
Student Number

SCEGGS Darlinghurst

2023 **PRELIMINARY COURSE SEMESTER 2 EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 1 hour and 30 minutes
- Write using black pen
- NESA approved calculators may be used
- A NESA Reference Sheet is provided on a separate sheet
- For Questions 8–10, show relevant mathematical reasoning and/or calculations

Total marks: 50

Section I – Multiple Choice – 7 marks

- Attempt Questions 1–7
- Allow about 12 minutes for this section

Section II – Short Answer – 43 marks

- Attempt Questions 8–10
- Allow about 1 hour and 18 minutes for this section

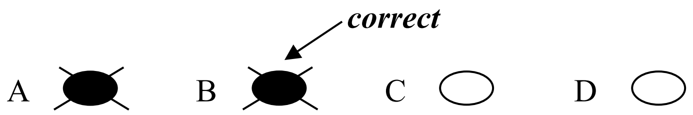
Section	Questions	Possible Marks	Awarded Marks
I Multiple Choice	1–7	7	
II Short Answer	8–10	43	
TOTAL		**50**	

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = A. 2 B. 6 C. 8 D. 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.



Section I – Multiple Choice

7 marks
Attempt Questions 1–7
Allow about 12 minutes for this section

Use the multiple choice answer sheet provided.

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1 The sum of two roots of $4x^4 - 56x^3 + 239x^2 - 316x + 105 = 0$ is 12. What is the sum of the other two roots?

A. $-\frac{14}{3}$

B. -2

C. 2

D. $\frac{14}{3}$

2 Which of the following is a possible x value of the intersection between the curve $f(x) = x^2 - 2x - 4$ and its inverse?

A. -4

B. 1

C. 2

D. 4

3 What are the solutions of $\frac{2x}{x-3} \geq 1$?

A. $x \geq -3$

B. $-3 \leq x \leq 3$

C. $x \leq -3$ or $x > 3$

D. $x \leq -3$ or $x \geq 3$

4 The letters of the word REPETITION are arranged at random in a row. How many arrangements will have vowels and consonants alternating?

A. 1800

B. 3600

C. 14400

D. 28800

5 The members of a club votes for a new president. There were 5 candidates for the position of president and 368 members voted. Each member voted for one candidate only.

One candidate received more votes than any other candidate and so became the new president. What is the smallest number of votes the new president could have received?

A. 73

B. 74

C. 75

D. 185

6 Given $\tan\theta = A$, what is the correct expression for $\tan\left(\dfrac{\pi}{4} - \theta\right)$?

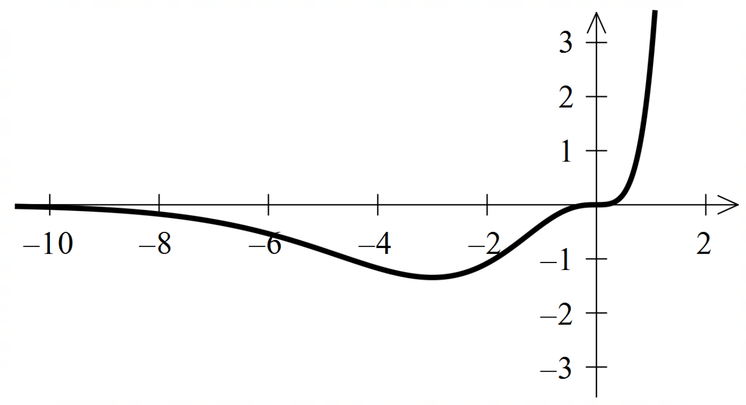
A. A

B. $\dfrac{1}{A}$

C. $1 - A$

D. $\dfrac{1-A}{1+A}$

7 Given the graph of $y = e^x x^3$, how many solutions are there to $e^{|x|}|x|^3 = 1$?



A. 0

B. 1

C. 2

D. 3

End of Section I

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Section II

43 marks
Attempt Questions 8–10
Allow about 1 hour and 18 minutes for this section

Answer in the writing booklet provided. Extra writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 8 (14 marks)

(a) Solve $|x-3| \leq 5$. **2**

(b) Using the substitution $t = \tan\left(\dfrac{\theta}{2}\right)$, express $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$ in terms of t. **2**

(c) The polynomial $P(x) = 2x^3 - x^2 - 16x + a$ has $(x+3)$ as a factor. **1**

(i) Find the value of a.

(ii) Hence, or otherwise, factorise $P(x)$ into linear factors. **1**

(d) Show that sin 165°−sin 75° has an exact value of $-\dfrac{1}{\sqrt{2}}$. **2**

(e) Consider the parametric equations below. **2**

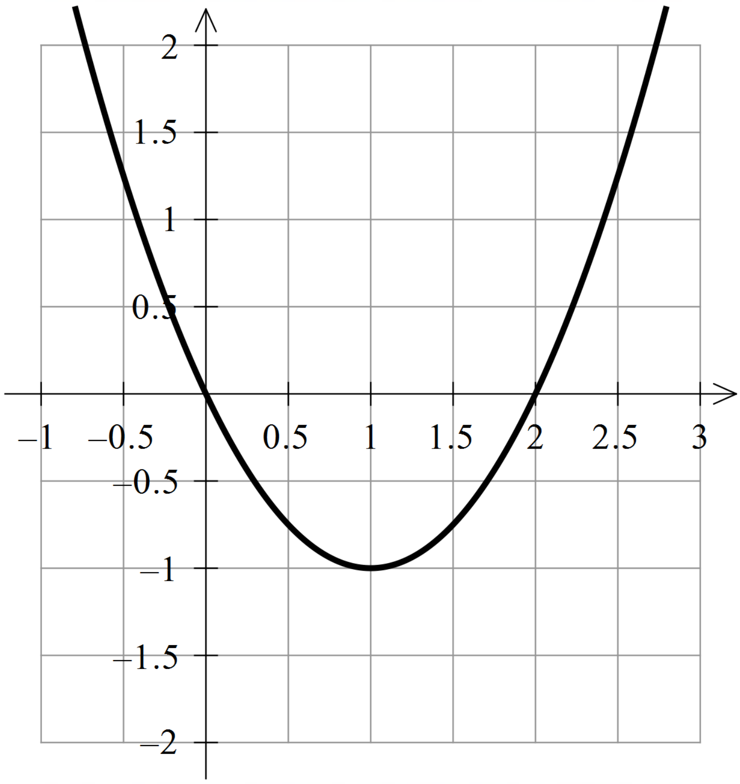
$$x = \sin t$$
$$y = \sin 2t$$

Find the Cartesian equation.

Question 8 continues on the next page

Question 8 (continued)

(f) The graph of $y = f(x)$ is shown below.
A separate sheet has been provided for your responses.



(i) Sketch the graph of $y = \dfrac{1}{f(x)}$. **2**

(ii) Sketch the graph of $y = f(x) + x^2$. **2**

End of Question 8

Question 9 (14 marks)

(a) Consider the function $y = 4 \arccos x$.

(i) Carefully sketch the graph of $y = 4\arccos x$, showing all important features. **1**

(ii) Hence, or otherwise, find the range of $y^2 = 4\arccos x$. **1**

(b) A basketball team has 15 players. They have 4 players that play the position of point guard.

(i) How many different teams of 5 can be picked which contains exactly 2 point guards? **1**

(ii) What is the probability that if a team of 5 is picked at random, it will contain one point guard? **2**

(iii) What is the probability that if a team of 5 is picked at random, it will contain at least 3 point guards? **2**

(c) Consider the function $f(x) = \sqrt{2x-5}$, which has an inverse function.

(i) State the domain and range of the function $y = f(x)$. **2**

(ii) Find the equation of the inverse function, stating any relevant restrictions. **2**

(d)

(i) Show that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$. **2**

(ii) Hence, using the result from (i), find the exact value of tan 15°. **1**

End of Question 9

Question 10 (15 marks)

(a) Prove Pascal's identity, ${}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$ given that n and r are natural numbers, where $n > r$. **2**

(b) Find the term independent of x in the expansion $\left(2x^2 - \dfrac{1}{x}\right)^{12}$. **2**

(c) An alien alphabet contains n letters, where α is one of them. If a three letter word is made (where there is no repetition of any of the letters) and α must be included, there are 918 possible words. How many letters are there in this alien alphabet including α? **3**

(d) Solve the following equation. **3**

$$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x - 1)$$

(e) Four letters of *A,A,B,C,D* and *E* are arranged in a circle. How many different arrangements are there? **3**

(f) It is given that **2**

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$$

Do not prove this.

There are 23 students in a class. All 23 students are to be randomly divided up into two groups. There must be at least one student in a group.

How many ways is this possible?

End of the paper

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MULTIPLE CHOICE

Q1) $\alpha+\beta+\gamma+\delta = -\left(\frac{-56}{4}\right)$

$12+\gamma+\delta = 14$

$\therefore \gamma+\delta = 2$ (C)

Q2) $f(x) = x^2-2x-4$

Solving simultaneously with $y = x$,

$x = x^2-2x-4$

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$\therefore x = 4, -1$ (D)

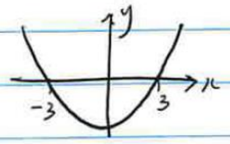
Q3) $\frac{2x}{x-3} \geq 1$

$2x(x-3) \geq (x-3)^2$

$2x(x-3)-(x-3)^2 \geq 0$

$(x-3)(2x-(x-3)) \geq 0$

$(x-3)(x+3) \geq 0$



$\therefore x \leq -3$ or $x \geq 3$

but $x \neq 3$

$\therefore x \leq -3$ or $x > 3$ (C)

Q4) REPTION

E TI

5 vowels, 5 consonants.

$\frac{5! \times 5!}{(2!)^3} \times 2 = 3600$ (B)

Q5) $368 \div 5 = 73$ r 3

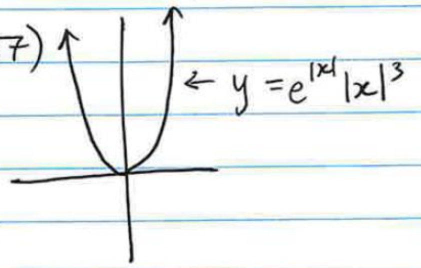
75 votes (C)

Q6) $\tan\left(\frac{\pi}{4}-\theta\right)$

$= \frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}$

$= \frac{1-A}{1+A}$ (D)

Q7)



So 2 solutions for $e^{|x|}|x|^3 = 1$

Question 8

method A

a) $|x-3| \le 5$

Think

$-5 \quad 0 \quad 5$

distance from origin.

$-5 \le x-3 \le 5$

$-2 \le x \le 8$ ✓✓

method B

If you split it, you must solve the cases and then combine them.

$x-3 \le 5$ when $x-3 \ge 0$

$x \le 8$ $\qquad$ $x \ge 3$

$3 \quad 8$

$x-3 \ge -5$ when $x-3 \le 0$

$x \ge -2$ $\qquad$ $x \le 3$

$-2 \quad 3$

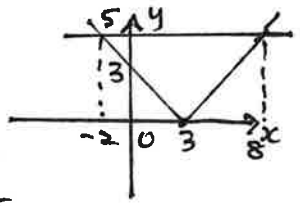
Now combine the solutions

$-2 \quad 3 \quad 8$

$\therefore -2 \le x \le 8$

Method C Graphically.

Draw
$\begin{cases} y = |x-3| \\ y = 5 \end{cases}$



solve for intersection

$x-3=5$

$-(x-3)=+5$ or $x-3=5$

$x=-2$ $\qquad$ $x=8$

$\therefore |x-3| \le 5$ (below horizontal line)

for $-2 \le x \le 8$

b) READ THE t-results from the REFERENCE SHEET.

$$\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$$

$$= \frac{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}$$

watch out for this sign.

Multiply top & bottom by $(1+t^2)$ is the quickest method.

$$= \frac{(1+t^2)+2t+1-t^2}{(1+t^2)+2t-(1-t^2)}$$

sign ✓

$$= \frac{1+t^2+2t+1-t^2}{1+t^2+2t-1+t^2}$$

$$= \frac{2+2t}{2t+2t^2}$$

factorise now & cancel.

$$= \frac{2(1+t)}{2t(1+t)}$$

$$= \frac{1}{t}$$ ✓

c) i) Since $(x+3)$ is a factor then solve $P(-3)=0$

$P(-3) = 2\times(-3)^3-(-3)^2-16(-3)+a = 0$

$-54-9+48+a=0$

$-15+a=0$

$a=15$ ✓

ii)

$$\begin{array}{r} 2x^2-7x+5 \\ x+3 \overline{)\,2x^3-x^2-16x+15} \\ \underline{2x^3+6x^2} \qquad\qquad \\ -7x^2-16x \qquad \\ \underline{-7x^2-21x} \qquad \\ 5x+15 \\ \underline{5x+15} \\ 0 \end{array}$$

must be zero since a factor

$\therefore P(x) = (x+3)(2x^2-7x+5)$

linear & quadratic factors so go further

$= (x+3)(2x-5)(x-1)$

these are called linear factors. ✓

Method A REF SHEET Change to a product

d) $\sin 165° - \sin 75°$

$= \sin(\overset{A}{120}+\overset{B}{45})° - \sin(\overset{A}{120}-\overset{B}{45})°$

$= 2\cos 120° \cos 45°$

$= 2 \times -\frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= -\frac{1}{\sqrt{2}}$ ✓✓

from Reference sheet.

$$\boxed{\cos A \cos B = \tfrac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]}$$

d) Method B Expansions.

$\sin 165° = \sin(120° + 45°)$
$= \sin 120° \cos 45° + \cos 120° \sin 45°$
$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + -\frac{1}{2} \times \frac{1}{\sqrt{2}}$
$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\sin 75° = \sin(30° + 45°)$
$= \sin 30° \cos 45° + \cos 30° \sin 45°$
$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$
$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

$\therefore \sin 165° - \sin 75°$
$= \frac{\sqrt{3}-1}{2\sqrt{2}} - \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$ negative sign!
$= \frac{\sqrt{3}-1-1-\sqrt{3}}{2\sqrt{2}}$
$= \frac{-2}{2\sqrt{2}}$
$= -\frac{1}{\sqrt{2}}$.

✓✓

e)
$x = \sin t$ ①
$y = \sin 2t$ ②

rearrange ①
$t = \sin^{-1} x$ where
$-1 \le x \le 1$
$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

sub in ②
$\therefore y = \sin(2\sin^{-1} x)$

This is a cartesian form but you should consider the domain too. and range is $-1 \le y \le 1$.

✓✓

Another version

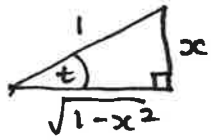
double angle
$\boxed{\sin 2A = 2\sin A \cos A}$

$y = \sin(2\sin^{-1} x)$
$= 2\sin(\sin^{-1} x)\cos(\sin^{-1} x)$
$= 2x\cos(\sin^{-1} x)$ ✓✓

Algebraic version.

$\frac{x}{1} = \sin t$

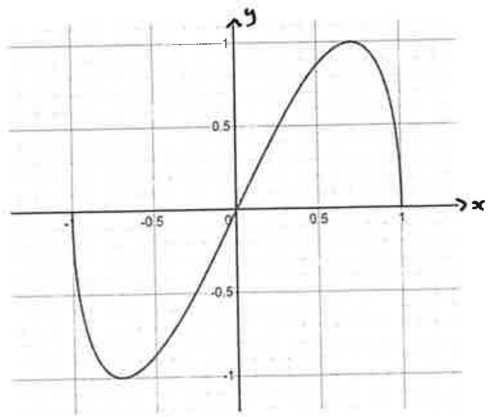
$\therefore \cos t = \sqrt{1-x^2}$



← By Pythagoras.

$\therefore y = \sin 2t$
$= 2\sin t \cos t$
$= 2x\sqrt{1-x^2}$ ✓✓

domain $-1 \le x \le 1$
range $-1 \le y \le 1$



This is what the curve looks like.

$y = \sin(2\sin^{-1} x)$
or
$y = 2x\sqrt{1-x^2}$

Student Number

Answer Page

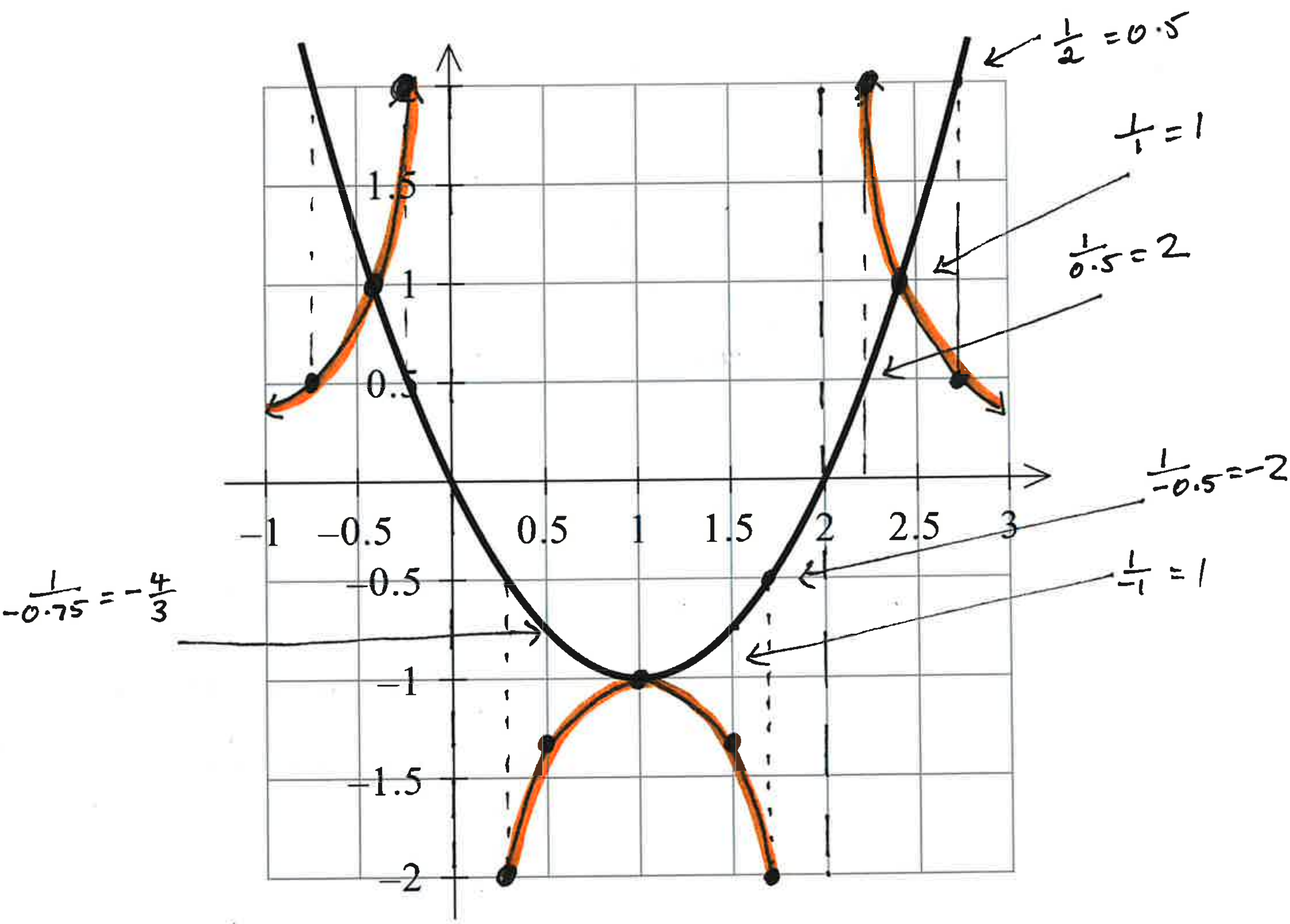
Write your Student Number at the top of this page.

Question 8

(f) Using the graph of $y = f(x)$

Vertical asymptotes at $x=0$ or $x=2$.

(i) Sketch the graph of $y = \dfrac{1}{f(x)}$. **2**



You must be accurate to gain 2 marks.

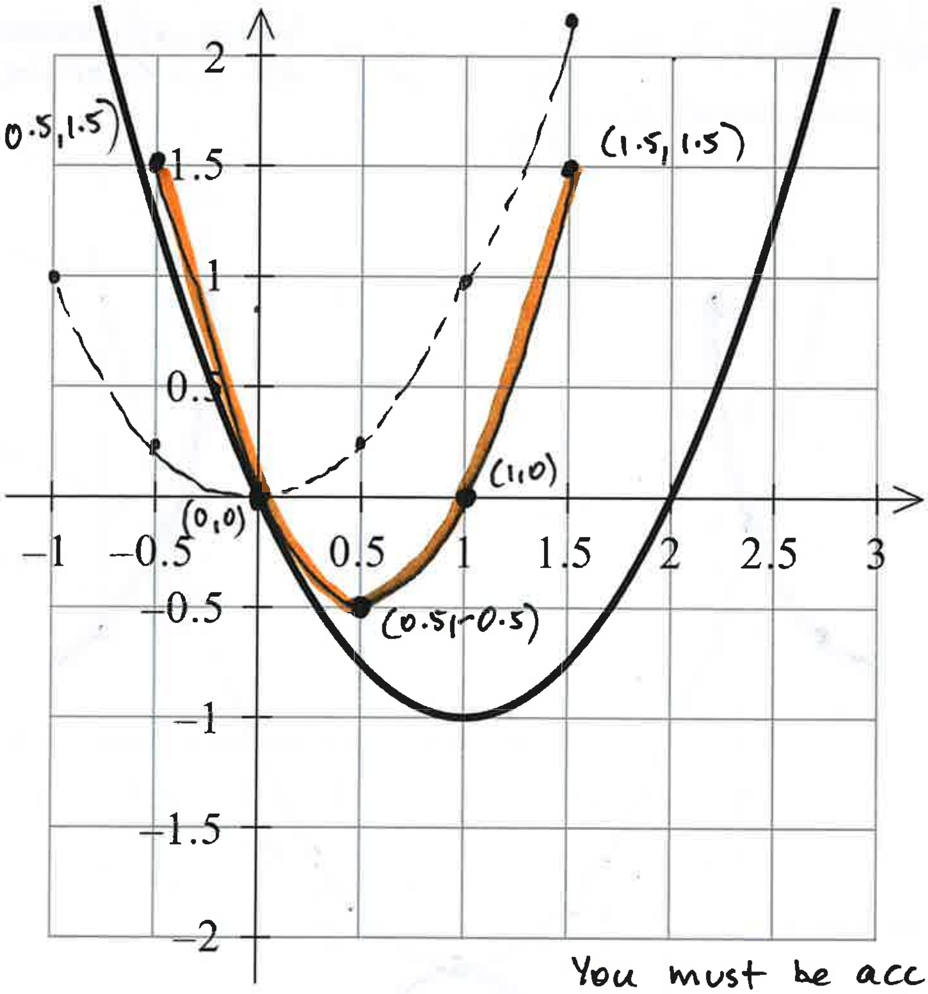
You must have more than 2 points correct for one mark.

Question 8 (f) continues on the next page

Question 8 (f) (continued)

(ii) Sketch the graph of $y = f(x) + x^2$. **2**

Draw $y = x^2$ first. Then add y values



You must be accurate.

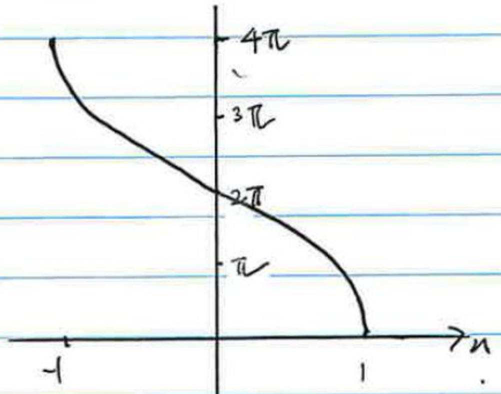
You must have more than 2 points correct for one mark.

End of Question 8 (f)

Question 9

(a) $y = 4\arccos x$

(i)



As this was only worth 1 mark, it was generously marked. Students are recommended to work on the shape of their inverse trigonometric graphs, making sure there is a vertical tangent at end point.

(ii) $-2\sqrt{\pi} \le y \le 2\sqrt{\pi}$

(b) (i) ${}^4C_2 \times {}^{11}C_3 = 990$

(ii) $\dfrac{{}^4C_1 \times {}^{11}C_4}{{}^{15}C_5} = \dfrac{40}{91}$

(iii) $\dfrac{{}^4C_3 \times {}^{11}C_2 + {}^4C_4 \times {}^{11}C_1}{{}^{15}C_5} = \dfrac{1}{13}$

(c) (i) $f(x) = \sqrt{2x-5}$

Domain is $[\frac{5}{2}, \infty)$

Range is $[0, \infty)$

(ii) The inverse has equation

$x = \sqrt{2y-5}$

$x^2 = 2y - 5$

$y = \frac{1}{2}(x^2+5)$

where $x \in [0, \infty)$

Some students wrote down the range. This is not enough to determine which part of the graph is the inverse. If you sketch it, the range where domain is [infinity, 0] is the same as the range where the domain is [0,infinity]. Some students wrong both, but is encouraged to review as to why only the domain will give the correct part of the graph.

(d) (i) $\text{LHS} = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

$= \dfrac{2\sin\theta\cos\theta}{1+\cos^2\theta-\sin^2\theta}$

$= \dfrac{2\sin\theta\cos\theta}{\cos^2\theta + \cancel{\sin^2\theta} + \cos^2\theta - \cancel{\sin^2\theta}}$

$= \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta = \text{RHS}$

(ii) let $\theta = 15^\circ$

$\tan 15^\circ = \dfrac{\sin 30}{1+\cos 30}$

$= \dfrac{(\frac{1}{2})}{1+(\frac{\sqrt{3}}{2})}$

$= \dfrac{1}{2+\sqrt{3}}$

Although some students answered the question correctly, they were not awarded full marks as they did not use the result from (i). Since the question say 'hence', the students are expected to use the result from (i).

Question 10

(a) RTP: ${}^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

$RHS = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

$= \frac{(n-1)!}{(k-1)!(n-1-k+1)!} + \frac{(n-1)!}{k!(n-1-k)!}$

$= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$

$= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)(n-k-1)!}$

$= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!}$

$= \frac{(n-1)!(k+n-k)}{k!(n-k)!}$

$= \frac{n!}{k!(n-k)!} = {}^{n}C_k = LHS$

(b) $(2x^2 - \frac{1}{x})^{12}$

General term $= \binom{12}{k}(2x^2)^{12-k}(-1)^k(\frac{1}{x})^k$

$= \binom{12}{k} 2^{12-k}(-1)^k x^{24-2k} x^{-k}$

$= \binom{12}{k} 2^{12-k}(-1)^k x^{24-3k}$

For independent term,

$24 - 3k = 0$

$k = 8$

$\therefore$ Term independent of x

$= \binom{12}{8} 2^{12-8}(-1)^8$

$= 7920$

(c) ${}^{n-1}C_2 \times 3! = 918$

$\frac{(n-1)!}{2!(n-1-2)!} \times 3! = 918$

$\frac{(n-1)!}{(n-3)!} = 306$

$\frac{(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 306$

$n^2 - 3n + 2 = 306$

$n^2 - 3n - 304 = 0$

$(n+16)(n-19) = 0$

$\therefore n = -16$ or 19

since $n \neq -16$

$n = 19$

$\therefore$ There are 19 letters in this alphabet.

(a) Well answered. Students that used an algebraic approach had greater success than those that used a combinatoric argument.

(b) There were a lot of algebraic errors in this question for those that chose to find the general term. For those that did not use this method, many struggled to determine how the constant term would be formed.

(c) More practice of harder combinatorics questions is required. Many students did not consider order or else considered it twice!

(d) $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-1)$

take sin on both sides

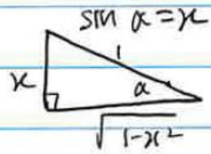
$$\sin(\sin^{-1}x - \cos^{-1}x) = \sin(\sin^{-1}(3x-1))$$
$$= 3x-1$$

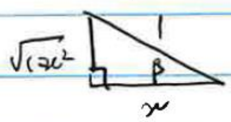
$$LHS = \sin(\sin^{-1}x - \cos^{-1}x)$$
$$= \sin(\sin^{-1}x)\cos(\cos^{-1}x) - \cos(\sin^{-1}x)$$
$$\wedge$$
$$\sin(\cos^{-1}x)$$

$$= x^2 - \cos(\sin^{-1}x)\sin(\cos^{-1}x)$$

let $\alpha = \sin^{-1}x$ $\quad$ $\beta = \cos^{-1}x$

$\sin\alpha = x$ $\quad$ $\cos\beta = x$





$$= x^2 - \cos\alpha \sin\beta$$
$$= x^2 - \left(\frac{\sqrt{1-x^2}}{1} \times \frac{\sqrt{1-x^2}}{1}\right)$$
$$= x^2 - 1 + x^2$$
$$= 2x^2 - 1$$

$$2x^2 - 1 = 3x - 1$$
$$2x^2 - 3x = 0$$
$$x(2x-3) = 0$$
$$\therefore x = 0, \frac{3}{2}$$

but $x \neq \frac{3}{2}$

$$\therefore x = 0$$

(e) 4 letters with no A's

$3!$

4 letters with 1 A

$^4C_3 \times 3!$

4 letters with 2 A's

$^4C_2 \times \frac{3!}{2}$

$$3! + {}^4C_3 \times 3! + {}^4C_2 \times \frac{3!}{2}$$
$$= 48$$

(d) Many students struggled to know how to start this problem. For those that made progress, it was necessary to consider the domain of the inverse functions to eliminate one of the solutions.

(e) Very few students considered the need to use cases to solve this problem. Given the allocation of 3 marks, it should be clear that one step of simple working will not be sufficient to solve the problem.

(f) $$\frac{\binom{23}{1} + \binom{23}{2} + \binom{23}{3} + \cdots \binom{23}{22}}{2}$$

$$= \frac{2^{23} - 2}{2} \quad \left(\text{since } \binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \cdots + \binom{23}{23} = 2^{23}\right)$$

$$= 4194303$$

(f) Given that there is at least one student in each group, the 23C0 and 23C23 cases must be discounted. We have overcounted by a factor of 2 since 23C1 = 23C22 (choosing a group of 1 automatically means the second group has 22, and these options would have been counted in both of the cases above) and so on.