Student Number: .... Teacher Name: ....

PENRITH SELECTIVE HIGH SCHOOL

MATHEMATICS DEPARTMENT

Year 11 Assessment Task 3
2024
Mathematics Extension 1

General Instructions:

- Reading time – 10 minutes
- Working time – 1 hour 30 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided with this examination paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- No correction tape/white out allowed.

Total marks: 58

Section I – 10 marks (pages 1–3)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 48 marks (pages 4–7)

- Attempt Questions 11–14
- Allow about 1 hour 15 minutes for this section

	Multiple Choice	Q11	Q12	Q13	Q14	Total
Further Work with Functions	1, 2, 4 /3	c /3	b /4	a /3		/13
Polynomials	9, 10 /2	a, b /8		b /3		/13
Inverse Trigonometric Functions	5, 6 /2	d /1			a /2	/5
Further Trigonometric Identities	3 /1		c /4	c /3	d, e /4	/12
Rates of Change	7, 8 /2		a /4	d /3	b, c /6	/15
Total	/10	/12	/12	/12	/12	/58

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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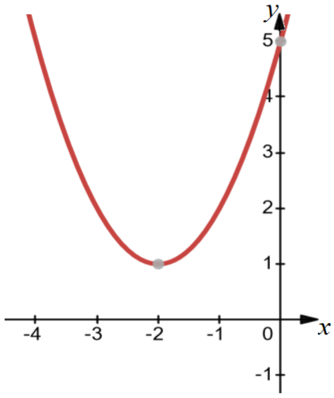
1 Which of the following pairs of parametric equations represents the parabola shown?

A. $x = t + 2,\ y = t^2 + 1$

B. $x = t + 2,\ y = t^2 - 1$

C. $x = t - 2,\ y = t^2 + 1$

D. $x = t - 2,\ y = t^2 - 1$



2 Which of the following represents the solution to the inequality $x^2 < 8 - 2x$?

A. $(-4, 2)$

B. $[-4, 2]$

C. $(-\infty, -4) \cup (2, \infty)$

D. $(-\infty, -4] \cup [2, \infty)$

3 If $\sin x = \frac{3}{5}$, where $\frac{\pi}{2} \le x \le \pi$, then what is the value of $\tan 2x$?

A. $\frac{24}{7}$

B. $-\frac{24}{7}$

C. $\frac{24}{25}$

D. $-\frac{24}{25}$

4 Solve $|3x+7|>5$.

A. $-4<x<-\frac{2}{3}$

B. $x<-4\cup x>-\frac{2}{3}$

C. $-\frac{2}{3}<x<-4$

D. $x<-\frac{2}{3}\cup x>-4$

5 What is the domain and range of the function $f(x)=\frac{1}{2}\cos^{-1}(2x)$?

A. Domain: $[-2,2]$, Range: $[0,2\pi]$

B. Domain: $[-2,2]$, Range: $\left[0,\frac{\pi}{2}\right]$

C. Domain: $\left[-\frac{1}{2},\frac{1}{2}\right]$, Range: $[0,2\pi]$

D. Domain: $\left[-\frac{1}{2},\frac{1}{2}\right]$, Range: $\left[0,\frac{\pi}{2}\right]$

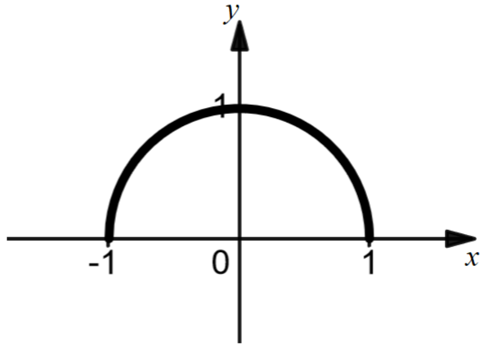
6 Which of the following inverse trigonometric functions is represented by the graph shown?

A. $y=\sin\left(\cos^{-1}x\right)$

B. $y=\sin\left(\sin^{-1}x\right)$

C. $y=\cos^{-1}(\sin x)$

D. $y=\sin^{-1}(\cos x)$



7 The half-life of radium is 1600 years. What percentage of radium would have decayed after 604 years?

A. 77%

B. 38%

C. 29%

D. 23%

8 The velocity of a certain particle, in metres per second is given by $v = 4x^2 - 2x + 1$, where x is its displacement in metres from the origin.

What is the acceleration of the particle at $x = 1$?

A. 6 m s^{-2}

B. 12 m s^{-2}

C. 15 m s^{-2}

D. 18 m s^{-2}

9 The two roots of the equation $x^3 - 3x^2 + kx + 15 = 0$ are reciprocals of each other. What is the value of k?

A. –69

B. –169

C. –269

D. –369

10 If $P(x) = ax^{11} + bx^7 + cx^5 + 19$ is divided by $x + 2024$, the remainder is 12 345. What is the remainder when $P(x)$ is divided by $x - 2024$?

A. –12 307

B. –12 326

C. –12 345

D. –12 364

Section II

48 marks
Attempt Questions 11–14
Allow about 1 hour and 15 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (12 marks) Use a separate Writing Booklet for Question 11.

(a) The product of two of the roots of the polynomial $x^4 + 2x^3 - 18x - 10 = 0$ is –5. Find the product of the two other roots. **1**

(b) Consider the polynomial $P(x) = x^3 - x^2 - 33x - 63$.

(i) Find the derivative, $P'(x)$, of this polynomial. **1**

(ii) Show that $P(-3) = P'(-3)$. **2**

(iii) What are the implications of your results from (ii)? **1**

(iv) Fully factorise $P(x)$ and sketch the graph of this polynomial showing all intercepts. **3**

(c) Solve $\dfrac{-3}{2x+5} < 2$. **3**

(d) It is given that $\sin\left(\dfrac{13\pi}{12}\right) = \dfrac{\sqrt{2}-\sqrt{6}}{4}$. **1**

Find the exact value of $\sin^{-1}\left(\dfrac{\sqrt{2}-\sqrt{6}}{4}\right)$.

End of Question 11

Question 12 (12 marks) Use a separate Writing Booklet for Question 12.

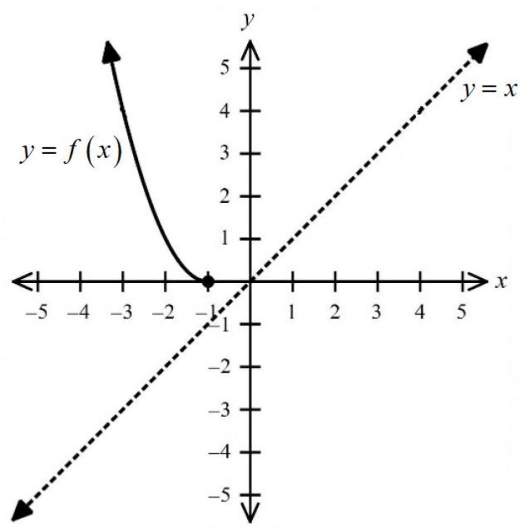
(a) A particle is moving along the x-axis and is initially at the origin. Its velocity, v metres per second, at time t seconds is given by $v = \dfrac{-3t}{t^2+25}$.

(i) What is the initial velocity of the particle? **1**

(ii) Find an expression for the acceleration of the particle. **2**

(iii) Find the time when the acceleration of the particle is zero. **1**

(b) The graph of $f(x) = (x+1)^2;\ (-\infty, -1]$ is shown below, along with the line $y = x$.



(i) Write the equation of the inverse function of $f(x)$ in the form $y = f^{-1}(x)$ and state its domain. **3**

(ii) Hence, sketch the inverse function, including the original function and the line $y = x$ in your sketch. **1**

(c) (i) Prove that $\dfrac{\cos 2\theta + 1}{\sin 2\theta} = \cot\theta$. **3**

(ii) Hence, show that the exact value of cot 15° is $2 + \sqrt{3}$. **1**

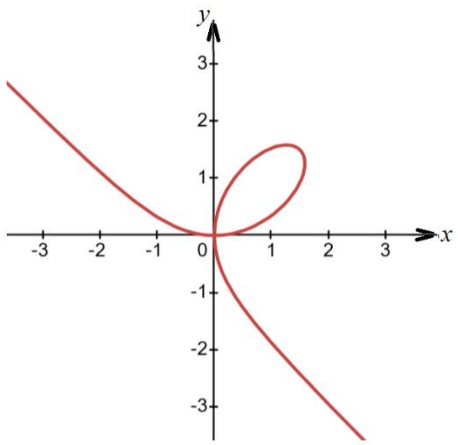
End of Question 12

Question 13 (12 marks) Use a separate Writing Booklet for Question 13.

(a) The graph shown to the right is known as the "Folium of Descartes".

This graph is defined by the parametric equations:

$$\begin{cases} x = \dfrac{3t}{1+t^3} \\ y = \dfrac{3t^2}{1+t^3} \end{cases}$$



(i) Find $\dfrac{y}{x}$ in terms of t, expressing your answer in simplest form. **1**

(ii) Deduce that the Cartesian equation of this curve is: $x^3 + y^3 = 3xy$. **2**

(b) The roots of $x^3 + 5x^2 + 7x - 1 = 0$ are α , β and γ .

(i) Find $\alpha(\beta+1) + \beta(\gamma+1) + \gamma(\alpha+1)$. **2**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **1**

(c) Show that the exact value of $\sin\left[\cos^{-1}\left(\dfrac{12}{13}\right) - \tan^{-1}\left(\dfrac{4}{3}\right)\right]$ is $-\dfrac{33}{65}$. **3**

(d) Detroit City is grappling with a troubling rise in pollution, with levels increasing exponentially. Over the past 2 years, from 2022, pollution has already surged by 10%. The city government has stated they will only take action once pollution has increased by 50% from 2022 levels. How long will it take for the city to reach this critical threshold? **3**

End of Question 13

Question 14 (12 marks) Use a separate Writing Booklet for Question 14.

(a) Sketch the graph of $y = 2\tan^{-1} x$ showing intercepts and asymptotes. **2**

(b) A vessel is shaped so that when the depth of water in it is x cm, the volume of the water is V cm^3 where $V = 108x + x^3$. Water is poured into the vessel at a constant rate of 30 cm^3/s. At what rate is the water level rising when its depth is 8 cm? **3**

(c) Newton's law of cooling states that the rate at which an exposed body changes temperature is proportional to the difference between its own temperature and the ambient temperature of its surroundings, provided the temperature difference is small and the nature of the body's surface does not change.

Little Johnny took an ice block with temperature –16°C from the fridge, ran outside and accidentally dropped the ice block on the lawn. The air temperature is a constant 24°C.

(i) If after 30 seconds, the ice block's temperature rose to –13°C, show that $T = 24 - 40e^{-0.0026t}$. **2**

(ii) Determine how many seconds it will take for the ice block to reach 0°C and begin to melt. **1**

(d) Use the substitution $t = \tan\dfrac{\theta}{2}$ to show that $\operatorname{cosec}\theta + \cot\theta = \cot\dfrac{\theta}{2}$. **2**

(e) Prove that $4\sin 3x \sin 2x \sin x = \sin 2x + \sin 4x - \sin 6x$. **2**

END OF PAPER

Student Number: .... Teacher Name: ....

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Total marks: 58

Section I – 10 marks (pages 1–3)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 48 marks (pages 4–7)

- Attempt Questions 11–14
- Allow about 1 hour 15 minutes for this section

Solutions with Teacher Feedback

	Multiple Choice	Q11	Q12	Q13	Q14	Total
Further Work with Functions	1, 2, 4 3/3	c 3/3	b 4/4	a 3/3		13/13
Polynomials	9, 10 2/2	a, b 8/8		b 3/3		13/13
Inverse Trigonometric Functions	5, 6 2/2	d 1/1			a 2/2	5/5
Further Trigonometric Identities	3 1/1		c 4/4	c 3/3	d, e 4/4	12/12
Rates of Change	7, 8 2/2		a 4/4	d 3/3	b, c 6/6	15/15
Total	10/10	12/12	12/12	12/12	12/12	58/58

Section I **Solutions**

1 Which of the following pairs of parametric equations represents the parabola shown?

A. $x = t + 2,\ y = t^2 + 1$

B. $x = t + 2,\ y = t^2 - 1$

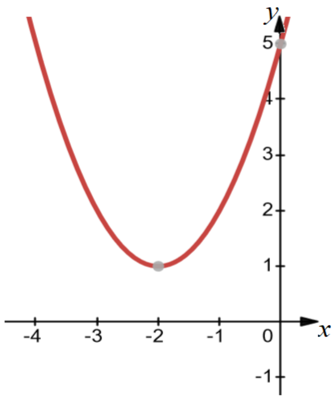
C. $x = t - 2,\ y = t^2 + 1$

D. $x = t - 2,\ y = t^2 - 1$

$y = (x+2)^2 + 1$

$\therefore t = x + 2, y = t^2 + 1$

$\Rightarrow$ C



2 Which of the following represents the solution to the inequality $x^2 < 8 - 2x$?

A. $(-4, 2)$

B. $[-4, 2]$

C. $(-\infty, -4) \cup (2, \infty)$

D. $(-\infty, -4] \cup [2, \infty)$

$x^2 < 8 - 2x$

$x^2 + 2x - 8 < 0$

$(x+4)(x-2) < 0$

$-4 < x < 2$

$\Rightarrow$ A

3 If $\sin x = \frac{3}{5}$, where $\frac{\pi}{2} \le x \le \pi$, then what is the value of $\tan 2x$?

A. $\frac{24}{7}$

B. $-\frac{24}{7}$

C. $\frac{24}{25}$

D. $-\frac{24}{25}$

For quadrant 2:

$\tan x = -\frac{3}{4}$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$= \frac{2 \times -\frac{3}{4}}{1 - \left(-\frac{3}{4}\right)^2} = -\frac{24}{7}$$

$\Rightarrow$ B

4 Solve $|3x+7|>5$.

A. $-4<x<-\frac{2}{3}$

B. $x<-4 \cup x>-\frac{2}{3}$

C. $-\frac{2}{3}<x<-4$

D. $x<-\frac{2}{3} \cup x>-4$

$$3x+7>5 \qquad 3x+7<-5$$
$$3x>-2 \qquad 3x<-12$$
$$x>-\frac{2}{3} \qquad x<-4$$
$$\therefore x<-4 \cup x>-\frac{2}{3}$$
$\Rightarrow$ B

5 What is the domain and range of the function $f(x)=\frac{1}{2}\cos^{-1}(2x)$?

A. Domain: $[-2,2]$, Range: $[0,2\pi]$

B. Domain: $[-2,2]$, Range: $\left[0,\frac{\pi}{2}\right]$

C. Domain: $\left[-\frac{1}{2},\frac{1}{2}\right]$, Range: $[0,2\pi]$

D. Domain: $\left[-\frac{1}{2},\frac{1}{2}\right]$, Range: $\left[0,\frac{\pi}{2}\right]$

$$-1\le 2x\le 1$$
$$-\frac{1}{2}\le x\le \frac{1}{2}$$
$$D:\left[-\frac{1}{2},\frac{1}{2}\right] \qquad R:\left[0,\frac{\pi}{2}\right]$$
$\Rightarrow$ D

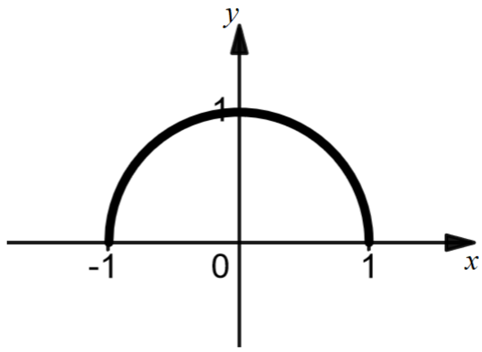
6 Which of the following inverse trigonometric functions is represented by the graph shown?

A. $y=\sin\left(\cos^{-1}x\right)$

B. $y=\sin\left(\sin^{-1}x\right)$

C. $y=\cos^{-1}(\sin x)$

D. $y=\sin^{-1}(\cos x)$



Consider A, then constructing a table of values for the intercepts:

x	-1	0	1
$\cos^{-1}x$	π	$\frac{\pi}{2}$	0
$y=\sin\left(\cos^{-1}x\right)$	$\sin\pi=0$	$\sin\frac{\pi}{2}=1$	$\sin 0=0$

$\Rightarrow$ A

7 The half-life of radium is 1600 years. What percentage of radium would have decayed after 604 years?

A. 77%

B. 38%

C. 29%

$\boxed{\text{D.}}$ 23%

$$N = Ae^{kt}$$

For half-life,

$$\frac{1}{2}A = Ae^{k(1600)}$$

$$\frac{1}{2} = e^{k(1600)}$$

$$\ln\frac{1}{2} = 1600k$$

$$k \approx -0.000433\ldots$$

When $t = 604$,

$$\frac{N}{A} = e^{k(604)} \approx 0.77$$

$\therefore$ percentage decayed

$= 0.23 = 23\%$

$\Rightarrow$ D

8 The velocity of a certain particle, in metres per second is given by $v = 4x^2 - 2x + 1$, where x is its displacement in metres from the origin.

What is the acceleration of the particle at $x = 1$?

A. 6 m s^{-2}

B. 12 m s^{-2}

C. 15 m s^{-2}

$\boxed{\text{D.}}$ 18 m s^{-2}

$$v = 4x^2 - 2x + 1$$

$$\frac{dv}{dx} = 8x - 2$$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

$$a = (8x-2)(4x^2 - 2x + 1)$$

when $x = 1$,

$$a = (8\times 1 - 2)(4\times 1^2 - 2\times 1 + 1)$$

$$a = 18$$

$\Rightarrow$ D

9 The two roots of the equation $x^3 - 3x^2 + kx + 15 = 0$ are reciprocals of each other. What is the value of k?

A. –69

B. –169

$\boxed{\text{C.}}$ –269

D. –369

Given: $x^3 - 3x^2 + kx + 15 = 0$

let the roots be α, β, γ

$\alpha\beta\gamma = -15$

$\alpha\beta = 1$ (since reciprocals of each other)

$\gamma = -15$

$$P(-15) = (-15)^3 - 3(-15)^2 - 15k + 15 = 0$$

$$k = 269$$

$\Rightarrow$ C

10 If $P(x) = ax^{11} + bx^7 + cx^5 + 19$ is divided by $x + 2024$, the remainder is 12 345. What is the remainder when $P(x)$ is divided by $x - 2024$?

$\boxed{\text{A.}}$ –12 307

B. –12 326

C. –12 345

D. –12 364

$$P(-2024) = a(-2024)^{11} + b(-2024)^7 + c(-2024)^5 + 19 = 12345$$

$$= -a(2024)^{11} - b(2024)^7 - c(2024)^5 + 19 = 12345$$

$$\therefore -\left[a(2024)^{11} + b(2024)^7 + c(2024)^5\right] = 12326$$

$$P(2024) = a(2024)^{11} + b(2024)^7 + c(2024)^5 + 19$$

$$= -12326 + 19$$

$$= -12307$$

$\Rightarrow$ A

Section II **Solutions**

Question 11 (12 marks) Use a separate Writing Booklet for Question 11.

(a) The product of two of the roots of the polynomial $x^4 + 2x^3 - 18x - 10 = 0$ is –5. Find the product of the two other roots. **1**

$\alpha\beta\gamma\delta = -10$

if $\alpha\beta = -5$

then $\gamma\delta = \dfrac{-10}{-5} = 2$

> This question was worth 1 mark, so any mistake resulted in 0 mark.
>
> Common mistake was writing –2.

(b) Consider the polynomial $P(x) = x^3 - x^2 - 33x - 63$.

(i) Find the derivative, $P'(x)$, of this polynomial. **1**

$P'(x) = 3x^2 - 2x - 33$

> This question was done well.
>
> Part (iv) graph was poorly drawn, some were too small, bad scaling. Marks were not deducted but may be deducted in future exams

(ii) Show that $P(-3) = P'(-3)$. **2**

$P(-3) = (-3)^3 - (-3)^2 - 33(-3) - 63 = 0$

$P'(-3) = 3(-3)^2 - 2(-3) - 33 = 0$

$\therefore P(-3) = P'(-3)$

(iii) What are the implications of your results from (ii)? **1**

<u>It implies that –3 is a (potential) double root.</u>

(iv) Fully factorise $P(x)$ and sketch the graph of this polynomial showing all intercepts. **3**

Method 1: By long division

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\; x - 7 \\ x^2 + 6x + 9 & x^3 - x^2 - 33x - 63 \\ & \underline{x^3 + 6x^2 + 9x} \\ & \quad -7x^2 - 42x - 63 \\ & \quad \underline{-7x^2 - 42x - 63} \\ & \quad\quad\quad \underline{} \end{array}$$

$P(x) = (x+3)^2(x-7)$

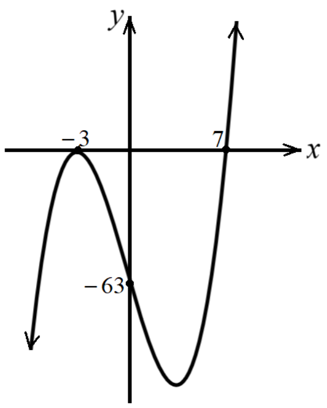
Method 2: by sum of the roots

$\alpha + \beta + \gamma = 1$

$-3 - 3 + \gamma = 1$

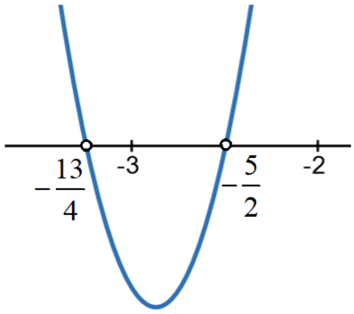
$\gamma = 7$

$P(x) = (x+3)^2(x-7)$



(c) Solve $\dfrac{-3}{2x+5} < 2$. **3**

$$\left(\frac{-3}{2x+5} < 2\right)(2x+5)^2$$
$$-3(2x+5) < 2(2x+5)^2$$
$$2(2x+5)^2 + 3(2x+5) > 0$$
$$(2x+5)(2(2x+5)+3) > 0$$
$$(2x+5)(4x+13) > 0$$
$$x < -\frac{13}{4} \cup x > -\frac{5}{2}$$
$$\text{or } \left(-\infty, -\frac{13}{4}\right) \cup \left(-\frac{5}{2}, \infty\right)$$



Common errors:
- Algebraic errors with expanding & factorising
- Some students did not multiply by the square of the denominator.

(d) It is given that $\sin\left(\dfrac{13\pi}{12}\right) = \dfrac{\sqrt{2}-\sqrt{6}}{4}$. **1**

Find the exact value of $\sin^{-1}\left(\dfrac{\sqrt{2}-\sqrt{6}}{4}\right)$.

$\sin\left(\dfrac{13\pi}{12}\right)$ is negative.

The domain of $y = \sin^{-1} x$ is $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ and the reference angle of $\dfrac{13\pi}{12}$ is $\dfrac{\pi}{12}$,

$$\therefore \sin^{-1}\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) = -\frac{\pi}{12}$$

This question was not done well like Q11a. Any mistake made resulted to 0 marks.

Students forgot to refer to the domain of $y = \sin^{-1}x$ and/or reference angle.

End of Question 11

Question 12 (12 marks) Use a separate Writing Booklet for Question 12.

(a) A particle is moving along the x-axis and is initially at the origin. Its velocity, v metres per second, at time t seconds is given by $v = \dfrac{-3t}{t^2+25}$.

(i) What is the initial velocity of the particle? **1**

When $t = 0$, $v = \dfrac{-3(0)}{(0)^2+25} = 0$.

The velocity of the particle is 0 m/s.

This question was done really well.

(ii) Find an expression for the acceleration of the particle. **2**

let $u = -3t$ $\quad v = t^2 + 25$

$u' = -3$ $\quad v' = 2t$

$$\frac{dv}{dt} = \frac{-3(t^2+25) - 2t(-3t)}{(t^2+25)^2}$$

$$= \frac{-3t^2 - 75 + 6t^2}{(t^2+25)^2}$$

$$\therefore a = \frac{3t^2 - 75}{(t^2+25)^2} \qquad \text{or } a = \frac{3(t^2-25)}{(t^2+25)^2}$$

Students are required to show the simplified form to achieve full marks.
Common error:

- Students using product rule and messed up in their simplification.
- Students simplifying the numerator into $-3t^2 - 75 + 6t$.

(iii) Find the time when the acceleration of the particle is zero. **1**

$$\frac{3t^2 - 75}{(t^2+25)^2} = 0$$

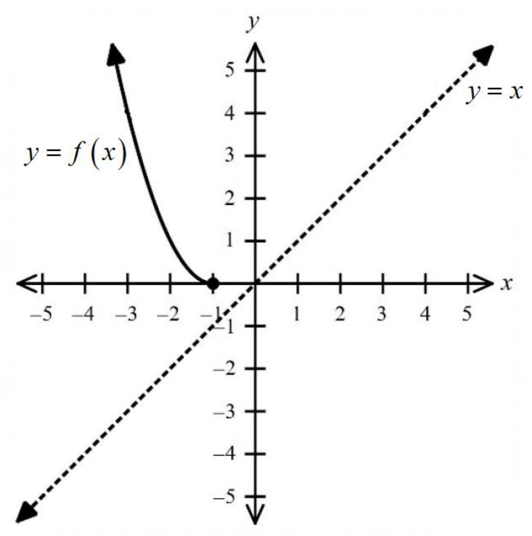
$$3t^2 - 75 = 0$$

$$t^2 - 25 = 0$$

$$\therefore t = 5 \text{ s} \quad (t \geq 0)$$

When students managed to get the numerator right, they would normally achieve the correct answer. However, if not, students messed up in using the quadratic formula.

(b) The graph of $f(x)=(x+1)^2;\ (-\infty,-1]$ is shown below, along with the line $y=x$.



(i) Write the equation of the inverse function of $f(x)$ in the form $y=f^{-1}(x)$ and state its domain. **3**

Common error:

- Forgetting to put the ±.
- Fogetting to write the domain.

Given $y=(x+1)^2$

For inverse function:

$$x=(y+1)^2$$

$$y+1=\pm\sqrt{x}$$

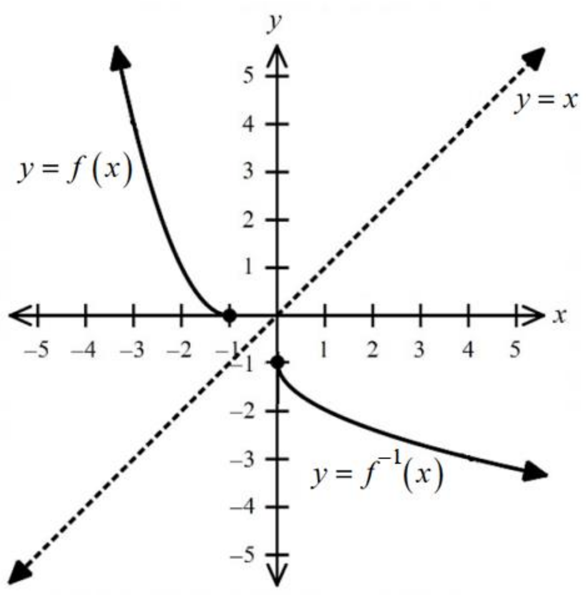
$$y=-1\pm\sqrt{x} \quad \rightarrow 1 \text{ mark}$$

Since for $f(x),\ x\in(-\infty,-1]$

$$f^{-1}(x)=-1-\sqrt{x} \quad \rightarrow 1 \text{ mark}$$

domain: $[0,\infty)$ → 1 mark

(ii) Hence, sketch the inverse function, including the original function and the line $y=x$ in your sketch. **1**



This question was done really well.

(c) (i) Prove that $\dfrac{\cos 2\theta + 1}{\sin 2\theta} = \cot\theta$. **3**

$$\text{LHS} = \frac{\cos 2\theta + 1}{\sin 2\theta}$$

$$= \frac{2\cos^2\theta - 1 + 1}{2\sin\theta\cos\theta}$$

→ 1 mark

→ 1 mark

$$= \frac{\cancel{2}\cos^{\cancel{2}}\theta}{\cancel{2}\sin\theta\cancel{\cos\theta}}$$

$$= \frac{\cos\theta}{\sin\theta}$$

→ 1 mark

$$= \cot\theta$$

$$= \text{RHS}$$

This question was done really well.

(ii) Hence, show that the exact value of cot 15° is $2+\sqrt{3}$. **1**

From (i):

$$\cot 15^\circ = \frac{\cos 2(15^\circ) + 1}{\sin 2(15^\circ)}$$

$$= \frac{\cos 30^\circ + 1}{\sin 30^\circ}$$

$$= \frac{\frac{\sqrt{3}}{2} + 1}{\frac{1}{2}}$$

$$= \sqrt{3} + 2$$

$$= 2 + \sqrt{3}$$

This question was done really well.

Although some students tried to do it the hard way for a 1-mark question.

When it is a 'hence' question, that should tell you to use the previous information.

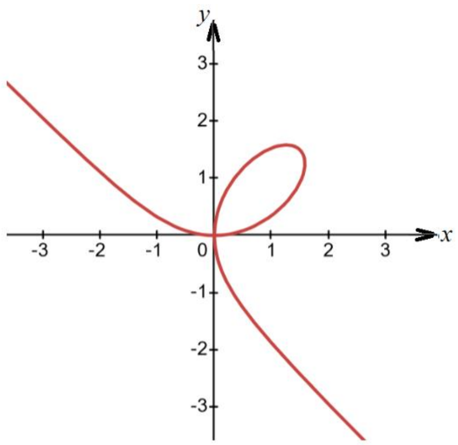
End of Question 12

Question 13 (12 marks) Use a separate Writing Booklet for Question 13.

(a) The graph shown to the right is known as the "Folium of Descartes".

This graph is defined by the parametric equations:

$$\begin{cases} x = \dfrac{3t}{1+t^3} \\ y = \dfrac{3t^2}{1+t^3} \end{cases}$$



(i) Find $\dfrac{y}{x}$ in terms of t, expressing your answer in simplest form. **1**

$$\frac{y}{x} = \frac{3t^2}{1+t^3} \div \frac{3t}{1+t^3}$$
$$= \frac{3t^2}{1+t^3} \times \frac{1+t^3}{3t}$$
$$= t$$

1 mark for arriving to the correct answer.

(ii) Deduce that the Cartesian equation of this curve is: $x^3 + y^3 = 3xy$. **2**

Given: $x = \dfrac{3t}{1+t^3}$

From (i), substitute $t = \dfrac{y}{x}$

Method 1:	Method 2:
$x = \dfrac{3\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^3}$ $x = \dfrac{\left(\frac{3y}{x}\right)(x^3)}{\left(1+\frac{y^3}{x^3}\right)(x^3)}$ $x = \dfrac{3x^2y}{x^3+y^3}$ $x^3 + y^3 = \dfrac{3x^2y}{x}$ $x^3 + y^3 = 3xy$	$y = \dfrac{3\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)^3}$ $y = \dfrac{\frac{3y^2}{\cancel{x^2}}}{\frac{x^3+y^3}{x^{\cancel{3}}}}$ $y\left(x^3+y^3\right) = 3xy^2$ $x^3 + y^3 = 3xy$

(b) The roots of $x^3+5x^2+7x-1=0$ are α, β and γ.

(i) Find $\alpha(\beta+1)+\beta(\gamma+1)+\gamma(\alpha+1)$. **2**

Given: $x^3+5x^2+7x-1=0$

$$\alpha+\beta+\gamma=-5$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=7$$ → 1 mark

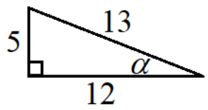
$$\alpha(\beta+1)+\beta(\gamma+1)+\gamma(\alpha+1)$$
$$=\alpha\beta+\beta\gamma+\gamma\alpha+(\alpha+\beta+\gamma)$$
$$=7+(-5)$$
$$=2$$ → 1 mark

(ii) Find $\alpha^2+\beta^2+\gamma^2$. **1**

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$=(-5)^2-2(7)$$
$$=11$$ → 1 mark

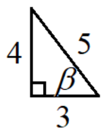
(c) Show that the exact value of $\sin\left[\cos^{-1}\left(\frac{12}{13}\right)-\tan^{-1}\left(\frac{4}{3}\right)\right]$ is $-\frac{33}{65}$. **3**

$$\alpha=\cos^{-1}\left(\frac{12}{13}\right)$$



$$\cos\alpha=\frac{12}{13} \quad \sin\alpha=\frac{5}{13}$$ → 1 mark

$$\beta=\tan^{-1}\left(\frac{4}{3}\right)$$



$$\tan\beta=\frac{4}{3} \quad \sin\beta=\frac{4}{5} \quad \cos\beta=\frac{3}{5}$$

↑ 1 mark

$$\sin(\alpha-\beta)$$
$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=\frac{5}{13}\times\frac{3}{5}-\frac{12}{13}\times\frac{4}{5}$$ → 1 mark
$$=\frac{15}{65}-\frac{48}{65}$$
$$=-\frac{33}{65}$$

(d) Detroit City is grappling with a troubling rise in pollution, with levels increasing exponentially. Over the past 2 years, from 2022, pollution has already surged by 10%. The city government has stated they will only take action once pollution has increased by 50% from 2022 levels. How long will it take for the city to reach this critical threshold? **3**

$$A=Pe^{kt}$$

when $t=2$, $A=1.1P$

1 mark→ $$1.1\cancel{P}=\cancel{P}e^{k(2)}$$
$$2k=\ln 1.1$$
$$k=\frac{1}{2}\ln 1.1$$

Find t, when $A=1.5P$

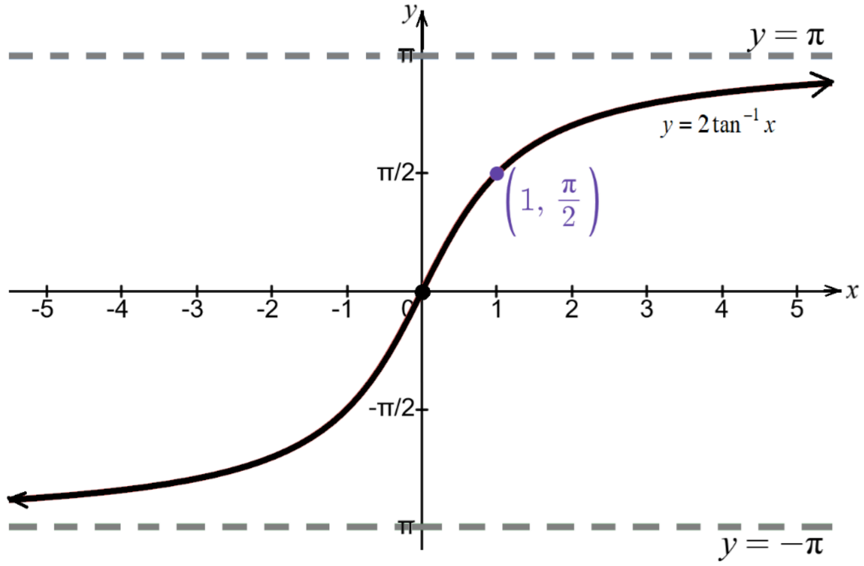
$$1.5\cancel{P}=\cancel{P}e^{kt}$$
$$kt=\ln 1.5$$
$$t=\frac{\ln 1.5}{\frac{1}{2}\ln 1.1}\approx 8.5 \text{ years}$$ → 1 mark

1 mark for some attempt to calculate *k*.

The city will reach this critical threshold in 8.5 years.

Question 14 (12 marks) Use a separate Writing Booklet for Question 14.

(a) Sketch the graph of $y = 2\tan^{-1} x$ showing intercepts and asymptotes. **2**



Common error:
Drawing the graph too quickly or roughly with the ends bending away from the asymptotes rather than getting closer to them.
Always take your time in sketching graphs to avoid losing marks.

(b) A vessel is shaped so that when the depth of water in it is x cm, the volume of the water is V cm^3 where $V = 108x + x^3$. Water is poured into the vessel at a constant rate of 30 cm^3/s. At what rate is the water level rising when its depth is 8 cm? **3**

$$\frac{dV}{dx} = 108 + 3x^2$$

$$\frac{dV}{dt} = 30 \text{ cm}^3/\text{s}$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$30 = \left(108 + 3x^2\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{30}{108 + 3x^2}$$

when $x = 8$

$$\frac{dx}{dt} = \frac{30}{108 + 3(8)^2}$$

$$= 0.1 \text{ cm/s}$$

Common error:
Students who did not start with $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ made mistakes.
But overall, this question was done well.

The water level is rising by 0.1 cm/s.

(c) Newton's law of cooling states that the rate at which an exposed body changes temperature is proportional to the difference between its own temperature and the ambient temperature of its surroundings, provided the temperature difference is small and the nature of the body's surface does not change.

Little Johnny took an ice block with temperature –16°C from the fridge, ran outside and accidentally dropped the ice block on the lawn. The air temperature is a constant 24°C.

(i) If after 30 seconds, the ice block's temperature rose to –13°C, show that $T = 24 - 40e^{-0.0026t}$. **2**

$$T = P + T_0 e^{kt}$$
when $t = 0,\ T = -16,\ P = 24$
$$-16 = 24 + T_0 e^{k(0)}$$
$$\therefore T_0 = -40$$
$$\therefore T = 24 - 40e^{kt}$$
when $t = 30,\ T = -13$
$$-13 = 24 - 40e^{k(30)}$$
$$40e^{30k} = 37$$
$$30k = \ln\left(\frac{37}{40}\right)$$
$$k = \frac{1}{30}\ln\left(\frac{37}{40}\right) \approx -0.00259\ldots$$
$$\therefore T = 24 - 40e^{-0.0026t}$$

Some students did not show how they got –40 such as shown on the left.

(ii) Determine how many seconds it will take for the ice block to reach 0°C and begin to melt. **1**

Using exact values:	Using the equation from part (i):
$T = 24 - 40e^{kt}$ $0 = 24 - 40e^{kt}$ $40e^{kt} = 24$ $kt = \ln\frac{24}{40}$ $t = \dfrac{\ln\frac{24}{40}}{\frac{1}{30}\ln\left(\frac{37}{40}\right)}$ $t \approx 196.6$ seconds	$T = 24 - 40e^{-0.0026t}$ $0 = 24 - 40e^{-0.0026t}$ $40e^{-0.0026t} = 24$ $-0.0026t = \ln\left(\frac{24}{40}\right)$ $t \approx 196.5$ s

It is always best to use the exact values in your working out. Using rounded numbers in working can sometimes lead to an incorrect final answer. In this case using k = 0.0026 was accepted since it was provided in part (i).

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to show that $\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$. **2**

$\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$ let $t = \tan\frac{\theta}{2}$ $\text{LHS} = \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$	$\text{LHS} = \frac{2}{2t}$ $= \frac{1}{t}$ $= \cot\frac{\theta}{2} = \text{RHS}$

This question was done well.

(e) Prove that $4\sin 3x \sin 2x \sin x = \sin 2x + \sin 4x - \sin 6x$. **2**

There are multiple ways to prove this identity. Two possible solutions are shown.

Method 1:	Method 2:
$\text{LHS} = 4\sin 3x \sin 2x \sin x$	$\text{LHS} = 4\sin 3x \sin 2x \sin x$
$= (4\sin 3x)\frac{1}{2}\left[\cos(2x - x) - \cos(2x + x)\right]$	$= 4\left[\frac{1}{2}\left[\cos(3x - 2x) - \cos(3x + 2x)\right]\right]\sin x$
$= 2\sin 3x(\cos x - \cos 3x)$	$= 2(\cos x - \cos 5x)\sin x$
$= [2\sin 3x \cos x] - [2\sin 3x \cos 3x]$	$= 2\cos x \sin x - 2\cos 5x \sin x$
$= \left[\sin(3x + x) + \sin(3x - x)\right] - \left[\sin 2(3x)\right]$	$= [\sin 2x] - \left[\sin(5x + x) - \sin(5x - x)\right]$
$= \sin 4x + \sin 2x - \sin 6x$	$= \sin 2x - \sin 6x + \sin 4x$
$= \sin 2x + \sin 4x - \sin 6x$	$= \sin 2x + \sin 4x - \sin 6x$
$= \text{RHS}$	$= \text{RHS}$

END OF PAPER