PENRITH HIGH SCHOOL

MATHEMATICS EXTENSION 1
2013

Preliminary Yearly

Assessor: Mr Ferguson

General Instructions:

- Reading time – 5 minutes
- Working time – **2 hours**
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 6 – 9
- Work on this question paper will not be marked.

Total marks – 45

SECTION 1 – Pages 2 -3
5 marks
- Attempt Questions 1 – 5
- Allow about 8 minutes for this section.

SECTION 2 – Pages 4 - 7
40 marks
- Attempt Questions 6 – 9
- Allow about 1 hours 52 minutes for this section.

Section1

Question	Mark
Total	/5

Section 2

Question	Mark
6a	/3
6bc	/7
7a	/2
7bc	/8
8	/10
9	/10
Total	/40

Total	/45
%	

This paper MUST NOT be removed from the examination room

Student Name: ………………………………

Section I

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

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1 What is the solution to the inequality $\dfrac{x^2-4}{2x} > 0$?

(A) $-2 < x < 0$ or $x > 2$

(B) $-2 < x < 0$ or $x > 4$

(C) $-4 < x < 0$ or $x > 2$

(D) $-4 < x < 0$ or $x > 4$

2 Which of the following is equivalent to the expression $\sin 3\theta$?

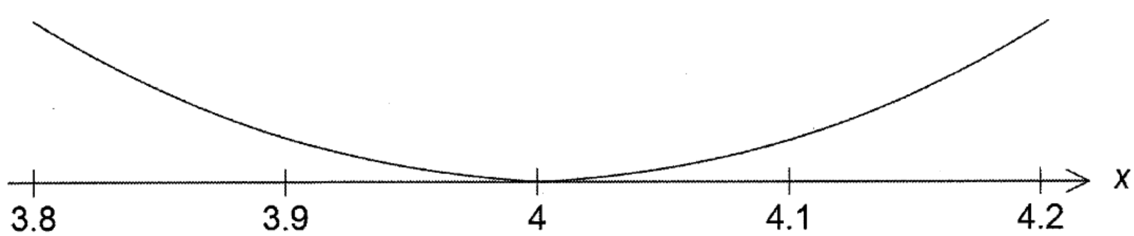
(A) $2\sin\theta - \sin^3\theta$

(B) $2\sin\theta - 4\sin^3\theta$

(C) $3\sin\theta - \sin^3\theta$

(D) $3\sin\theta - 4\sin^3\theta$

3 Part of the graph of $y = P(x)$, where $P(x)$ is a polynomial of degree 3 is shown below



Which of the following could be the polynomial $P(x)$

(A) $(x-4)^3$

(B) $(x-5)(x+4)^2$

(C) $(x-1)(x-4)^2$

(D) $(x-1)(x+2)(x-4)$

4 The point $P(2,5)$ lies on the graph of the polynomial $y = P(x)$. This polynomial is an odd function. The remainder when $P(x)$ is divided by $\left(x+2\right)$ is:

(A) -5

(B) 0

(C) 5

(D) 2

5 Which of the following is true for the polynomial $P(x) = x^3 - x$

(A) It has a point of inflexion at $x = 0$

(B) It has a minimum at $x = \frac{1}{3}$

(C) It has only one turning point.

(D) It has a stationary point at $x = -\frac{1}{3}$

Section II

40 marks
Attempt Questions 6–9
Allow about 1 hours and 52 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 6–9, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (10 marks) Use a SEPARATE writing booklet.

(a) Solve the inequality $\frac{6}{x^2} \le \frac{x-5}{x}$ **3**

(b) A and B are the points $(-5, 12)$ and $(4, 9)$ respectively, P is the point which divides AB externally in the ratio $5:2$. Find the co-ordinates of P . **2**

(c) (i) Show that the acute angle between the lines $mx - y = 0$ and $x - my = 0$ can be given by $\tan\theta = \left|\frac{m^2-1}{2m}\right|$ where θ is the angle between the lines. **2**

(ii) Find the values of the real number m such that the angle θ is 45^0. **3**

Question 7 (10 marks) Use a SEPARATE writing booklet.

(a) If $\cos A = \frac{7}{9}$ and $\sin B = \frac{1}{3}$ where $0 \le A \le 90^\circ$, and $0 \le B \le 90^\circ$. **4**

(i) Find the value of $\tan\left(A+B\right)$ in simplest surd form.

(ii) Show that $A = 2B$

(b) (i) Express the polynomial $P(x) = 2x^3 - x^2 - 13x - 6$ as the product of three linear factors. **2**

(ii) Hence solve the inequality $2x^3 - x^2 - 13x - 6 \le 0$ **1**

(d) (i) Show that $x^3 - 3x + 1 = 0$ has a root between $x = 1$ and $x = 2$ **1**

(ii) Using $x = 1.5$ as a first approximation, obtain a better approximation of the root using Newton's Method once. [Answer to 2 decimal places] **2**

Question 8 (10 marks) Use a SEPARATE writing booklet.

(a) Expand the expression $(2x+5)^5$ **2**

(b) For what value of n is ${}^nC_5 = {}^nC_9$ **1**

(c) Find, as a rational number, the coefficient of x in the expansion of $\left(x^2 + \frac{1}{2x}\right)^8$ **2**

(d) Suppose $(5+2x)^{15} = \sum_{k=0}^{15} t_k x^k$

(i) Use the binomial theorem to write an expression for t_k , $0 \le k \le 15$ **1**

(ii) Show that $\frac{t_{k+1}}{t_k} = \frac{2(15-k)}{5(k+1)}$ **2**

(iii) Hence or otherwise find the largest coefficient of t_k . **2**

You may leave your answer in the form ${}^{15}C_k 5^c 2^d$

Question 9 (10 marks) Use a SEPARATE writing booklet.

(a) Consider the curve given by $y = 1 + 3x - x^3$, for $-2 \le x \le 3$

(i) Find the stationary points and determine their nature. **2**

(ii) Determine if there exists any points of inflexion. **1**

(iii) Sketch the curve for $-2 \le x \le 3$. **2**

(iv) What is the minimum value of y for $-2 \le x \le 3$? **1**

(b) The sum of the height h of a cylinder and the circumference of its base is 20 metres.

(i) Show that the volume of the cylinder is $V = \pi r^2 (20 - 2\pi r)$, **2**
where r is the radius of the cylinder r.

(ii) Hence find the maximum volume of the cylinder **2**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

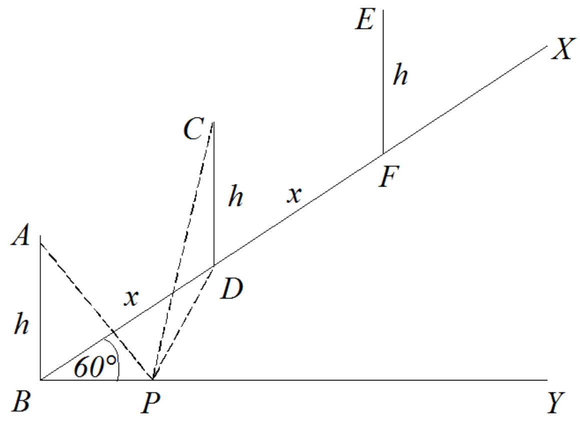
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$



(a) In the above diagram, BX and BY represent two roads intersecting at an angle of 60°. On the road BX are situated three telegraph poles AB, CD and EF, all of equal height, the same distance, x metres apart (i.e $BD = DF = x$). P is a point on the road BY and the angles of elevation of A and C from P are 45° and 30° respectively.

(i) Show that $BP = h$ and $DP = h\sqrt{3}$

(ii) By the use of the Sine rule in triangle BDP, show that $BDP = 30^\circ$ and hence that triangle BDP is right angles at P.

(iii) Prove that $x = 2h$

(iv) By the use of the cosine rule in triangle PDF, show that $PF = h\sqrt{13}$ and hence show that the angle of elevation of E from P is approximately 15.5°.

(e) A monic cubic polynomial when divided by $\left(x^2+4\right)$ leaves a remainder of $x+8$ and when divided by x leaves a remainder of -4. Find the polynomial in the form ax^3+bx^2+cx+d

6 Consider the polynomial $P(x) = 3x^3 + 3x + a$.
If $x-2$ is a factor of $P(x)$, what is the value of a?

(A) -30
(B) -18
(C) 18
(D) 30

7 Let α, β and γ be the roots of $4x^3 - 2x^2 + 3x - 2 = 0$.

What is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

(A) $-\frac{3}{2}$

(B) $-\frac{2}{3}$

(C) $\frac{2}{3}$

8 What are the coordinates of the point that divides the interval joining the points $A(7,1)$ and $B(0,-6)$ internally in the ratio 4:3?

(A) $(3,-3)$

(B) $(3,-2)$

(C) $(4,-2)$

9 Which of the following is equivalent to the expression $\sqrt{\frac{4+4\cos 2x}{1-\cos 2x}}$?

(A) $2\cot^2 x$

(B) $2\cot x$

(C) $2\tan x$

10 What is the acute angle to the nearest degree between the lines $y = 2x - 1$ and $x - 3y + 6 = 0$?

(A) 45°

(B) 54°

(C) 79°

(D) 82°

(d) and show that if Q is the point $(0, 2)$,

then the triangle APQ is both right-angled and isosceles.

(e) Simplify $\frac{4^n - 1}{4^n - 2^n}$ **2**

Solve for θ in the range $0 \le \theta \le 360^\circ$: $6\cot^2\theta - 4\cos^2\theta = 1$ **3**

(e) In the binomial expansion $\left(x^2 + \frac{a}{x}\right)^6$ the term independent of x is 240. **3**

Find the value of a.

(f) Show that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ **2**

(g) Given $y = x^3 - x^2 + x$ show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2x(3x-5) = 0$ **2**

(h) Show that if $y = -3x^2 + 1$ then $\frac{dy}{dx} = -6x$ through differentiation by first principles **3**

(i)

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Multiple choice.

1) A
2) D
3) C
4) A
5) A

6)

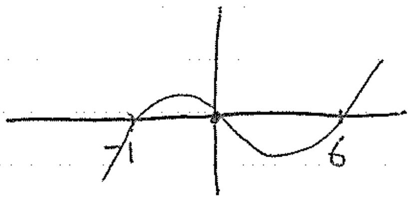
a) $\frac{6}{x^2} \le \frac{x-5}{x}$

$\boxed{\text{for } x>0}$ $6x \le x^3 - 5x^2$

$0 \le x^3 - 5x^2 - 6x$

$0 \le x(x^2 - 5x - 6)$

$0 \le x(x-6)(x+1)$



$\therefore -1 \le x \le 0 \quad x \ge 6.$

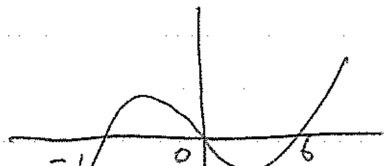
$\therefore \boxed{x \ge 6}$

$\boxed{\text{for } x<0}$

$6x \ge x^3 - 5x^2$

$0 \ge x^3 - 5x^2 - 6x$

$0 \ge x(x-6)(x+1)$



$x \le -1 \quad 0 \le x \le 6$

$\therefore \boxed{x \le -1}$

$\therefore x \ge 6$ or $x \le -1$

b) $m:n = 5:-2$

$x = \frac{-2(-5)+5(4)}{5-2}$ $\qquad$ $y = \frac{-2(12)+5(9)}{5-2}$

$= \frac{10+20}{3}$ $\qquad$ $= \frac{-24+45}{3}$

$= 10$ $\qquad$ $= 7$

$P(10, 7)$

c(i)

$mx - y = 0$ $\qquad$ $x - my = 0$

$-y = -mx$ $\qquad$ $-my = -x$

$y = mx$ $\qquad$ $y = \frac{x}{m}.$

$\tan\theta = \left|\frac{m - \frac{1}{m}}{1 + m.\frac{1}{m}}\right|$

$= \left|\frac{\frac{m^2-1}{m}}{2}\right|$

$= \left|\frac{m^2-1}{m} \times \frac{1}{2}\right|$

$= \left|\frac{m^2-1}{2m}\right|$

(ii) $\frac{m^2-1}{2m} = 1$ and $\frac{m^2-1}{2m} = -1$

$m^2 - 1 = 2m$ $\qquad$ $m^2 - 1 = -2m$

$m^2 - 2m - 1 = 0$ $\qquad$ $m^2 + 2m - 1 = 0$

$\Delta = (-2)^2 - 4 \times 1 \times -1$ $\qquad$ $\Delta = (2)^2 - 4 \times 1 \times -1$

$= 8$ $\qquad$ $= 8$

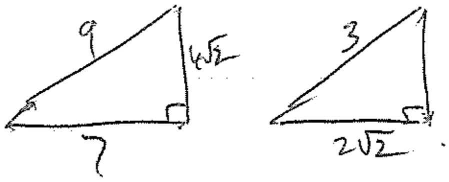
$m = \frac{2 \pm \sqrt{8}}{2}$ $\qquad$ $m = \frac{-2 \pm \sqrt{8}}{2}$

$= 1 \pm \sqrt{2}$ $\qquad$ $= -1 \pm \sqrt{2}$

$\therefore m = \pm 1 \pm \sqrt{2}$

7a)

(i)



$$\therefore \sin 2B = 2\sin B\cos B$$
$$= 2\cdot\tfrac{1}{3}\cdot\tfrac{2\sqrt{2}}{3}$$
$$= \frac{4\sqrt{2}}{9}$$
$$\sin A = \frac{4\sqrt{2}}{9}$$
$$\therefore A = 2B$$

(ii)
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$= \frac{\frac{4\sqrt{2}}{7} + \frac{1}{2\sqrt{2}}}{1 - \frac{4\sqrt{2}}{7}\times\frac{1}{2\sqrt{2}}}$$
$$= \frac{\frac{16+7}{14\sqrt{2}}}{\frac{5}{7}}$$
$$= \frac{23}{14\sqrt{2}}\times\frac{7}{5}$$
$$= \frac{23}{10\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{23\sqrt{2}}{20}$$

b) $P(x) = 2x^3 - x^2 - 13x - 6$

$P(1) = 2-1-13-6$
$P(-1) = -2-1+13-6$
$P(2) = 16-4-26-6$
$P(-2) = -16-4+26-6 = 0$
$\therefore (x+2)$ is factor

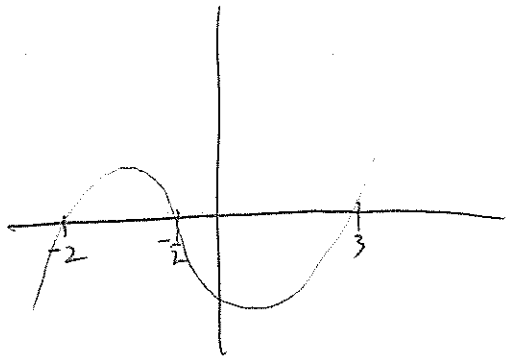
$$\begin{array}{r} 2x^2 - 5x - 3 \\ x+2 \overline{\smash{)}\, 2x^3 - x^2 - 13x - 6} \\ \underline{2x^3 + 4x^2} \\ -5x^2 - 13x \\ \underline{-5x^2 - 10x} \\ -3x - 6 \\ -3x - 6 \end{array}$$

$(x+2)(2x^2-5x-3)$

$(x+2)\ \frac{(2x-6)(2x+1)}{2}$

$\therefore (x+2)(x-3)(2x+1) = P(x).$

(ii) $2x^3 - x^2 - 13x - 6 \leq 0$



$x \leq -2$ and $-\frac{1}{2} \leq x \leq 3$

d(i) $P(1) = 1^3 - 3(1) + 1 = 1-3+1 = -2$
$P(2) = 2^3 - 3(2) + 1 = 8-6+1 = 3$
$\therefore$ curve $y = x^2 - 3x + 1$ cuts the x axis between -2 and 3.

(ii) $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$ $\qquad f'(x) = 3x^2 - 3.$

$f(1.5) = (1.5)^3 - 3(1.5) + 1 = -0.125$
$f'(1.5) = 3(1.5)^2 - 3 = 3.75.$

$$a_2 = 1.5 - \frac{-0.125}{3.75}$$
$$\doteqdot 1.53.$$

8a) $(2x+5)^5$

$= \binom{5}{0}(2x)^0 5^5 + \binom{5}{1}(2x)^1 5^4 + \binom{5}{2}(2x)^2 5^3$
$+ \binom{5}{3}(2x)^3 5^{\cancel{2}} + \binom{5}{4}(2x)^4 5^1 + \binom{5}{5}(2x)^5 5^0$ ✓

$= 3125 + 6250x + 5000x^2 + 2000x^3 + 400x^4 + 32x^5$ ✓

(b) $^{14}C_5 = {}^{14}C_9$ through symmetry ✓

(c) $T_{k+1} = {}^{8}C_k (x^2)^{8-k} \left(\frac{1}{2x}\right)^k$

$= \frac{^{8}C_k}{2^k} \cdot \frac{x^{16-2k}}{x^k}$

$= \frac{^{8}C_k}{2^k} \cdot x^{16-3k}$

$\therefore 16 - 3k = 1$

$k = 5$ ✓

$\therefore T_6 = \frac{^{8}C_5}{2^5} \cdot x$

$= \frac{56}{32}$

$= \frac{7}{4}$ ✓

d) ~~$\therefore {}^{6}C_k (x^2)(\ldots)$~~

$(5+2x)^{15} = \sum_{t=0}^{15} t_k x^k$

$T_{k+1} = {}^{15}C_k 5^{15-k} (2x)^k$

$\therefore t_k = {}^{15}C_k 5^{15-k} \cdot 2^k \cdot x^k$ ✓

(ii) $\frac{t_{k+1}}{t_k} = \frac{^{15}C_{k+1} 5^{15-(k+1)} 2^{k+1}}{^{15}C_k 5^{15-k} 2^k}$ ✓

$= \frac{\frac{15!}{[15-(k+1)]!(k+1)!} \cdot 5^{14-k} 2^{k+1}}{\frac{15!}{(15-k)!(k!)} \cdot 5^{15-k} 2^k}$ ✓

$= \frac{2(15-k)}{5(k+1)}$

$\frac{t_{k+1}}{t_k} > 1 \qquad \frac{2(15-k)}{5(k+1)} > 1$

$2(15-k) > 5(k+1)$

$30 - 2k > 5k + 5$

$25 > 7k$

$3\frac{4}{7} > k$

$\therefore k = 3$ ✓

$\therefore t_4 > t_3$

$t_4 = {}^{15}C_4 5^{11} 2^4$ ✓

9a) $y = 1 + 3x - x^3$

$-2 \le x \le 3$

(i) $\frac{dy}{dx} = 3 - 3x^2$

$= 3(1 - x^2)$

$\frac{d_2y}{dx^2} = -6x$

$\frac{dy}{dx} = 0 \quad 0 = 3(1 - x^2)$

$= (1+x)(1-x)$

$\therefore \quad x = \pm 1$

for $x = 1 \quad y'' = -6 \times 1 < 0 \therefore$ max

$x = -1 \quad y'' = -6 \times -1 > 0 \therefore$ min

must do needs to be determined

$\therefore (1, 3)$ is max

$(-1, 3)$ is min

(ii) $\frac{d_2y}{dx^2} = 0 \quad \therefore -6x = 0$

$x = 0$

test

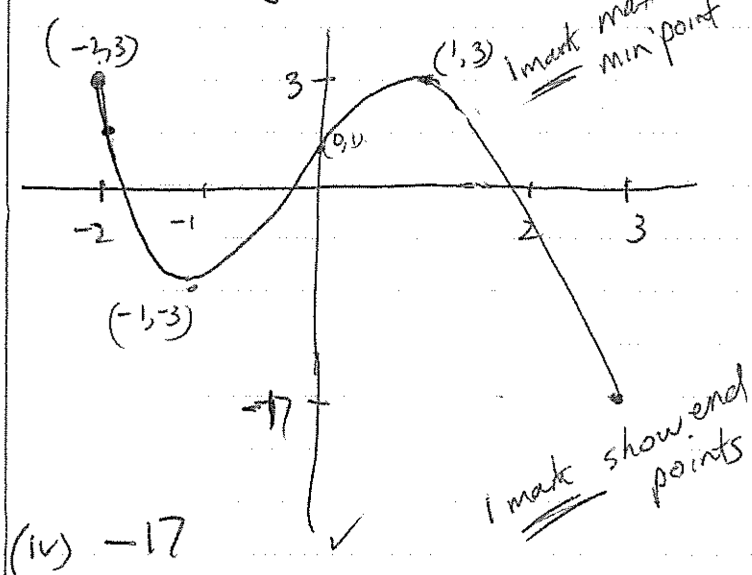
x	$-\frac{1}{2}$	0	$\frac{1}{2}$
y''	3	0	-3

change in concavity

$\therefore (0, 1)$ is inflexion

iii) $x = -2 \quad y = 1 - 6 + 8 = 3$

$x = 3 \quad y = 1 + 9 - 27 = -17$



(iv) -17

b) $V = \pi r^2 h$

h

$c = 20$

$h + c = 20$

$h + 2\pi r = 20$

$h = 20 - 2\pi r$

$= \pi r^2 (20 - 2\pi r)$

$= 20\pi r^2 - 2\pi^2 r^3$

(ii) $\frac{dV}{dr} = 40\pi r - 6\pi^2 r^2$

$0 = 40\pi r - 6\pi^2 r^2$

$0 = 2\pi r(20 - 3\pi r)$

$r = 0 \quad -3\pi r = -20$

$r = \frac{20}{3\pi}$

$V'' = 40\pi - 12\pi^2 r$

$= 40\pi - 12\pi^2 \frac{20}{3\pi}$

$= -40\pi$

< 0

$\therefore$ max.

$V = 20\pi \left(\frac{20}{3\pi}\right)^2 - 2\pi^2 \left(\frac{20}{3\pi}\right)^3$

$= \frac{8000}{27\pi}$ cubic units