PLC Sydney Maths Department

Solutions for exams and assessment tasks

Ver 1

Academic Year	11	Calendar Year	2021
Course	Ext	Name of task/exam	Yearly

Section I

1. $|x-3|>4$

$x-3>4 \qquad -(x-3)>4$

$x>7 \qquad x-3<-4$

$x<-1$

$\therefore x<-1,\ x>7$ (B)

2. $y=|x|$ fails the horizontal line test. (D)

3. zeroes at 1 and -2

Equation is of the form

$y=a(x-1)^2(x+2)^3$

Two options : A or B.

a must be negative

so (B).

4. $\dfrac{7!}{2!\,3!}=420$ (A)

2 D's 3 E's

5. Eliminate A and D as these graphs need to be above/on the x-axis.

Eliminate B as the section above/on the x-axis of $f(x)$ should remain the same

$\therefore$ (C)

Section II

6. a) $P(x)=x^3-2x^2+kx+24$

(i) $\alpha+\beta+\gamma=\dfrac{-b}{a}$

$=\dfrac{--2}{1}$

$=2$

(ii) $\alpha\beta\gamma=\dfrac{-d}{a}$

$=\dfrac{-24}{1}$

$=-24$

(iii) let the roots be $\alpha, -\alpha, \beta$

$\alpha+-\alpha+\beta=2$

$\beta=2$

$\therefore$ one of the roots is 2.

$P(2)=0$

$0=2^3-2(2)^2+k(2)+24$

$0=2k+24$

$-24=2k$

$\therefore k=-12$

b) $\dfrac{3}{x-2}\cdot(x-2)^2 \leq 1\cdot(x-2)^2$

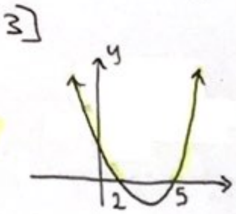
$3(x-2)\leq(x-2)^2$

$0\leq(x-2)^2-3(x-2)$

$0\leq(x-2)[x-2-3]$

$0\leq(x-2)(x-5)$

$(x-2)(x-5)\geq 0$



$x<2,\ x\geq 5$

$(x\neq 2)$

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6. cont.

c) (i) $5+7=12$ students in total.

$${}^{12}C_7 = 792$$

(ii) 4 high school students

This means 3 primary school students.

$${}^{7}C_4 \cdot {}^{5}C_3 = 350$$

Probability is $\frac{350}{792} = \frac{175}{396}$

d) $x = t+4 \quad y = 2+t^2$

(i) $t = x-4$

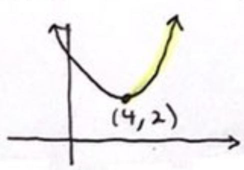
sub $t=x-4$ into $y=2+t^2$

$$y = 2+(x-4)^2$$

$$y = (x-4)^2+2$$

$$f(x) = (x-4)^2+2$$

(ii)



Largest possible domain for when $f(x)$ is monotonically increasing: $x \geq 4$

(iii) $f: y=(x-4)^2+2$

Inverse relation:

$$x = (y-4)^2+2$$

$$x-2 = (y-4)^2$$

$$y-4 = \pm\sqrt{x-2}$$

$$y = 4 \pm \sqrt{x-2}$$

$\therefore f^{-1}(x) = 4+\sqrt{x-2}$ as $y \geq 4$

(Domain of $f(x)$ is $x \geq 4$ so range of $f^{-1}(x)$ is $y \geq 4$).

6. cont.

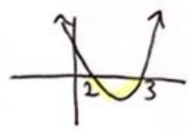
e) $x = \frac{t^3}{3} - \frac{5t^2}{2} + 6t + 4$

$$\frac{dx}{dt} = \frac{3t^2}{3} - \frac{10t}{2} + 6$$

$$= t^2 - 5t + 6$$

$$t^2 - 5t + 6 < 0$$

$$(t-3)(t-2) < 0$$



$$\therefore 2 < t < 3$$

7. a) $P(x) = x^3 - 3x^2 + kx - 5$

$$P(-2) = 7$$

$$7 = (-2)^3 - 3(-2)^2 + k(-2) - 5$$

$$7 = -2k - 25$$

$$32 = -2k$$

$$k = -16$$

b) $y = 3\cos^{-1} 2x$

D: $-1 \leq 2x \leq 1$

$$-\tfrac{1}{2} \leq x \leq \tfrac{1}{2}$$

R: $0 \leq \cos^{-1} 2x \leq \pi$

$$0 \leq 3\cos^{-1} 2x \leq 3\pi$$

$$0 \leq y \leq 3\pi$$

7. cont.

c) $^{n+1}C_r \div {^nC_{r-1}} = \frac{n+1}{r}$

$LHS = {^{n+1}C_r} \div {^nC_{r-1}}$

$= \frac{(n+1)!}{r!(n+1-r)!} \div \frac{n!}{(r-1)!(n-(r-1))!}$

$= \frac{(n+1)\cancel{n!}}{r\cancel{(r-1)!}\cancel{(n+1-r)!}} \times \frac{\cancel{(r-1)!}\cancel{(n-r+1)!}}{\cancel{n!}}$

$= \frac{n+1}{r}$

$= RHS$

d) (i) $(9-1)! = 8! = 40320$

(ii) Total arrangements − Number of arrangements they do sit together

$= 8! - (2\times 7!)$

$= 40320 - 10080$

$= 30240$

OR

$\underline{1} \times \underline{6} \times \underline{7!} = 30240$

1 seat Steve or Danny.

6 the other has 6 seats he can select where they are not sitting next to each other.

7! arrangements for the remaining 7 people.

e) (i) $\tan 2A$

$= \tan(A+A)$

$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$

$= \frac{2\tan A}{1-\tan^2 A}$

(ii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ from (i)

$\tan 2(67.5°) = \frac{2\tan 67.5°}{1-\tan^2 67.5°}$

$\tan 135° = \frac{2\tan 67.5°}{1-\tan^2 67.5°}$

$-1 = \frac{2\tan 67.5°}{1-\tan^2 67.5°}$

let $u = \tan 67.5°$

$-1 = \frac{2u}{1-u^2}$

$-(1-u^2) = 2u$

$-1+u^2 = 2u$

$u^2 - 2u - 1 = 0$

$u^2 - 2u + 1^2 = 1 + 1^2$

$(u-1)^2 = 2$

$u - 1 = \pm\sqrt{2}$

$u = 1 \pm \sqrt{2}$

$\tan 67.5° = 1+\sqrt{2}$ as $\tan 67.5° > 0$

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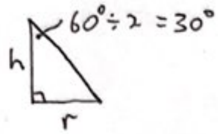
7. cont.

f) (i)



$$\tan 30° = \frac{r}{h}$$
$$h \cdot \frac{1}{\sqrt{3}} = \frac{r}{h} \cdot h$$
$$\frac{h}{\sqrt{3}} = r$$
$$\therefore r = \frac{h}{\sqrt{3}}$$

(ii) $V = \frac{1}{3}\pi r^2 h$
$$= \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$$
$$= \frac{1}{3}\pi \cdot \frac{h^2}{3} \cdot h$$
$$\therefore V = \frac{\pi h^3}{9}$$

(iii) $\frac{dh}{dt} = ?$ when $h = 3$ m

$$\frac{dV}{dt} = 0.5 \text{ m}^3/\text{s}$$
$$V = \frac{\pi h^3}{9}$$
$$\frac{dV}{dh} = \frac{3\pi h^2}{9}$$
$$= \frac{\pi h^2}{3}$$

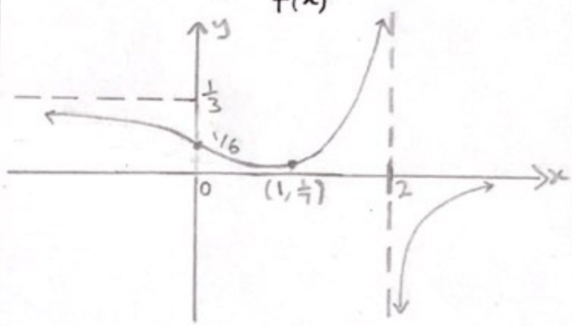
when $h = 3$, $\frac{dV}{dh} = \frac{\pi \times 3^2}{3} = 3\pi$

7. f) (iii) cont.

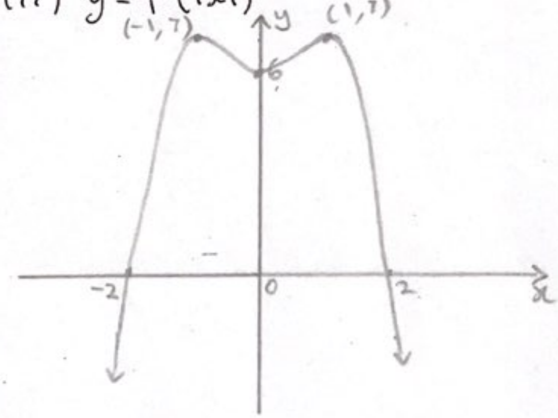
$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$
$$= 0.5 \times \frac{1}{3\pi}$$
$$= \frac{1}{2} \times \frac{1}{3\pi}$$
$$= \frac{1}{6\pi}$$

$\therefore$ height of pile is increasing at a rate of $\frac{1}{6\pi}$ m/s.

8. a) (i) $y = \frac{1}{f(x)}$



(ii) $y = f(|x|)$



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8. cont.

b) (i) $T = A + Be^{kt}$

$\frac{dT}{dt} = k.Be^{kt}$, $Be^{kt} = T - A$

$= k(T-A)$

(ii) $A = 25°C$ (surrounding air temperature)

$t=0$, $T = 5°C$

$5 = 25 + Be^{0}$

$5 = 25 + B$

$B = -20$

So $T = 25 - 20e^{kt}$

$t = 20$, $T = 15°C$

$15 = 25 - 20e^{k(20)}$

$-10 = -20e^{20k}$

$e^{20k} = \frac{1}{2}$

$20k = \ln\left(\frac{1}{2}\right)$

$k = \frac{1}{20}\ln\left(\frac{1}{2}\right)$

a further 50 minutes,
$t = 20 + 50 = 70$ minutes

$t = 70$, $T = 25 - 20e^{\left(\frac{1}{20}\ln\frac{1}{2}\right)(70)}$

$= 23.2322...$

$\doteqdot 23°C$

8. cont.

c) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (property of inverse trig functions).

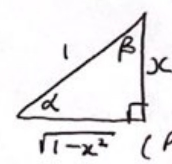
Let $\alpha = \sin^{-1}x$ and $\beta = \cos^{-1}x$.

From the property above, α and β are complementary angles.

From above,

$\alpha = \sin^{-1}x$ and $\beta = \cos^{-1}x$

$\sin\alpha = x$ $\cos\beta = x$



(Pythagoras' theorem)

$LHS = \sin(\sin^{-1}x - \cos^{-1}x)$

$= \sin(\alpha - \beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= x.x - \left(\sqrt{1-x^2}.\sqrt{1-x^2}\right)$

$= x^2 - (1-x^2)$

$= x^2 - 1 + x^2$

$= 2x^2 - 1$

$= RHS$

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8. cont.

d) (i) $t = \tan\frac{\theta}{2}$

$\sin\theta = \frac{2t}{1+t^2}$ $\quad \cos\theta = \frac{1-t^2}{1+t^2}$

(reference sheet)

$LHS = \frac{1}{2}(3\cos\theta + 4\sin\theta + 5)$

$= \frac{1}{2}\left(3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) + 5\right)$

$= \frac{1}{2}\left[\frac{3(1-t^2) + 4(2t) + 5(1+t^2)}{1+t^2}\right]$

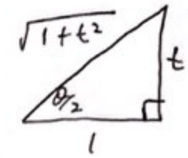
$= \frac{(3 - 3t^2 + 8t + 5 + 5t^2)}{2(1+t^2)}$

$= \frac{(2t^2 + 8t + 8)}{2(1+t^2)}$

$= \frac{2(t^2 + 4t + 4)}{2(1+t^2)}$

$= \frac{(t+2)^2}{1+t^2}$

(Pythagoras' theorem)



$t = \tan\frac{\theta}{2}$

$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$

$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$

$RHS = \left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)^2$

$= \left(\frac{t}{\sqrt{1+t^2}} + 2\left(\frac{1}{\sqrt{1+t^2}}\right)\right)^2$

$= \left(\frac{t+2}{\sqrt{1+t^2}}\right)^2$

$= \frac{(t+2)^2}{(\sqrt{1+t^2})^2}$

$= \frac{(t+2)^2}{\sqrt{1+t^2}}$

$= LHS$

d) (ii) $\theta = \frac{\pi}{3}$

$\frac{1}{2}\left(3\cos\frac{\pi}{3} + 4\sin\frac{\pi}{3} + 5\right) = \left(\sin\left(\frac{\frac{\pi}{3}}{2}\right) + 2\cos\left(\frac{\frac{\pi}{3}}{2}\right)\right)^2$

$\frac{1}{2}\left(3\left(\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right) + 5\right) = \left(\sin\frac{\pi}{6} + 2\cos\frac{\pi}{6}\right)^2$

$\frac{1}{2}\left(\frac{13}{2} + 2\sqrt{3}\right) = \left(\frac{1}{2} + 2\frac{\sqrt{3}}{2}\right)^2$

$\frac{13 + 4\sqrt{3}}{2} = 2\left(\frac{1}{2} + \sqrt{3}\right)^2$

$13 + 4\sqrt{3} = 4\left(\frac{1 + 2\sqrt{3}}{2}\right)^2$

$13 + 4\sqrt{3} = \frac{4(1 + 2\sqrt{3})^2}{2^2}$

$13 + 4\sqrt{3} = (1 + 2\sqrt{3})^2$

$\therefore$ the two square roots of $13 + 4\sqrt{3}$ are

$\pm\sqrt{(1+2\sqrt{3})^2}$

$= \pm(1 + 2\sqrt{3})$

PRESBYTERIAN LADIES' COLLEGE
PLC SYDNEY
1888

2021 Year 11 Yearly Examination
Weighting 40%

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 1.5 hours
- Write using black pen
- Calculators approved by NESA may be used
- The HSC reference sheet may be used
- For questions in Section II, show relevant mathematical reasoning and/or calculation
- **This is an open book examination.**

Total marks: 50

Section I – 5 marks

- Attempt Questions 1–5
- Allow about 10 minutes for this section

Section II – 45 marks

- Attempt Questions 6–8
- Allow about 1 hour and 20 minutes for this section

Marks awarded

Section I	/5
Section II	/45
Total	**/50**

Section I: Multiple Choice (5 marks)

Marks

Start a new writing booklet.

Attempt questions 1 – 5.

Write the correct alternative (A, B, C or D) for each question.

Allow about 10 minutes for this section.

1. Which of the following is the solution to $|x-3|>4$? **1**

(A) $-1<x<7$

(B) $x<-1$ or $x>7$

(C) $x<-7$ or $x>7$

(D) $x>1$ or $x<-1$

2. Which one of these functions has an inverse relation that is not a function? **1**

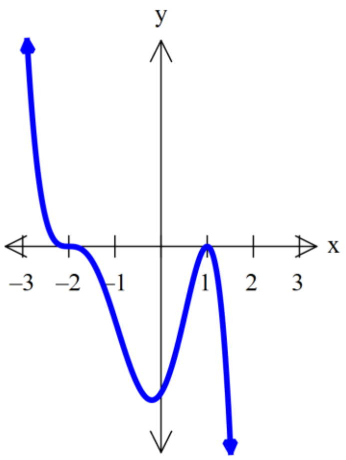
(A) $y=x^3$

(B) $y=\sqrt{x}$

(C) $y=e^x$

(D) $y=|x|$

3. Which equation best represents the following graph? **1**



(A) $y=(x-1)^2(x+2)^3$

(B) $y=-(x-1)^2(x+2)^3$

(C) $y=(x+1)^2(x-2)^3$

(D) $y=-(x+1)^2(x-2)^3$

4. How many seven-letter arrangements can be made using the letters of the word DELETED? **1**

(A) 420

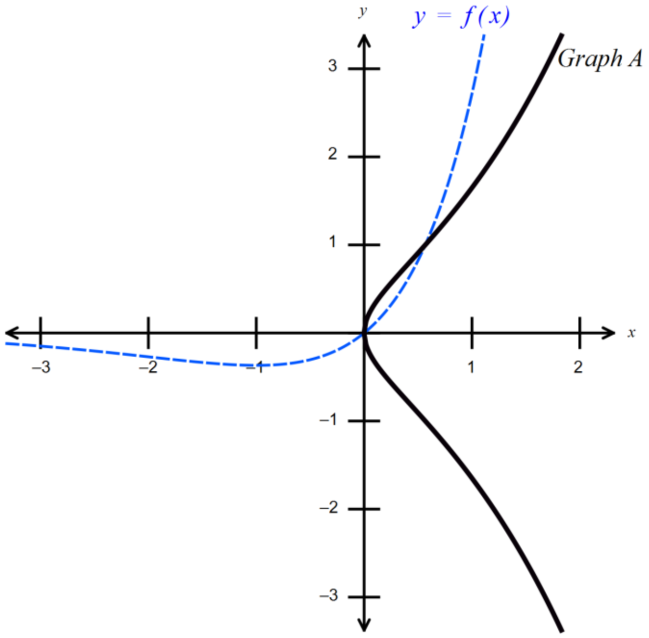
(B) 840

(C) 2520

(D) 5040

5. The graph of $y = f(x)$ is shown on the diagram below in blue. **1**

Which equation best represents the transformation that produced *Graph A* shown below?



(A) $y = |f(x)|$

(B) $|y| = f(x)$

(C) $y^2 = f(x)$

(D) $y = \sqrt{f(x)}$

END OF SECTION I

Section II (45 marks)

Start a new writing booklet for each question.

Attempt questions 6 – 8.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Allow about 1 hour and 20 minutes for this section.

Question 6 (15 marks) **Start a new writing booklet.**

a) The polynomial $P(x) = x^3 - 2x^2 + kx + 24$ has roots α, β and γ.

(i) Find the value of $\alpha + \beta + \gamma$. **1**

(ii) Find the value of $\alpha\beta\gamma$. **1**

(iii) It is known that two of the roots are equal in magnitude but opposite in sign. Find the value of k. **2**

b) Solve $\frac{3}{x-2} \leq 1$. **3**

c) A seven-person committee is to be chosen at random from 5 primary school students and 7 high school students.

(i) In how many ways can this committee be chosen? **1**

(ii) What is the probability that the committee will consist of exactly 4 high school students? **1**

Question 6 continues on the next page

d) A parabola is defined by the parametric equations $x = t + 4$ and $y = 2 + t^2$.

(i) Show that the Cartesian equation of the parabola is $y = (x - 4)^2 + 2$. **1**

Rewrite the equation of the parabola as $f(x) = (x - 4)^2 + 2$.

(ii) Write the largest possible restricted domain of $f(x)$ such that $f(x)$ is monotonically increasing. **1**

(iii) Find the equation of the inverse function $f^{-1}(x)$ on the domain found in part (ii). Justify your answer. **2**

e) The displacement x metres from the origin at time t seconds of a particle travelling in a straight line is given by $x = \frac{t^3}{3} - \frac{5t^2}{2} + 6t + 4$. **2**

For what values of t is the velocity of the particle negative?

END OF QUESTION 6

Question 7 (16 marks) **Start a new writing booklet.**

a) Let $P(x) = x^3 - 3x^2 + kx - 5$, where k is a rational number. **2**

When $P(x)$ is divided by $(x + 2)$, the remainder is 7. Find k.

b) State the domain and range of $y = 3\cos^{-1}(2x)$. **2**

c) Show that ${}^{n+1}C_r \div {}^{n}C_{r-1} = \frac{n+1}{r}$. **2**

d) Nine people are to be seated at a round table.

(i) How many seating arrangements are possible? **1**

(ii) Two people, Steve and Danny, refuse to sit next to each other. **2**

How many seating arrangements are then possible?

e) (i) Write $\tan 2A$ in terms of $\tan A$. **1**

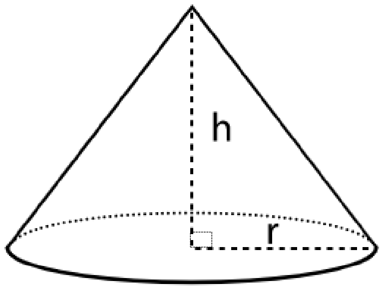
(ii) Hence find the exact value of $\tan\left(67\frac{1}{2}\right)°$. **2**

Question 7 continues on the next page

f) Grain is poured at a constant rate of 0.5 cubic metres per second. It forms a conical pile, with the angle at the apex of the cone equal to $60°$.

The height of the pile is h metres, and the radius of the base is r metres.

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$, where V is the volume, r is the radius and h is the height.



(i) Show that $r = \frac{h}{\sqrt{3}}$. **1**

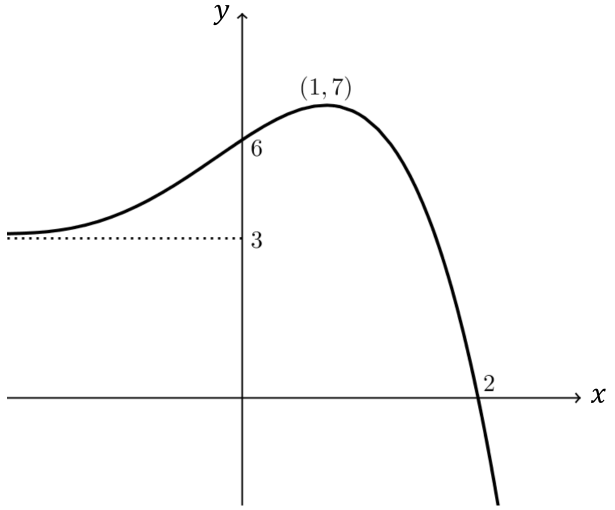
(ii) Show that V, the volume of the pile, is given by $V = \frac{\pi h^3}{9}$. **1**

(iii) Hence find the rate at which the height of the pile is increasing when the height of the pile is 3 metres. Leave your answer in exact form. **2**

END OF QUESTION 7

Question 8 (14 marks) **Start a new writing booklet.**

a) The following diagram shows the graph of $y = f(x)$.



On separate diagrams, draw graphs of each of the following, carefully showing any asymptotes and all key points.

(i) $y = \frac{1}{f(x)}$ **2**

(ii) $y = f(|x|)$ **2**

b) The rate at which a body warms in air is proportional to the difference between its temperature, T, and the constant temperatue, A, of the surrounding air.

This rate can be expressed by the differential equation $\frac{dT}{dt} = k(T - A)$, where t is the time in minutes and k is a constant.

(i) Show that $T = A + Be^{kt}$, where B is a constant, is a solution of the differential equation. **1**

(ii) An object warms from 5°C to 15°C in 20 minutes. The temperature of the surrounding air is 25°C. **3**

Find the temperature of the object after a further 50 minutes have elapsed. Give your answer to the nearest degree.

Question 8 continues on the next page

Question 8 continued

c) Given that $\sin^{-1} x$ and $\cos^{-1} x$ are acute, show that **2**

$$\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$$

d) (i) Use the substitution $t = \tan\frac{\theta}{2}$ to prove the identity **3**

$$\frac{1}{2}(3\cos\theta + 4\sin\theta + 5) = \left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)^2$$

(ii) Hence use the substitution $\theta = \frac{\pi}{3}$ to find the two square roots of **1**

$(13 + 4\sqrt{3})$. Leave your answers in exact form.

END OF EXAM