n}{n-r}$ $= 2\left[\binom{2002}{0}^2 - \binom{2002}{1}^2x + \cdots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2\right] + \binom{2002}{1001}^2$ Let $\sum_{k=0}^{1001}(-1)^k\binom{2002}{k}^2 = \left[\binom{2002}{0}^2 - \binom{2002}{1}^2x + \cdots + \binom{2002}{1000}^2 - \binom{2002}{1001}^2\right] = S$, From (ii) LHS = $-1\binom{2002}{1001}$, thus $-1\binom{2002}{1001} = 2S + \binom{2002}{1001}^2$ $2S = -1\binom{2002}{1001} - \binom{2002}{1001}^2$ $S = -\frac{1}{2}\binom{2002}{1001}[1 + \binom{2002}{1001}]$ Thus $\sum_{k=0}^{1001}(-1)^k\binom{2002}{k}^2 = -\frac{1}{2}\binom{2002}{1001}[1 + \binom{2002}{1001}]$ as required	1 1 1