NORTH
SYDNEY
GIRLS HIGH
SCHOOL

2019	Preliminary Assessment Task 3

Preliminary Mathematics Extension

General Instructions

- Reading Time – 5 minutes
- Working Time – 90 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 6 – 10, show relevant mathematical reasoning and/or calculations

Total marks – 57

Section I – 5 marks (pages 2 – 4)

- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II – 52 marks (pages 5 – 9)

- Attempt Questions 6 – 10
- Allow about 82 minutes for this section

NAME:________________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

QUESTION	MARK
1-5	/5
6	/11
7	/10
8	/10
9	/10
10	/11
Total	/57

Section I

5 marks
Attempt Questions 1-5
Allow about 8 minutes for this section

Use the multiple choice answer sheet for Questions 1-5.

---

1 What is the remainder when $8x^3 - 2x + 3$ is divided by $2x + 3$?

(A) -216

(B) 213

(C) -21

(D) 27

2 Given that α, β and γ are the roots of $3x^3 - 2x^2 + x - 1 = 0$, what is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

(A) 2

(B) -1

(C) 1

(D) -2

3 Given that α and β are acute, what is the value of $\sin\left(\alpha+\beta\right)$ if $\sin\alpha = \frac{8}{17}$ and $\sin\beta = \frac{4}{5}$?

(A) $\frac{108}{85}$

(B) $\frac{84}{85}$

(C) $\frac{36}{85}$

(D) $\frac{28}{85}$

4 If $f\left(x\right) = \frac{1}{x+1}+2$, what is the equation of the inverse function $y = f^{-1}\left(x\right)$?

(A) $f^{-1}\left(x\right) = \frac{1}{x-1}-2$

(B) $f^{-1}\left(x\right) = \frac{1}{x-2}-1$

(C) $f^{-1}\left(x\right) = \frac{1}{x+1}-2$

(D) $f^{-1}\left(x\right) = \frac{1}{x-2}+1$

5. f and g are increasing, continuous functions.

f maps the interval $[0, 5]$ to the interval $[3, 7]$.
g maps the interval $[1, 4]$ to the interval $[2, 6]$.

Further, $f(2) = 4$, $g(3) = 4$ and $g(3.5) = 5$.

What is the domain of $f \circ g$?

(A) $[0, 2]$

(B) $[2, 5]$

(C) $[1, 3.5]$

(D) $[3, 3.5]$

BLANK PAGE

Section II

52 marks
Attempt Questions 6-10
Allow about 82 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 6-10, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks) Use a SEPARATE writing booklet

(a) Solve $\left|2x-3\right| \leq 6$. **2**

(b) Solve $x+2 \leq \dfrac{4}{x-1}$. **3**

(c) Find the exact value of $\sin\dfrac{\pi}{8}\cos\dfrac{\pi}{8}$. Show working. **1**

(d) Solve $\sin x+\cos x=-1$ for $0 \leq x \leq 2\pi$, by using the substitution $t=\tan\dfrac{x}{2}$ or otherwise. **3**

(e) The point $P(10,\,-3)$ lies on the curve defined by the parametric equations: **2**

$$x = 3t^2 - 2$$
$$y = 2t + 1$$

What are the coordinates of the point Q whose parameter value is three less than the parameter value for P?

Question 7 (10 marks) Use a SEPARATE writing booklet

(a) (i) Factorise $P(x) = 4x^3 + 8x^2 - 3x - 9$. **2**

(ii) Sketch $y = P(x)$ and hence solve $4x^3 + 8x^2 - 3x - 9 \geq 0$. **2**

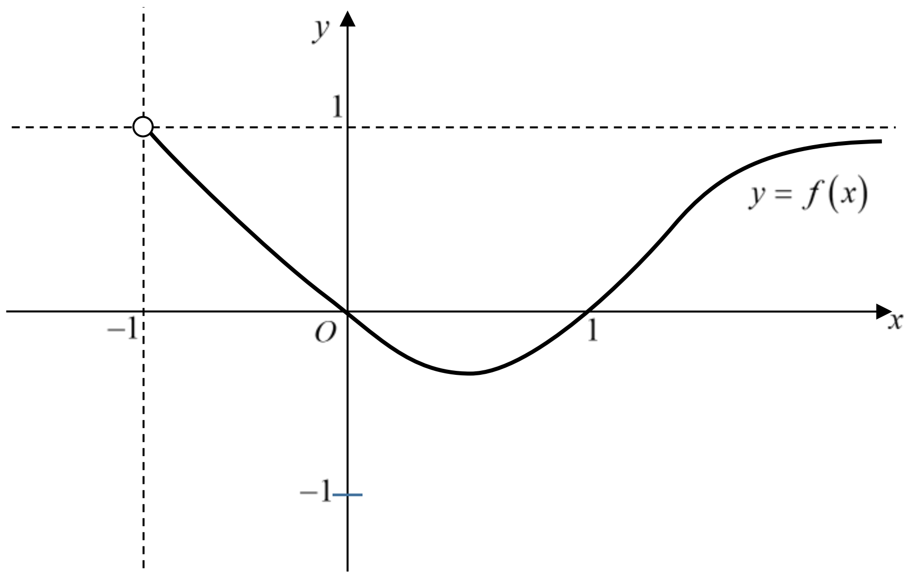
(b) When the polynomial $P(x)$ is divided by $1 - x^2$, the remainder is $4 - x$. **2**
What is the remainder when $P(x)$ is divided by $1 + x$?

(c) (i) Express $\sin 3x \cos 2x$ as the sum of two sine functions. **1**

(ii) Hence solve $\sin 5x + \sin x = 0$ in the interval $[0, \pi]$. **3**

Question 8 (10 marks) Use a SEPARATE writing booklet

(a) The diagram shows a graph of the function $y = f(x)$ defined on the interval $(-1, \infty)$.



<u>On the response sheet provided</u>, draw graphs for each of the following relations, showing all important features:

(i) $y = f(|x|)$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(b) (i) Show that $\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2} = -\dfrac{h(2x+h)}{x^2(x+h)^2}$. **2**

(ii) Hence differentiate $f(x) = \dfrac{1}{x^2}$ by first principles. **2**

Question 9 (10 marks) Use a SEPARATE writing booklet

(a) The roots of the equation $x^2 - 3x + 4 = 0$ are α and β.

(i) Find the value of $\alpha^2 + \beta^2$. **2**

(ii) Hence find the quadratic equation which has roots α^2 and β^2. **1**

(b) Consider the function $f(x) = 2^x - 2^{-x}$.

(i) Sketch $y = 2^x$ and $y = -2^{-x}$ on the same set of axes, and use the method of addition of ordinates to sketch the graph of $y = f(x)$. **2**

(ii) Explain, with reference to the graphs of $y = 2^x$ and $y = -2^{-x}$, why the function $f(x)$ has an inverse function. **1**

(iii) Replacing $f(x)$ by y, show that this equation can be written as: **1**

$$2^{2x} - y \cdot 2^x - 1 = 0.$$

(iv) Hence find the equation of the inverse function $y = f^{-1}(x)$. **3**

Question 10 (11 marks) Use a SEPARATE writing booklet

(a) (i) Find the values of k such that $x^3 + 3x^2 - 24x + k = 0$ has a double root. **2**

(ii) Hence solve this equation given that the product of all roots is positive. **2**

(b) Sydney's average daily maximum temperature reaches a high of $27.8\,°\mathrm{C}$ on January 21, and a low of $17.6\,°\mathrm{C}$ six months later. **4**

Assume that this average maximum temperature can be modelled by an equation of the form:

$$T = A + B\cos nt$$

where A, B and n are constants, and t represents time.

By defining a scale for t, and finding values of A, B and n consistent with this choice, determine on which date after January 21 the average temperature first falls below $25\,°\mathrm{C}$.

(c) The quadratic equation $ax^2 + bx + c = 0$ has roots $\tan\alpha$ and $\tan\beta$, where $\alpha < \beta$. **3**

Find a fully simplified expression involving a, b and c for $\tan(\alpha - \beta)$.

End of Exam

BLANK PAGE

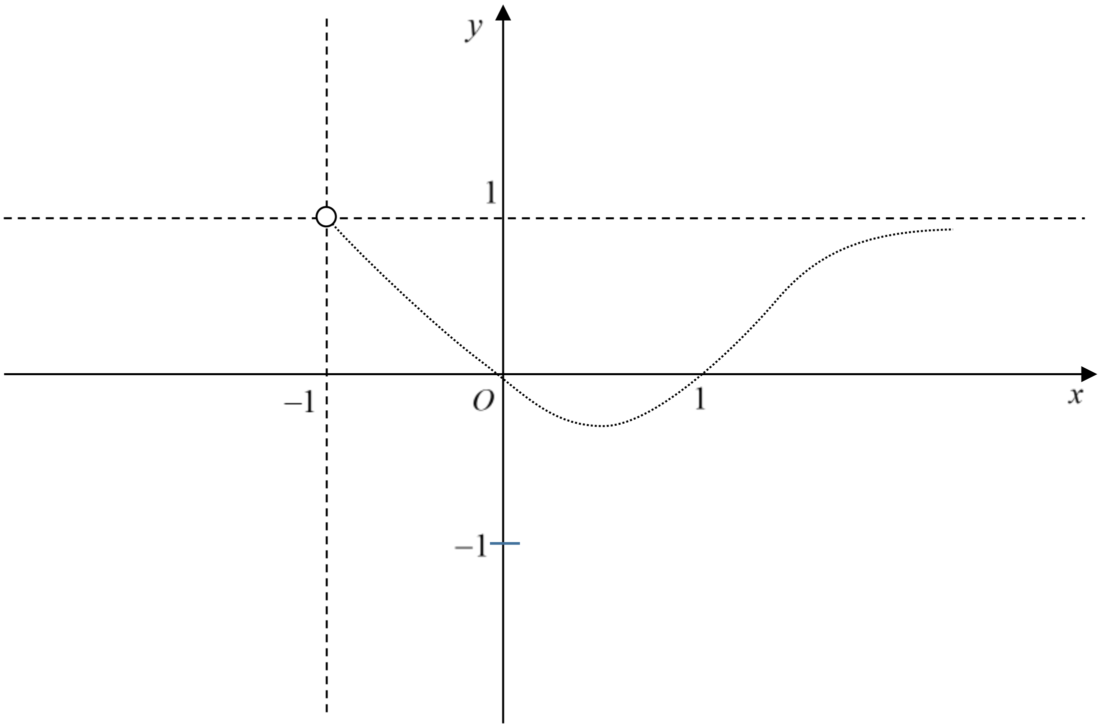
BLANK PAGE

Response sheet for Question 8 (a)

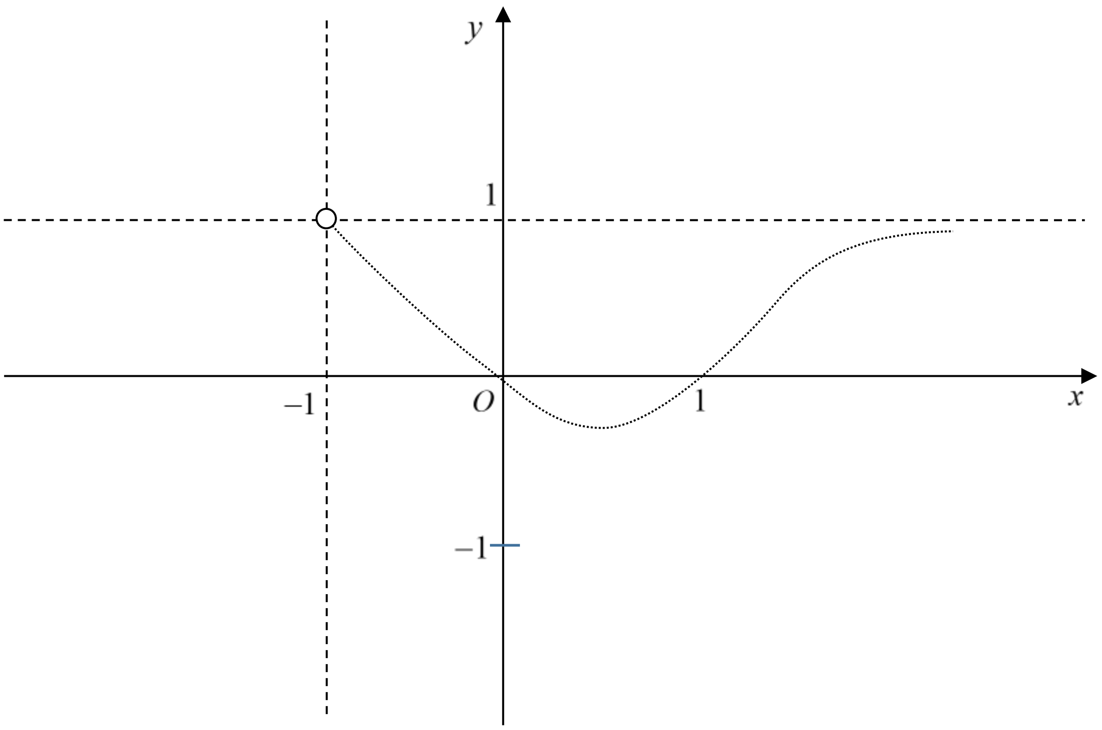
Student Number: ________________

If your response includes part of the original curve, make this clear by darkening it.

(i) $y = f\left(|x|\right)$ **2**

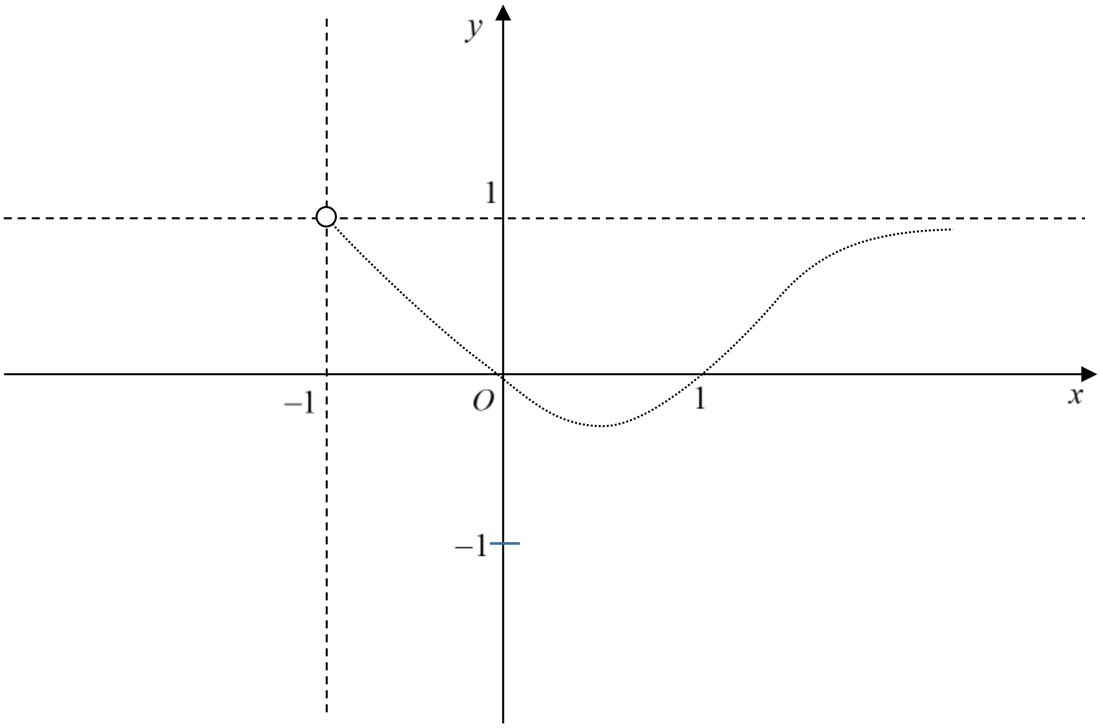


(ii) $y = \dfrac{1}{f(x)}$ **2**



PTO

(iii) $y^2 = f(x)$ **2**



Year 11 Ext 1 Yearly Solutions

I - Multiple Choice

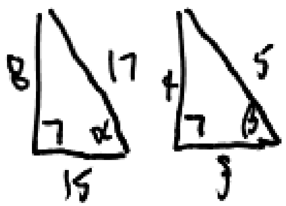
Answer Summary:

1. C 2. C 3. B 4. B 5. C

Worked Solutions

1. $\text{Rem} = P(-3/2)$
$$= 8\left(-\tfrac{3}{2}\right)^3 - 2\left(-\tfrac{3}{2}\right) + 3$$
$$= -21 \qquad [C]$$

2. $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$
$$= \frac{1/3}{1/3}$$
$$= 1 \qquad [C]$$

3. 

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \frac{8}{17}\cdot\frac{3}{5} + \frac{15}{17}\cdot\frac{4}{5}$$
$$= \frac{84}{85} \qquad [B]$$

4. Inverse: $$x = \frac{1}{y+1} + 2$$
$$x - 2 = \frac{1}{y+1}$$
$$y + 1 = \frac{1}{x-2}$$
$$y = \frac{1}{x-2} - 1 \qquad [B]$$

5. Range of g: $[2, 6]$
Domain of f: $[0, 5]$

Overlap: $[2, 5]$

Domain of g for this range: $[1, 3.5]$ [C]

Question 6

(a) $|2x-3| \leq 6$

$-6 \leq 2x-3 \leq 6$

$-3 \leq 2x \leq 9$

$-\frac{3}{2} \leq x \leq \frac{9}{2}$ [2]

(b) $x+2 \leq \frac{4}{x-1}$

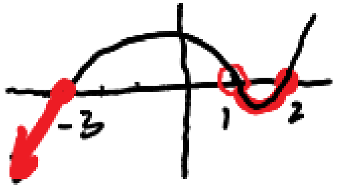
$(x+2)(x-1)^2 \leq 4(x-1), \ x \neq 1$

$(x+2)(x-1)^2 - 4(x-1) \leq 0$

$(x-1)[(x+2)(x-1)-4] \leq 0$

$(x-1)(x^2+x-6) \leq 0$

$(x-1)(x+3)(x-2) \leq 0, \ x \neq 1$



$x \leq -3, \ 1 < x \leq 2$

OR

$(-\infty, -3] \cup (1, 2]$ [3]

(c) $\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

$= \frac{1}{2}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)$

$= \frac{1}{2}\sin\frac{\pi}{4}$

$= \frac{1}{2\sqrt{2}}$ [1]

(d) Hence

let $t = \tan\frac{x}{2}$

$\sin x + \cos x = -1$

$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$

$2t + 1 - t^2 = -1 - t^2$

$2t = -2$

$t = -1$

$\tan\frac{x}{2} = -1$

$\frac{x}{2} = \frac{3\pi}{4}$

$x = \frac{3\pi}{2}$

Test $x = \pi$:

$\sin\pi + \cos\pi = 0 - 1$

$= -1$ ✓

$\therefore x = \pi, \frac{3\pi}{2}$

Otherwise:

Let $\sin x + \cos x = R\sin(x+\alpha)$, $R>0$, $0\leq\alpha\leq 2\pi$

$= R(\sin x\cos\alpha + \cos x\sin\alpha)$

$= (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

Equating coefficients of $\sin x$ & $\cos x$:

$R\cos\alpha = 1$ - ① $R\sin\alpha = 1$ - ②

①^2 + ②^2: $R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2+1^2$

$R^2(\cos^2\alpha+\sin^2\alpha) = 2$

$R^2 = 2$

$R = \sqrt{2}$

② ÷ ①: $\tan\alpha = 1$

$\alpha = \pi/4$

$\therefore \sqrt{2}\sin(x+\frac{\pi}{4}) = -1$

$\sin(x+\pi/4) = -\frac{1}{\sqrt{2}}$

$x+\pi/4 = \pi+\frac{\pi}{4},\ 2\pi-\frac{\pi}{4}$

$= \frac{5\pi}{4},\ \frac{7\pi}{4}$

$x = \pi,\ \frac{3\pi}{2}$ [3]

(e) $10 = 3t^2-2$

$12 = 3t^2$

$t^2 = 4$

$t = \pm 2$

$-3 = 2t+1$

$-4 = 2t$

$t = -2$

New value of parameter: $t = -5$

$x = 73$

$y = -9$

$\therefore (73, -9)$ [2]

Question 7

(a) (i) $P(1) = P\left(-\frac{3}{2}\right) = 0$

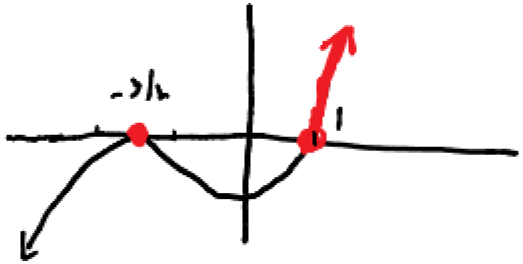
$\therefore (x-1)$ and $(2x+3)$ are factors

Let 3rd root be α.

Sum of roots: $1 - \frac{3}{2} + \alpha = -2$

$\alpha = -3/2$

$\therefore \underline{P(x) = (x-1)(2x+3)^2}$ [2]



$\therefore x \geq 1,\ x = -\frac{3}{2}$ [2]

Done well. However, many students did not answer the question fully including $x = -\frac{3}{2}$.

(b) Let $P(x) = (1-x^2)Q(x) + (4-x)$

remainder $= P(-1)$

$= 0 \cdot Q(-1) + 4 + 1$

$= 5$ [2]

Done well.

(c)(i) Using formula on reference sheet:

$\sin 3x \cos 2x = \frac{1}{2}[\sin(3x+2x) + \sin(3x-2x)]$

$= \frac{1}{2}(\sin 5x + \sin x)$ [1]

(11) $\sin 5x + \sin x = 0$

$2\sin 3x \cos 2x = 0$

$\sin 3x = 0, \quad 3x \in [0, 3\pi]$

$3x = 0, \pi, 2\pi, 3\pi$

$x = 0, \frac{\pi}{3}, \frac{3\pi}{3}, \pi$

$\cos 2x = 0, \quad 2x \in [0, 2\pi]$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}$

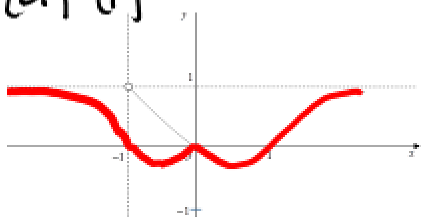
$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$\therefore x = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$ [3]

Most students did not solve $\sin 3x = 0$ and $\cos 2x = 0$ but they tried to write $\sin 3x$ in terms of $\sin x$ and $\cos x$ then solved.

Question 8

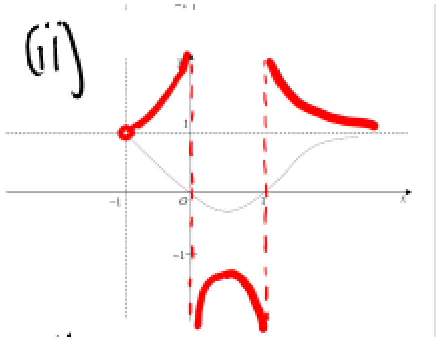
(a) (i)



[2]

A large number of students sketched $y = |f(x)|$.

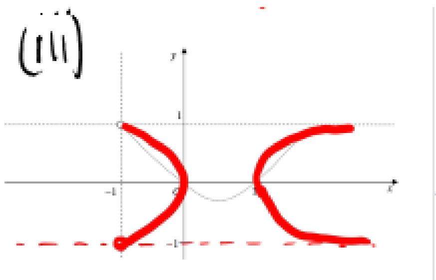
(ii)



[2]

Many students did not get the correct horizontal asymptotes. Argue it to yourself: if $f(x)$ is slightly less than 1, then $\frac{1}{f(x)}$ is slightly greater than 1. Many couldn't get the right sign – reciprocating doesn't change the sign.

(iii)



[2]

Many students did not lose the negative portion of the graph – a negative cannot be square rooted. Further, as the original graph did not exist for $x \le -1$, then neither does any derivative graph exist in that domain. ('Derivative' not used here in the calculus sense.) The new curve does not have any sharp points.

(b) (i) $\frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{x^2-(x+h)^2}{x^2(x+h)^2}$

$$= \frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}$$

$$= \frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}$$

$$= \frac{-2xh-h^2}{x^2(x+h)^2}$$

$$= -\frac{h(2x+h)}{x^2(x+h)^2}$$

This was a SHOW question. Show ALL steps.

(ii) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h(2x+h)}{x^2(x+h)^2} \quad \text{(part i)}$$

$$= \lim_{h\to 0} -\frac{2x+h}{x^2(x+h)^2}$$

$$= \frac{-2x+0}{x^2(x+0)^2} = -\frac{2}{x^3} \quad [2]$$

You had to show the substitution of the function before using part (i) to simplify. The rule begins " $f'(x)=$ ". You had to show what it was you were calculating. Many students did not include the limit in the rule, so did not know how to finish off the question. Others snuck it in as an afterthought. A large number wasted time expanding instead of substituting immediately. Quite a few thought that $\frac{h}{h} = h^2$.

Question 9

(a) (i) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= 3^2 - 2(4)$
$= 1$ [2]

Done well

(ii) Roots of new equation: α^2, β^2

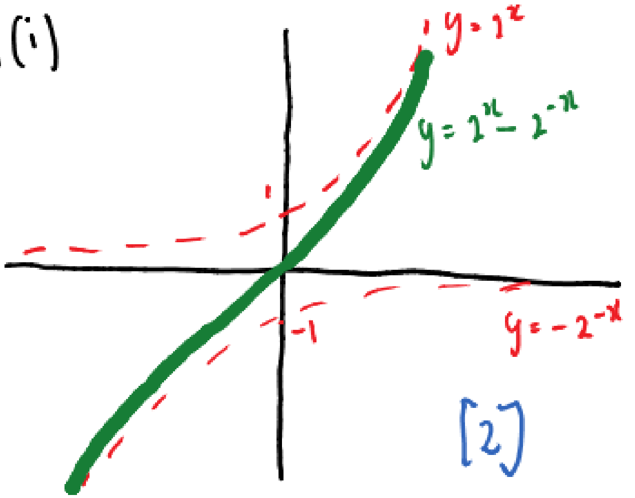
Sum of roots: $\alpha^2+\beta^2 = 1$ (part i)

Product of roots $= \alpha^2\beta^2$
$= (\alpha\beta)^2$
$= 4^2$
$= 16$

$\therefore$ new equation: $x^2 - x + 16 = 0$ [1]

Students who had difficulty here need to learn the form of a quadratic equation given the roots: $x^2 - (\text{sum of roots})\,x + \text{product of roots} = 0$

(b) (i)



Mostly done well

(ii) $y = 2^x$ and $y = -2^{-x}$ are increasing functions, so the sum is increasing

$\therefore$ one-to-one [1]

Many students had difficulty with this. Some students incorrectly wrote that since both $y = 2^x$ and $y = -2^{-x}$ are one-to-one functions then so is their sum. Some students also incorrectly thought if it passed the vertical line test then it had an inverse.
Students who referred to the graph of $y = f(x)$ had most success. Better solutions stated that since both functions were monotonically increasing then so was their sum, hence the function is one-to-one and has an inverse. Many students stated that $f(x)$ passed the horizontal line test and were awarded the mark.

(iii) $y = 2^x - 2^{-x}$

$(\times 2^x)$: $y \cdot 2^x = 2^{2x} - 1$

$2^{2x} - y \cdot 2^x - 1 = 0$

Many students struggled with the algebra here. A few misread the question and replaced y with $f(x)$.

(iv) Inverse: $2^{2y} - x \cdot 2^y - 1 = 0$

$$2^y = \frac{x \pm \sqrt{x^2 + 4}}{2} \quad \text{(quadratic formula)}$$

$$y = \log_2 \left[\frac{x \pm \sqrt{x^2 + 4}}{2} \right]$$

But $\sqrt{x^2 + 4} > \sqrt{x^2} = x$

$\therefore x - \sqrt{x^2 + 4} < 0$ (can't find its log)

$\therefore y = \log_2 \left[\frac{x + \sqrt{x^2 + 4}}{2} \right]$ is the equation of the inverse function

Many students made a good start with this part and could form a quadratic equation. Some students were able to continue and apply the quadratic formula to solve. The solution had to be changed from exponential to log form. The last mark awarded here required that one solution be discarded, and this must be justified algebraically (see solutions). It was disappointing to see some students ignore the word "hence" and try to find the inverse from an earlier equation. They had no success.

Question 10

(a) (i) **Method I**

Let $P(x) = x^3 + 3x^2 - 24x + k$

As $P(x) = 0$ has a double root,

$P'(x) = 0$ has the same root.

$3x^2 + 6x - 24 = 0$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4, 2$

If the double root is -4:

$P(-4) = 0$

$(-4)^3 + 3(-4)^2 - 24(-4) + k = 0$

$\underline{k = -80}$

If the double root is 2:

$2^3 + 3(2)^2 - 24(2) + k = 0$

$\underline{k = 28}$

Method II

Let the roots be α, α, β.

Sum: $2\alpha + \beta = -3 \Rightarrow \beta = -3 - 2\alpha$

Sum in pairs: $\alpha^2 + 2\alpha\beta = -24$

$\alpha^2 + 2\alpha(-3 - 2\alpha) = -24$

$3\alpha^2 + 6\alpha - 24 = 0$

$\alpha^2 + 2\alpha - 8 = 0$

$(\alpha + 4)(\alpha - 2) = 0$

$\alpha = -4, 2$

This question was done quite well. Most students understood that they needed to solve the second derivative which gave them two solutions which then gave two possible values of k. Students needed to solve for both values of k to get both marks.

(ii) If the product of the roots is positive then $k < 0$, ie. $k = -80$ and the double root is -4.
Let the 3rd root be β.
Product of roots: $-4 \cdot -2 \cdot \beta = 80$

$\beta = 10$

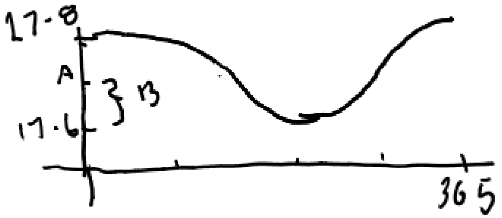
ie. the solution to the equation is $x = -4, 10$

Overall done well. However many students confused the formula for the product of roots, and selected the wrong value of k to solve the equation. 1 mark was given for the right value of k with justification and 1 mark was for solving the equation.

(b) Let $t = 0$ represent Jan 21, where t is measured in days.
Assume 365 days in a year.

Period $= 365$ days $= \frac{2\pi}{n}$

$\therefore n = \frac{2\pi}{365}$



$A = \frac{27.8 + 17.6}{2}$

$= 22.7$

$B = \frac{27.8 - 17.6}{2}$

$= 5.1$

$T = 22.7 + 5.1 \cos \frac{2\pi}{365} t$

$(T = 25)$: $25 = 22.7 + 5.1 \cos \frac{2\pi}{365} t$

$5.1 \cos \frac{2\pi}{365} t = 2.3$

$\cos \frac{2\pi}{365} t = \frac{23}{51}$

$\frac{2\pi}{365} t = \cos^{-1} \frac{23}{51}$ (1st occurrence)

$t = \frac{365}{2\pi} \cos^{-1} \frac{23}{51}$

$= 64$

ie. March 26

This question was done poorly. The majority of students could not attempt this question at all and struggled to develop the equation of the model.

Many students did not draw a diagram to represent the question given. Students are reminded again that drawing a diagram, particularly in a modelling question such as this one will help them to visualise and enable them to find the amplitude and period of the graph more easily.

The majority of students defined t to be the number of months after January 21st. As the question asked for days, use t as the number of days after January 21st. Students who used months and then subsequently made the assumption that there are 30 or 31 days in a month lost half mark for doing this. You **cannot** assume that each month has 30 or 31 days as this is approximating your final answer.

Many students are still making silly errors while solving trig equations and are not being consistent with their use of degrees or radians. If the period of the equation was developed using radians, then the solution to the equation should also be calculated in radians and not degrees.

(c) $(\tan\alpha - \tan\beta)^2 = (\tan\alpha + \tan\beta)^2 - 4\tan\alpha\tan\beta$

$$= \left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)$$

$$= \frac{b^2 - 4ac}{a^2}$$

$$\tan\alpha - \tan\beta = \pm\frac{\sqrt{b^2-4ac}}{|a|}$$

but $0 < \alpha < \beta < \frac{\pi}{2}$

$\therefore\ 0 < \tan\alpha < \tan\beta$ (as $\tan x$ is increasing fn)

$\therefore\ \tan\alpha - \tan\beta < 0$

$\therefore\ \tan\alpha - \tan\beta = -\dfrac{\sqrt{b^2-4ac}}{|a|}$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{-\sqrt{b^2-4ac}/|a|}{1 + c/a}$$

$$= \frac{-\sqrt{b^2-4ac}}{|a| + c\cdot\frac{|a|}{a}}$$

$$= \begin{cases} \dfrac{-\sqrt{b^2-4ac}}{a+c} & \text{if } a > 0 \\[2ex] \dfrac{\sqrt{b^2-4ac}}{a+c} & \text{if } a < 0 \end{cases}$$

This question was also done poorly. The majority of students could access the first ½ mark which was simply to write down the value of the sum and product of the roots. To gain the subsequent marks, students should have been able to find **both** values of $\tan(\alpha-\beta)$ and then justify why they selected the negative value of $\tan(\alpha-\beta)$ using the information that $0 < \alpha < \beta < \frac{\pi}{2}$.

The main area where students struggled was to find the value of $\tan\alpha - \tan\beta$. Some students tried to use the quadratic formula but then failed to adequately justify why they chose a particular solution (need to take into consideration the signs of a, b, c).