SYDNEY BOYS HIGH SCHOOL

2019

YEAR 11
TERM 3
ASSESSMENT
TASK #3

Mathematics Extension

General Instructions

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–16, show ALL relevant mathematical reasoning and/or calculations

Total Marks:

71

Section I – 10 marks (pages 2 - 6)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 61 marks (pages 8 - 19)
- Attempt Questions 11-16
- Allow about 1 hour and 15 minutes for this section

Examiner:

AMG

Section I – Multiple Choice

10 Marks
Attempt questions 1−10
Allow approximately 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1−10

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1 Which expression is the complete factorisation of $2x^2 - 32$?

A. $2(x^2 - 16)$

B. $2x(x - 16)$

C. $(2x - 16)(2x + 16)$

D. $2(x + 4)(x - 4)$

2 What is 36.1984 written in scientific notation, correct to 4 significant figures?

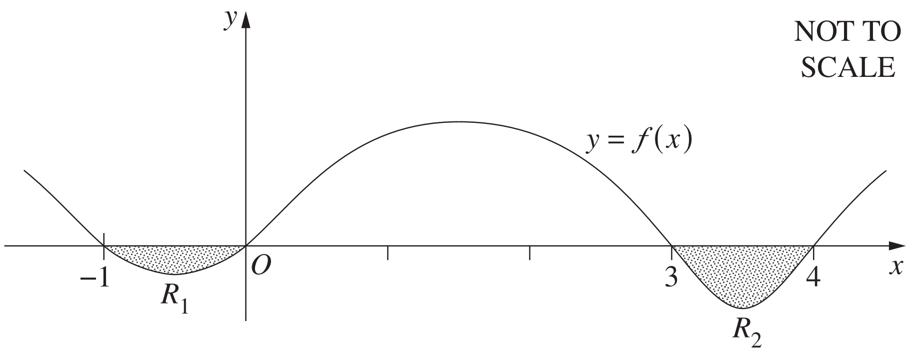
A. 3.620×10^{1}

B. 3.62×10^{1}

C. 3.620×10^{-1}

D. 3.6198×10^{-1}

3 The diagram shows the graph of $y = f(x)$ with intercepts at $x = -1$, 0, 3 and 4.



The area of the shaded region R_1 is 1 and the area of the shaded region R_2 is 2.

It is given that $\int_0^4 f(x)\,dx = 11$. What is the value of $\int_{-1}^{3} f(x)\,dx$?

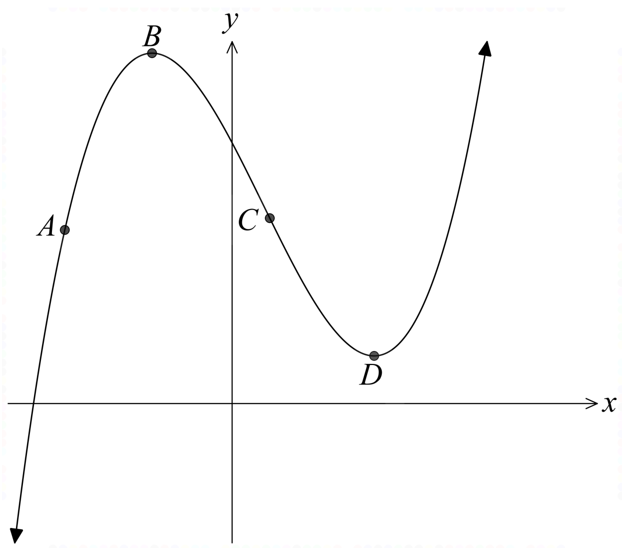
A. 8

B. 10

C. 12

D. 14

4

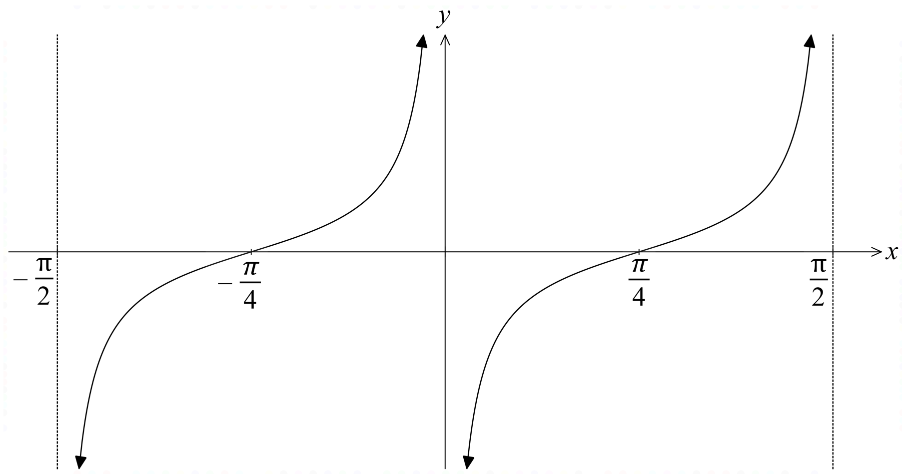


The diagram above shows the points A, B, C and D on a curve.

At which point is $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$?

A. A
B. B
C. C
D. D

5 Part of the graph of $y = f(x)$ is shown below



Which of the following could be an equation of $f(x)$?

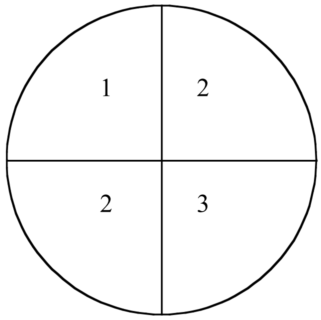
A. $f(x) = \tan\left(2x - \dfrac{\pi}{2}\right)$

B. $f(x) = \tan x$

C. $f(x) = \tan\left(2x - \dfrac{\pi}{4}\right)$

D. $f(x) = \tan\left(x + \dfrac{\pi}{4}\right)$

6 A spinner is marked as shown



If X is the number indicated by the spinner, which of the following shows the probability distribution of X?

A.

x	1	2	3
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

B.

x	1	2	3
$P(X=x)$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

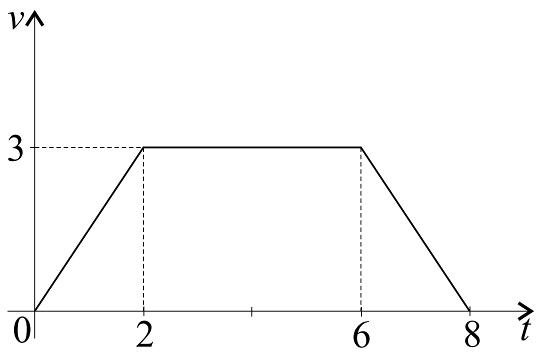
C.

x	1	2	3
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

D.

x	1	2	3
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

7



The diagram above shows the velocity–time graph of a particle moving in a straight line.
Which of the following statements is true?

A. The velocity is constant throughout the motion.

B. The particle does not return to its starting point.

C. The particle cannot move because its initial velocity is zero.

D. The distance travelled by the particle cannot be determined.

8 What is the value of $\lim_{x \to \frac{2\pi}{3}} \dfrac{\sin x - \frac{\sqrt{3}}{2}}{x - \frac{2\pi}{3}}$?

A. $-\dfrac{\sqrt{3}}{2}$

B. $-\dfrac{1}{2}$

C. $\dfrac{1}{2}$

D. $\dfrac{\sqrt{3}}{2}$

9 Let $f(x)=\dfrac{1}{x+2}$ and D_1 be the largest domain for which f is defined.

Let $g(x)=e^{2x}$ and D_2 be the largest domain for which g is defined.

Let $h(x)=\dfrac{1}{x+2}-e^{2x}$ and D_3 be the largest domain for which h is defined.

Which one of the following is true?

(A) $D_1 = D_3$ and the range of f is the same as the range of h.

(B) $D_1 = D_3$ and the range of g is NOT the same as the range of h.

(C) $D_1 \neq D_3$ and the range of g is the same as the range of h.

(D) $D_1 \neq D_3$ and the range of f is NOT the same as the range of h.

10 The average rate of change of the function $f(x)=x^2-2x$ over the interval $[1, a]$ is 8, where $a > 1$.

What is the value of a?

A. 4

B. 7

C. 8

D. 9

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Section II

61 marks
Attempt Questions 11–16
Allow about 1 hour and 15 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra pages are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (10 Marks) Start a SEPARATE Writing Booklet

(a) Factorise $a^3 - 27$. **1**

(b) Rationalise the denominator of $\dfrac{10}{3-\sqrt{3}}$, leaving your answer in simplest form. **2**

(c) Solve $\dfrac{x}{2} - \dfrac{x-5}{3} = 5$. **2**

(d) If $f'(x) = -\dfrac{3}{x}$ and $f(e) = 1$, find an expression for $f(x)$. **2**

(e) Find $\displaystyle\int e^{2-5x}\, dx$. **1**

(f) Differentiate $\dfrac{x}{x^2+1}$. **2**

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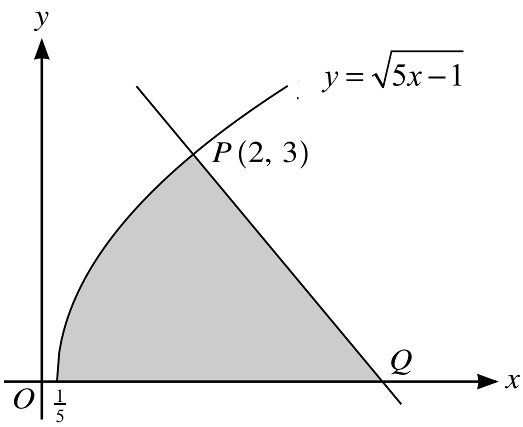
Question 12 (10 Marks) Start a SEPARATE Writing Booklet

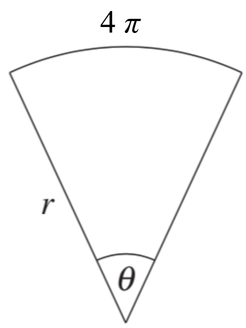
(a) The diagram below shows part of the curve $y=\sqrt{5x-1}$ and the normal to the curve at the point $P(2, 3)$. This normal meets the x-axis at Q.



(i) Show that the equation of the normal is given by $6x + 5y - 27 = 0$. **2**

(ii) Hence, find the exact area of the shaded region. **2**

(b)



The diagram shows a sector with radius r and angle θ where $0 < \theta \leq 2\pi$.
The arc length is 4π .

(i) Show that $r \geq 2$ **1**

(ii) Find the exact area of the sector when $r = 4$. **1**

Question 12 continues of page 11

Question 12 (continued)

(c) Given that $\int_0^a \frac{2}{3x+1}\,dx = \ln 16$, find the value of the positive constant a. **2**

(d) It is given that x satisfies the equation $|\,x+1\,| = 4$. Find the possible values of **2**

$$|\,x+4\,| - |\,x-4\,|.$$

End of Question 12

Question 13 (10 Marks) Start a SEPARATE Writing Booklet

(a) A function $g(x)$ is defined by $g(x) = (x-2)(x^2-4)$.

(i) Solve $g(x) = 0$ **1**

(ii) Find the coordinates of the turning points of the graph of $y = g(x)$, and determine their nature. **3**

(iii) Hence sketch the graph of $y = g(x)$, showing the stationary points and the points where the curve meets the coordinate axes. **2**

(iv) For what values of x is the graph of $y = g(x)$ concave up and increasing? **1**

(b) The discrete random variable X has the following probability distribution. **3**

x	1	2	3	6
$P(X = x)$	0.15	p	0.4	q

Given that $E(X) = 3.05$, find the values of p and q.

End of Question 13

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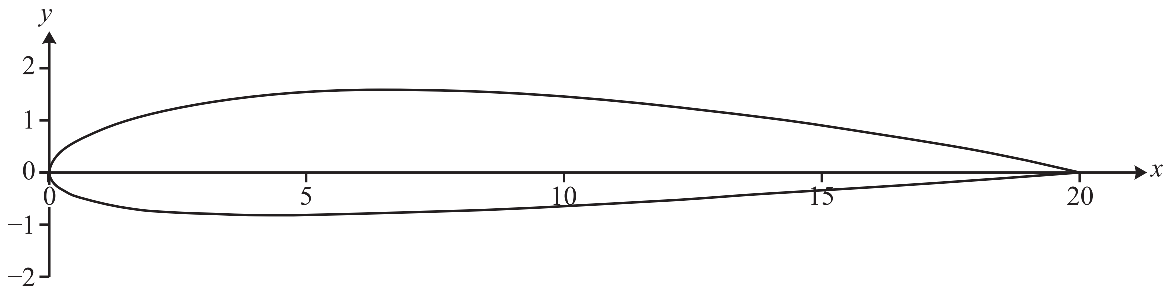
Question 14 (10 Marks) Start a SEPARATE Writing Booklet

(a) Gaino is designing a model aeroplane. The diagram shows his sketch of the wing's cross-section. **2**



The table below are some of the coordinates that Gaino has used to draw the cross-section.

Upper surface		Lower surface	
x	y	x	y
0	0	0	0
4	1.45	4	−0.85
8	1.56	8	−0.76
12	1.27	12	−0.55
16	1.04	16	−0.30
20	0	20	0

Use the trapezium rule with 6 function values to calculate an estimate of the area of this cross-section.

(b) The curve with equation $y = \dfrac{2 - \sin x}{\cos x}$ has one stationary point in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

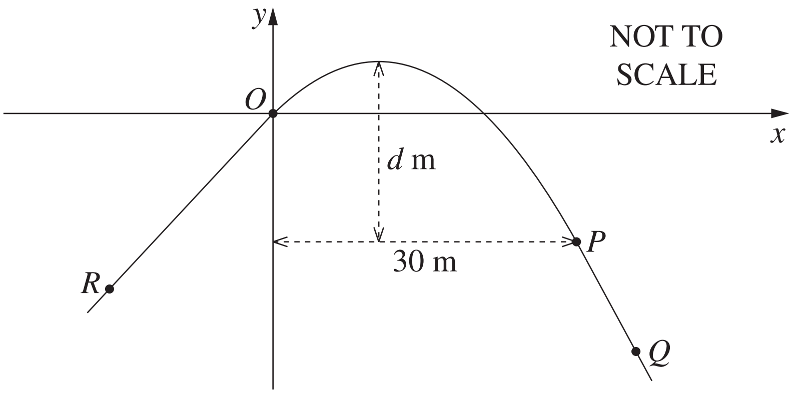
(i) Find the exact coordinates of this point. **2**

(ii) Determine whether this point is a maximum or a minimum point. **1**

Question 14 continues on page 15

Question 14 (continued)

(c) The diagram illustrates the design for part of a roller-coaster track. The section RO is a straight line with slope 1.2, and the section PQ is a straight line with slope -1.8. The section OP is a parabola $y = ax^2 + bx$. The horizontal distance from the y-axis to P is 30 m.



In order that the ride is smooth, the straight line sections must be tangent to the parabola at O and at P.

(i) Find the values of a and b so that the ride is smooth. **3**

(ii) Find the distance d, from the vertex of the parabola to the horizontal line through P, as shown on the diagram. **2**

End of Question 14

Question 15 (10 Marks) Start a SEPARATE Writing Booklet

(a) What is the derivative of $\cos(\ln x)$, where $x > 0$? **1**

(b) In November 1923, 18 koalas were introduced on Kangaroo Island. By November 1993, the number of koalas had increased to 5000. **3**

Assume that the number N of koalas on the island is increasing at a rate proportional to the number of koalas present.

Using this model, predict the number of koalas that will be present on Kangaroo Island in November 2019.

(c) Khaled is allowed two attempts to pass an examination. If he succeeds on his first attempt, he does not make a second attempt. The probability that he passes at the first attempt is 0.4 and the probability that he passes on either the first or second attempt is 0.58. **2**

Find the probability that he passes, given that he failed on the first attempt.

$$\left[\text{You may use } \mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}\right]$$

(d) The function $f(x)$ is defined by **2**

$$f(x) = \frac{1-x}{1+x},\ x \neq -1$$

Given that $f^2(x) = f\left(f(x)\right)$, by first finding the simplest expression for $f^2(x)$, show that

$$f^{2020}(x) = x\ ,\ x \neq -1.$$

(e) The curve C has equation **2**

$$y = \frac{2x^2 - 3x - 2}{x^2 - 2x + 1}$$

Show that $y \leq 2\tfrac{1}{12}$ at all points on C.

End of Question 15

(Question 16 starts on page 18)

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Question 16 (11 Marks) Start a SEPARATE Writing Booklet

(a) A particle starts from a point O and moves in a straight line.
The velocity of the particle at time t seconds after leaving O is v m/s where

$$v = \begin{cases} 1.5 + 0.4t, & 0 \le t \le 5 \\ \dfrac{100}{t^2} - 0.1t, & t > 5 \end{cases}$$

(i) Find the acceleration of the particle during the first 5 seconds of motion. **1**

(ii) Find when the particle is at rest. **1**

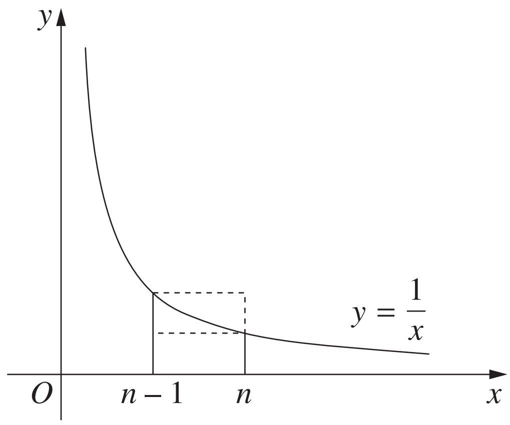
(iii) Find the total distance travelled by the particle in the first 12 seconds of motion. **3**

(b) Evaluate $\displaystyle\int_0^2 3^{x+1}\,dx$. **2**

Question 16 continues on page 19

Question 16 (continued)

(c) Let n be a positive integer greater than 1.



The diagram above shows the curve

$$y = \frac{1}{x}, x > 0$$

and two rectangles from $x = n - 1$ to $x = n$.

(a) Show that **3**

$$e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}.$$

(b) Hence, evaluate $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$. **1**

End of paper

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SYDNEY
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2019

YEAR 11 COHORT TASK 3
YEARLY

Mathematics Extension

SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1. D
2. A
3. C
4. B
5. A
6. D
7. B
8. B
9. B
10. D

Ext Y11 Yearly 2019 Multiple choice solutions

Mean (out of 10): 7.08

1. $2x^2 - 32$

$= 2(x^2 - 16)$

$= 2(x+4)(x-4)$ D

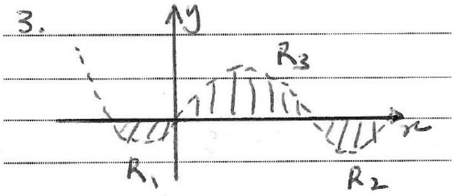
A	4
B	0
C	0
D	169

2. $36.1984 = 3.61984 \times 10^1$

$\simeq 3.620 \times 10^1$ A

A	155
B	3
C	15
D	0

3.



$\int_0^4 f(x)\,dx = R_3 - R_2 = 11$

$\therefore R_3 - 2 = 11$

$\therefore R_3 = 13$

$\int_{-1}^{3} f(x)\,dx = R_3 - R_1$

$= 13 - 1$

$= 12$ C

A	17
B	37
C	112
D	6

Q4 A: $\frac{dy}{dx} > 0,\ \frac{d^2y}{dx^2} < 0$ ✗

B: $\frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} < 0$ ✓

C: $\frac{dy}{dx} < 0,\ \frac{d^2y}{dx^2} = 0$ ✗

D: $\frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} > 0$ ✗

B

A	1
B	156
C	3
D	12

5. The graph of $y = \tan x$ has a period of π. This graph has a period of $\frac{\pi}{2} \Rightarrow y = \tan(2x + c)$

This graph has been shifted $\frac{\pi}{4}$ units to the right $\Rightarrow y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$

$= \tan\left(2x - \frac{\pi}{2}\right)$

A

— OR —

Substitute $x = \frac{\pi}{4}$ into each equation.

A: $\tan\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = \tan 0 = 0$ ✓

B: $\tan \frac{\pi}{4} = 1$ ✗

C: $\tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1$ ✗

D: $\tan\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \tan \frac{\pi}{2} \rightarrow$ undefined ✗

A

A	77
B	7
C	61
D	27

6. $P(1) = \frac{1}{4}$

$P(2) = \frac{2}{4} = \frac{1}{2}$

$P(3) = \frac{1}{4}$

D

A	0
B	1
C	0
D	172

7. A: The velocity is only constant for $2 \le t \le 6$ ✗

B: As the particle only has non-negative velocity, the particle does not return to its starting point ✓

C: The particle is moving for $0 < t < 8$ ✗

D: The distance travelled can be found by finding the area under the curve. Distance $= \frac{1}{2} \times 3 \times (4 + 8)$

$= 18$ units ✗

B

A	8
B	152
C	2
D	11

8. $\lim\limits_{x \to \frac{2\pi}{3}} \dfrac{\sin x - \frac{\sqrt{3}}{2}}{x - \frac{2\pi}{3}}$

$= \lim\limits_{x \to \frac{2\pi}{3}} \dfrac{\sin x - \sin \frac{2\pi}{3}}{x - \frac{2\pi}{3}}$

$= \frac{d}{dx}(\sin x)$ at $x = \frac{2\pi}{3}$

$= \cos \frac{2\pi}{3}$

$= -\frac{1}{2}$ B

A	33
B	57
C	54
D	26

9. $f(x) = \frac{1}{x+2}$ $D_1: x \neq -2$ $R_1: y \neq 0$

$g(x) = e^{2x}$ $D_2: x \in \mathbb{R}$ $R_2: y > 0$

$h(x) = \frac{1}{x+2} - e^{2x}$ $D_3: x \neq -2$ $R_3: y \in \mathbb{R}$ (maybe)

y can be 0.

$\therefore D_1 = D_3$, $R_1 \neq R_3$ B

A	28
B	98
C	24
D	22

10. $\frac{f(a) - f(1)}{a-1} = 8$

$\therefore \frac{a^2 - 2a - (-1)}{a-1} = 8$

$\therefore \frac{(a-1)^2}{a-1} = 8$

$\therefore a - 1 = 8$

$a = 9$ D

A	41
B	22
C	32
D	77

Year 11 Term 3 Task

QUESTION 11 (10 marks)

General Comment: Candidates are not using their reference sheet. They must familiarise themselves with it. There simple mistake of incorrect use of factorising cubes, product, quotient and power rules. All of these are printed on the reference sheet.

(a) Factorise a^3-27. **1**

$=a^3-3^3$

$=(a-3)(a^2+3a+9)$

Criteria	**Marks**
Provides the factorisation	**1**
One error in applying the formula to factorising cubes	**0.5**

(b) Rationalise the denominator of $\dfrac{10}{3-\sqrt{3}}$, leaving your answer in simplest form. **2**

$$=\frac{10}{(3-\sqrt{3})}\times\frac{(3+\sqrt{3})}{(3+\sqrt{3})}$$

$$=\frac{10(3+\sqrt{3})}{9-3}$$

$$=\frac{5(3+\sqrt{3})}{3}$$

Acceptable for $\dfrac{15+5\sqrt{3}}{3}$

Criteria	**Marks**
Provides the correct answer	**2**
Provides the answer but not fully simplified	**1.5**
Correct expansion with the multiplication of the conjugate	**1**
Attempt to multiply by the conjugate	**0.5**

(c) Solve $\dfrac{x}{2}-\dfrac{(x-5)}{3}=5$ **2**

$3x-2x+10=30$

$x=20$

Criteria	**Marks**
Provides the correct solution	**2**
Incorrect expansion of the numerator but no subsequent error.	**1.5**
An attempt to simplify the numerator.	**1**

(d) If $f'(x) = -\frac{3}{x}$ and $f(e) = 1$, find an expression for $f(x)$. **2**

$$f(x) = \int -\frac{3}{x}\,dx$$
$$= -3\int \frac{1}{x}\,dx$$
$$= -3\ln x + c$$
$$f(e) = 1$$
$$\therefore 1 = -3\ln e + c$$
$$\therefore c = 4$$
$$f(x) = -3\ln x + 4$$

Criteria	Marks
Provides the correct answer	**2**
Incorrect value of *c*	**1.5**
Correct integral using ln	**1**
Attempt to evaluate *c*, with incorrect integral	**0.5**

(e) Find $\int e^{2-5x}\,dx$ **1**

$$= -\frac{1}{5}e^{2-5x} + c$$

Criteria	Marks
Provides the correct answer	**1**
Incorrect coefficient for the final answer.	**0.5**

(f) Differentiate $\frac{x}{x^2+1}$ **2**

$$= \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$
$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$
$$= \frac{1-x^2}{(x^2+1)^2}$$

Some candidates attempted to use combination of product and power rules. The main problem is differentiating $(x^2 + 1)^{-1}$ where trying to use the product rule. It will be far easier if candidates learn how to use the quotient rule. Other problems including confusion between the application of quotient and product rule.

Criteria	Marks
Provides the correct answer	**2**
Correctly apply the quotient rule but made an error with the numerator OR any subsequent error due to factorisation.	**1.5**
Attempted to apply the quotient rule or product rule	**1**

Question 12

a) i) $y = (5x-1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \times 5 \times (5x-1)^{-\frac{1}{2}}$

$= \frac{5}{2\sqrt{5x-1}}$ (1)

When $x = 2$,

$\frac{dy}{dx} = \frac{5}{2} \times \frac{1}{\sqrt{10-1}} = \frac{5}{6}$ ($\frac{1}{2}$)

$m_n = \frac{-6}{5}$

$y - 3 = \frac{-6}{5}(x-2)$ ($\frac{1}{2}$)

$5y - 15 = -6x + 12$

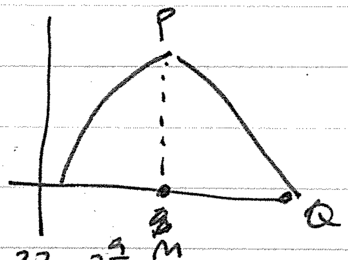
$6x + 5y - 27 = 0 \Rightarrow y = \frac{-6}{5}x + \frac{27}{5}$

ii) Finding Q, when $y = 0$

$6x = 27$, $x = \frac{9}{2}$

$\therefore$ Triangle $PMQ = \frac{1}{2} \times 3 \times \frac{5}{2}$

$= \frac{15}{4}$



OR $\int_2^{\frac{9}{2}} \left(\frac{-6}{5}x + \frac{27}{5}\right)dx = \left[\frac{-3x^2}{5} + \frac{27}{5}x\right]_2^{\frac{9}{2}}$

$= \frac{15}{4}$ (1)

Area $OPM = \int_{\frac{1}{5}}^{2} (5x-1)^{\frac{1}{2}}dx = \left[\frac{2}{3} \times \frac{1}{5}(5x-1)^{\frac{3}{2}}\right]_{\frac{1}{5}}^{2}$

$= \frac{2}{15} \times 9^{\frac{3}{2}} = \frac{54}{15} = \frac{18}{5}$ (1)

$\therefore$ Total area $= \frac{15}{4} + \frac{18}{5} = \frac{147}{20} = 7\frac{7}{20} = 7.35$

b i) Arc length $= r\theta = 4\pi$

since $\theta \leq 2\pi$, $r \geq 2$

OR $\theta = \frac{4\pi}{r}$ $\quad 0 \leq \frac{4\pi}{r} \leq 2\pi$

$0 \leq 4\pi \leq 2\pi r$

$0 \leq 2 \leq r$

1

ii) When $r = 4$, $\theta = \frac{4\pi}{4} = \pi$

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 4^2 \times \pi = 8\pi$ 1

c) $\int_0^a \frac{2}{3x+1}\,dx = \frac{2}{3}\int_0^a \frac{3}{3x+1}\,dx$

$= \frac{2}{3}\left[\ln(3x+1)\right]_0^a$ 1

$= \frac{2}{3}\ln(3a+1) = \ln(16)$

$\ln(3a+1) = \ln(16^{\frac{3}{2}})$

$= \ln(64)$

$3a+1 = 64$

$3a = 63$

$a = 21$ 1

d) $|x+1| = 4$

$x = 3, -5$ 1

when $x = -5$, $|x+4| - |x-4| = |-5+4| - |-5-4|$

$= 1 - 9 = -8$

when $x = 3$, $|x+4| - |x-4| = |3+4| - |3-4|$

$= 7 - 1 = 6$. 1

Comments - Q12

a) i) Generally done well, as students knew what the final expression should be.

However, some students couldn't successfully differentiate $y=\sqrt{5x-1}$, or attempted to get an expression for gradient without plugging in $x=2$.

ii) Very poorly done.

- Many students unable to integrate $\int\sqrt{5x-1}\,dx$.
- Quite a few attempted to integrate $\int_{\frac{1}{5}}^{\frac{9}{2}}\sqrt{5x-1}-\left(\frac{-6}{5}x+\frac{27}{5}\right)\partial x$ or similar.

- The question was successfully done as an integral in dy by some students

i.e. $\int_0^3\left(\frac{-5}{6}y+\frac{9}{2}\right)-\frac{y^2+1}{5}\,dy$

b) i) Generally poorly done.

Some students thought that demonstrating $r\neq 1$ or similar and that $r=2,3$ works is sufficient to say $r\geq 2$.

ii) Also not done well. Many students left their answer as 8θ, which is insufficient.

c) Many students had difficulty evaluating $\int_0^a \frac{2}{3x+1}\,dx$, either not recognising it gives a logarithm, or not getting the $\frac{2}{3}$ at the front.

Some students made errors such as $\frac{3}{2}\ln(16)\Rightarrow\ln(24)$

d) Most students found that possible values of $x=3,-5$.

Many thought that the required expression could be $|3+4|-|-5-4|$ or similar.

Some struggled to evaluate expressions with absolute value, eg $|-5-4|=-9$.

Question 13

(a) $g(x) = (x-2)(x^2-4)$

(i) $g(x) = 0$ when $(x-2)(x^2-4) = 0$

$x - 2 = 0 \qquad x^2 - 4 = 0$

$x = 2 \qquad x = \pm 2$

$\therefore x = 2, -2.$ [1]

(ii) $g(x) = (x-2)(x^2-4)$

$= x^3 - 2x^2 - 4x + 8$

$g'(x) = 3x^2 - 4x - 4$

$g''(x) = 6x - 4$

stationary point $g'(x) = 0$

$\therefore 3x^2 - 4x - 4 = 0$

$\left.\begin{array}{l} P: -12 \\ S: -4 \end{array}\right\} (-6, 2)$

$\dfrac{(3x-6)(3x+2)}{3} = 0$

$\dfrac{3(x-2)(3x+2)}{3} = 0$

$\therefore (x-2)(3x+2) = 0$

$\therefore x = 2, -\dfrac{2}{3}$

when $x = 2$; $g(2) = 0$ (from (i)).

$g''(2) = 6(2) - 4$

$= 12 - 4$

$= 8 > 0$ (concave up)

$\therefore$ minimum turning point at $(2, 0)$.

when $x = -\frac{2}{3}$; $g\left(-\frac{2}{3}\right) = \left(-\frac{2}{3} - 2\right)\left(\left(-\frac{2}{3}\right)^2 - 4\right)$

$$= 9\frac{13}{27}$$

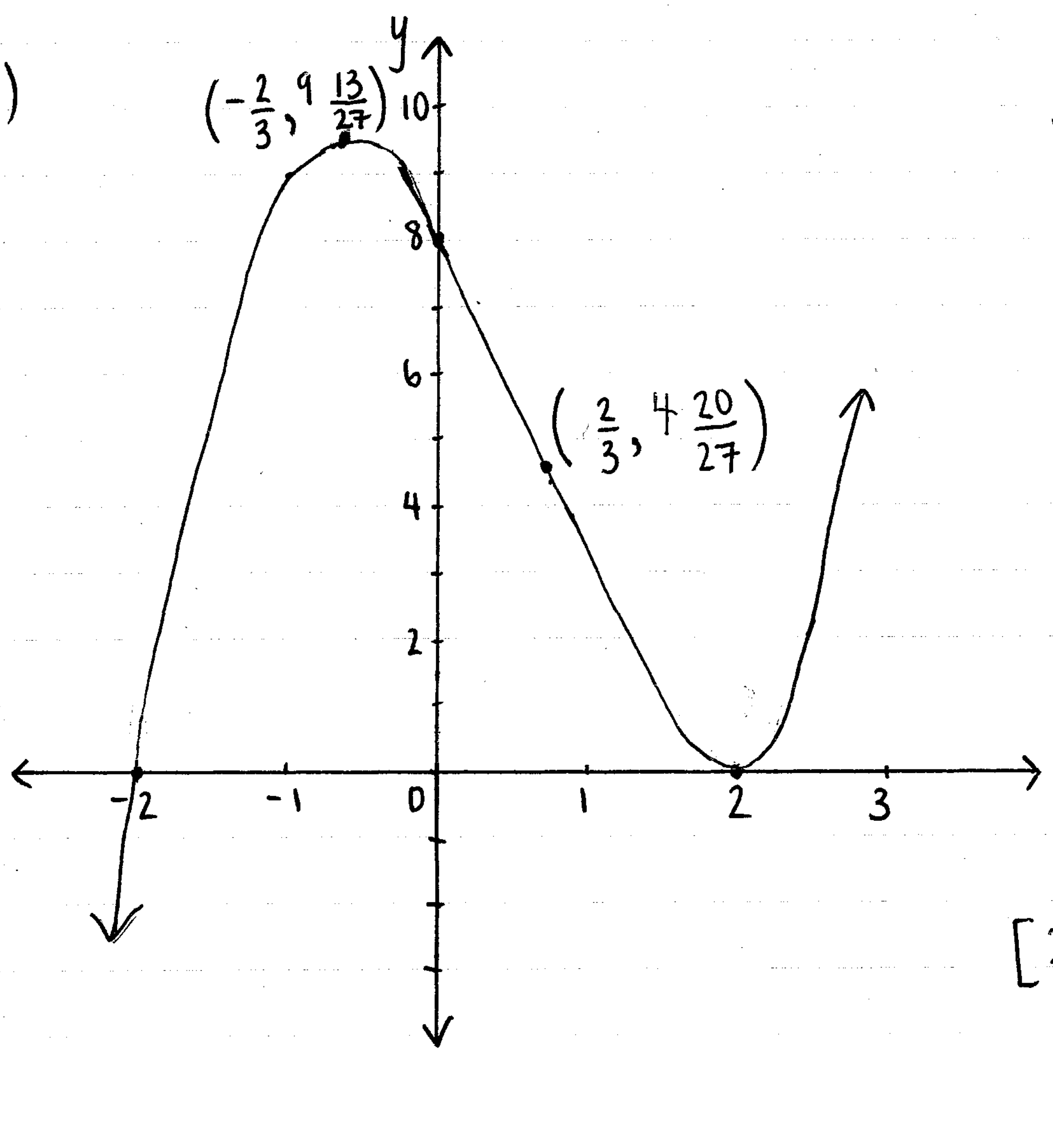
$$(\doteq 9.\dot{4}8\dot{1})$$

$$g''\left(-\frac{2}{3}\right) = 6\left(-\frac{2}{3}\right) - 4$$

$$= -8 < 0 \text{ (concave down)}$$

$\therefore$ maximum turning point at $\left(-\frac{2}{3}, 9\frac{13}{27}\right)$ [3]

(iii)



[2]

(iv) concave up $\Rightarrow x > \frac{2}{3}$

increasing $\Rightarrow x < -\frac{2}{3} \cap x > 2$

$\therefore$ concave up and increasing

$x > 2$. [1]

Generally well done

(b)

x	1	2	3	6
$P(X=x)$	0.15	p	0.4	q

$E(X) = 3.05$

Probability distribution: $\sum(P(X=x)) = 1$

$0.15 + p + 0.4 + q = 1$

$p + q = 0.45 \quad (1)$

$E(X) = 1 \times 0.15 + 2 \times p + 3 \times 0.4 + 6 \times q$

$\therefore 0.15 + 2p + 1.2 + 6q = 3.05$

$2p + 6q = 1.7$

$p + 3q = 0.85 \quad (2)$

$(2) - (1)$: $2q = 0.4$

$\therefore q = 0.2$

sub $q = 0.2$ into (1): $p + 0.2 = 0.45$

$\therefore p = 0.25$ [3]

Generally well done

Solutions for Question 14 for the Extension 1 Yearly (2019)

Part a) Since there are 6 function values for each section, we are able to make 5 trapeziums each of width 4. Therefore *h* = 4. For each trapezium $\int_a^b f(x)\ \ dx\ \approx \frac{h}{2}\left(f(b)+f(a)\right)$

Then for upper section (as an approximation):

$$\begin{aligned} A_1 \approx \tfrac{4}{2}(1.45+0)+\tfrac{4}{2}(1.56+1.45)+\tfrac{4}{2}(1.27+1.56)+\tfrac{4}{2}(1.04+1.27)+\tfrac{4}{2}(0+1.04) \\ \approx 4(1.45+1.56+1.27+1.04) \\ \approx 21.28 \end{aligned}$$

Similarly for the lower section:

$$\begin{aligned} A_2 &\approx |4(-0.85-0.76-0.55-0.3)| \\ &\approx 9.84 \end{aligned}$$

And finally the total approximate area is 31.12 squared units.

Notes on marks awarded:

Calculation errors were a half mark deduction.

The wrong value for *h* was also a half mark deduction.

If one only calculated 1 area (upper or lower – usually only the upper in that case) or one subtracted the lower area from the upper, one could only get a maximum of 1 mark. This is because the student failed to fundamentally understand the question or how to go about it.

If you showed by your working that you don't know how to approximate by this method you either got zero or just half a mark – depending on your level of misunderstanding.

Since the data was given to two decimal places, one should return a value that is at least as accurate. If one dropped figures after the decimal, then half a mark was deducted.

(The biggest mistake on this question was not understanding that the value *h* for this question was the equidistance (in the *x* direction) of each trapezium section)

Part b) i) To find a stationary point we can solve the derivate for the equation when we force it to be zero.

$y = \frac{2-\sin x}{\cos x}$ let $u = 2-\sin x$ and $v = \cos x$

Then, $u' = -\cos x$ and $v' = -\sin x$

$$y' = \frac{u'v - v'u}{v^2}$$

$$\begin{aligned} y' &= \frac{-\cos x \times \cos x - (2-\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{-\cos^2 x - (-2\sin x + \sin^2 x)}{\cos^2 x} \end{aligned}$$

$$= \frac{2\sin x - 1}{\cos^2 x}$$

Otherwise one could use the product rule:

$$\text{let } u = 2 - \sin x \quad \text{and} \quad v = (\cos x)^{-1}$$
$$\text{Here} \quad v' = -1 \times -\sin x \times (\cos x)^{-2}$$
$$= \frac{\sin x}{\cos^2 x}$$
$$y' = uv' + vu'$$
$$= (2 - \sin x)\frac{\sin x}{\cos^2 x} + (\cos x)^{-1} \times (-\cos x)$$
$$= (2 - \sin x)\frac{\sin x}{\cos^2 x} - 1$$
$$= \frac{2\sin x - 1}{\cos^2 x}$$

Or changing *y* to secant and tangent functions:

$$y = 2\sec x - \tan x$$
$$y' = 2\sec x \times \tan x - \sec^2 x$$

This simplifies to the same solution as previously of course.

For *y′* to be zero the numerator must be equal to zero (as long as the denominator is not).

$$2\sin x - 1 = 0$$
$$\sin x = \frac{1}{2}$$
$$x = \sin^{-1}\frac{1}{2}$$

In our restricted domain of $-\frac{\pi}{2} < x < \frac{\pi}{2}$ the only solution is $x = \frac{\pi}{6}$

To find the *y* coordinate for this *x* we substitute it back into our original equation.

$$y\left(\frac{\pi}{6}\right) = \frac{2 - \sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}$$
$$= \frac{2 - \frac{1}{2}}{\frac{\sqrt{3}}{2}}$$
$$= \sqrt{3}$$

The coordinates of the stationary point: $\left(\frac{\pi}{6}, \sqrt{3}\right)$

Notes on marks awarded for this question.

1 full mark was awarded for getting the correct derivative (half a mark deducted for each small mistake) – no marks when it was clear that the student does not know how to find the derivative within reason.

Half a mark each was awarded for finding *x* and *y*.

(Only about 25% of students found the correct derivative function of *y*)

Part b) ii) This could be done in a few ways. Using a table of values of either *y* or *y*′ and interpreting the results. Or finding *y*′′ and interpreting that result properly.

$$y'' = -2 \times \frac{\cos(x)^2 + \sin(x) - 2}{\cos(x)^3}$$

Substituting $\frac{\pi}{6}$ into *y*′′ we get $\frac{4}{3}\sqrt{3}$ and therefore the stationary point is a local minimum (global in fact over the restricted domain).

Using the derivative: substituting any *x*-value in the domain that is lower than the stationary point should have given you a negative value. And conversely and value higher would have produced a positive number. It may be concluded that the slope goes from negative to positive around the stationary point and it is therefore a minimum turning point.

Using the original function: substituting points around the stationary point would have shown that the stationary point is lower than both and thus it is a minimum turning point.

Notes on marks awarded for this question:

The examiner gave no room to move in terms of what the solution should look like – it is a turning point (the question was to determine if it is a max or min turning point only). If you got something else you should of realised you'd done something wrong. I did not award marks for another type of point.

If you just wrote an answer with no working then you got no marks. You would not know the answer unless you did some work to find out (ie: a sketch, the second derivative or a table of values).

If you did some working but made a silly mistake (like using degrees on your calculator instead of radians) then you were deducted half a mark.

If your working appeared to be wrong and not just a slight error then you weren't awarded any marks.

(Students doing a table of values often took values far away from the point they were testing. Since their answer was often (sadly) wrong, this produced the wrong result around it. I often pointed this out but I didn't necessarily deduct any marks since the question implied there was only one turning point within the domain. I recommend as a matter of course that you make it a habit of choosing values close to the point you are testing, since often you will not know if another critical value is within the interval you are checking)

Part c) i) The line OR meet the parabola at the origin. To be 'smooth' the derivatives at the meeting point must be equal.

The equation of the line OR is $y = \frac{6}{5}x$ its derivative is $\frac{6}{5}$

The equation of the parabola is $y = ax^2 + bx$ and its derivative is $2ax + b$

$$\text{The slope of the parabola at } x = 0 \text{ is } \quad 2a \times 0 + b = b$$

$$\text{Equating the two slopes of these equations gives us } \quad b = \frac{6}{5}$$

The equation of the line PQ is $y = -\frac{9}{5}x + c$ we are not told what c is.

It has a constant slope of $-\frac{9}{5}$, substituting in $b = \frac{6}{5}$ we can compare the slopes at the point where the parabola meets the line PQ and solve for them being equal.

The point they meet at is x=30. The slope of the parabola at this point is $2a \times 30 + \frac{6}{5}$

$$\text{Equating the slopes at } x = 30 \quad \text{we get } \; 2a \times 30 + \frac{6}{5} = -\frac{9}{5}$$

$$2a \times 30 = -3$$

$$a = -\frac{3}{\mathrm{L}60}$$

$$\text{Therefore} \quad a = -\frac{1}{20} \quad \text{and } b = \frac{6}{5}$$

Part c) ii) The vertical distance from the x-axis to point P is the ordinate of P.
This is:

$$y(30) = -\frac{30^2}{20} + \frac{6 \times 30}{5}$$
$$= -45 + 36$$
$$= -9$$

Distance can not be negative so it is 9.
We need to find the ordinate of the vertex to find the vertical distance of the vertex from the x-axis.
The abscissa of the vertex can be found by either finding the stationary point of the parabola or finding the midpoint of the zeroes of the parabola.
Using the second method, the zeroes of the parabola are found by solving:

$$0 = -\frac{x^2}{20} + \frac{6x}{5}$$
$$0 = \frac{x}{5}\left(6 - \frac{x}{4}\right)$$
$$x = 0 \text{ and } 24$$

The midpoint is 12 which is the abscissa of the vertex.
The ordinate of the vertex is:

$$y(12) = \frac{12}{5}\left(6 - \frac{12}{4}\right)$$
$$= 7\tfrac{1}{5}$$

The total vertical distance from the vertex to point P is the sum of these two distances.

$$\text{distance} = 7.2 + 9$$
$$= 16.2$$

Notes on marks awarded in this question:

Often in this question, students found the abscissa of the vertex but used it as the vertical distance. This received a one mark penalty. Similarly some students used the abscissa of P for the vertical distance. This is the same penalty.
Silly errors in working received a half mark deduction.
Using the equations incorrectly (like using $y = -\frac{9x}{5}$ for PQ which is an incorrect assumption) is a one mark deduction.

Question 15 Solutions

(a)	What is the derivative of $\cos(\ln x)$, where $x > 0$?	**1**

$$\frac{d}{dx}\left(\cos(\ln x)\right) = -\sin(\ln x) \times \frac{d(\ln x)}{dx}$$

$$= -\frac{\sin(\ln x)}{x}$$

Comment

A few students saw this question as a product rule question. However, it was generally done well by most students.

Question 15 Solutions

(b) In November 1923, 18 koalas were introduced on Kangaroo Island. By November 1993, the number of koalas had increased to 5000. **3**

Assume that the number N of koalas on the island is increasing at a rate proportional to the number of koalas present.

Using this model, predict the number of koalas that will be present on Kangaroo Island in November 2019.

Since the number N of koalas on the island is increasing at a rate proportional to the number of koalas present then $N = N_0 e^{kt}$.

$t = 0, N = 18; t = 70, N = 5000$: $\quad t = 96, N = ?$

Using $t = 0, N = 18$ then $N_0 = 18$

$\therefore N = 18e^{kt}$

Using $t = 70, N = 5000$:

$$\therefore 5000 = 18e^{70k}$$
$$\therefore 277\tfrac{7}{9} = e^{70k}$$
$$\therefore 70k = \ln 277\tfrac{7}{9}$$
$$\therefore k = \tfrac{1}{70}\ln 277\tfrac{7}{9}$$

$\therefore t = 96, N = ?$

$$\begin{aligned} N &= 18e^{96k} \\ &\doteqdot 40\,423.04969 \\ &\doteqdot 40\,423 \end{aligned}$$

There will be approximately 40 423 koalas

Alternative Solution

Consider a geometric series with $a = 18$ and so in November 1993: $T_{71} = 18r^{70} = 5000$

$\therefore r^{70} = \frac{5000}{18}$

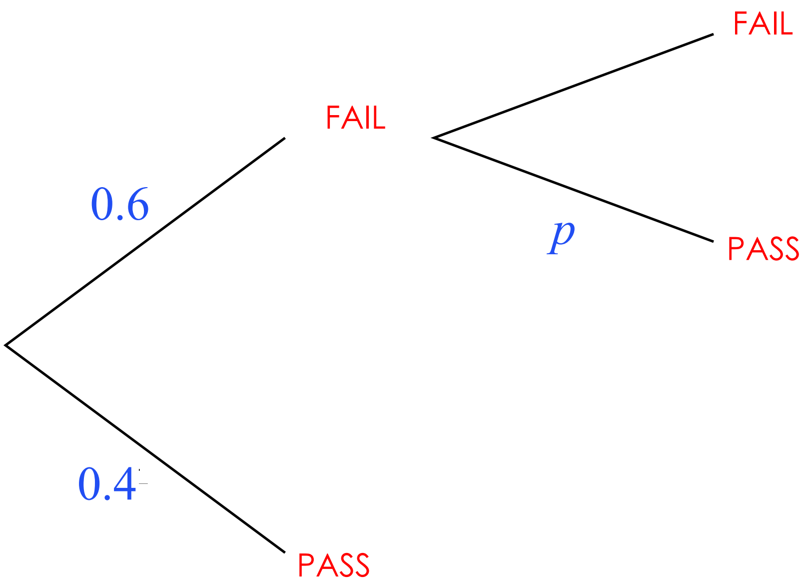
In November 2019: $\quad T_{97} = 18r^{96} = 18 \times \left[\left(\frac{5000}{18}\right)^{\frac{1}{70}}\right]^{96} \doteqdot 40\,423$ koalas

Comment

Rounding was a crucial problem here: either students rounded too early and/or rounded up at the end.
Or students didn't indicate what the rounding was, for example to the nearest 100.

Also, according to this model, the koalas don't die, so reasonable answers are expected.

(c)	Khaled is allowed two attempts to pass an examination. If he succeeds on his first attempt, he does not make a second attempt. The probability that he passes at the first attempt is 0.4 and the probability that he passes on either the first or second attempt is 0.58. Find the probability that he passes, given that he failed on the first attempt.	**2**



P(passes first or second attempt) = 0.58
$\therefore\ 0.4 + 0.6p = 0.58$
$\therefore\ 0.6p = 0.18$
$\therefore\ p = 0.3$

$\therefore$ the probability that he passes given that he fails on the first go is 0.3

Alternative Solution

$$\begin{aligned} \mathrm{P}(\text{passes} \mid \text{fails 1st go}) &= \frac{\mathrm{P}(\text{passes} \cap \text{fails 1st go})}{\mathrm{P}(\text{fails 1st go})} \\ &= \frac{0.6 \times 0.3}{0.6} \\ &= 0.3 \end{aligned}$$

Comment

Draw a tree diagram! The world needs more trees.

Too many students confused themselves by trying to use the formula. If you are going to use the formula, then clearly define A and B.

Many students couldn't understand the statement that "he passes on either the first or second attempt".
This caused a significant problem for them.

(d) The function $f(x)$ is defined by **2**

$$f(x)=\frac{1-x}{1+x}, x\neq -1$$

Given that $f^2(x)=f\left(f(x)\right)$, by first finding the simplest expression for $f^2(x)$, show that

$$f^{2020}(x)=x\ ,\ x\neq -1.$$

$$\begin{aligned}
f^2(x)&=f\left(\frac{1-x}{1+x}\right)\\
&=\frac{1-\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}}\times\frac{1+x}{1+x}\\
&=\frac{1+x-(1-x)}{1+x+(1-x)}\\
&=\frac{2x}{2}\\
&=x
\end{aligned}$$

$$\therefore f^2(x)=x$$

$$\begin{aligned}
\therefore f^{2020}(x)&=f^{2018}(f^2(x))\\
&=f^{2018}(x)\\
&=f^{2016}(f^2(x))\\
&=f^{2016}(x)\\
&\ \vdots\\
&=f^2(x)\\
&=x
\end{aligned}$$

Comment

Despite the definition of $f^2(x)$ being given to them, there were too many students not using it.

Generally well done by those who understood $f^2(x)$. That said, there were several students who didn't see the last part of the question

(e)	The curve C has equation $$y=\frac{2x^2-3x-2}{x^2-2x+1}$$ Show that $y \le 2\frac{1}{12}$ at all points on C.	**2**

$$y=\frac{2x^2-3x-2}{x^2-2x+1} \Rightarrow (x^2-2x+1)y=2x^2-3x-2$$

$$\therefore (x^2-2x+1)y=2x^2-3x-2$$
$$\therefore (y-2)x^2+(3-2y)x+y+2=0$$

$$\begin{aligned}\Delta &= (3-2y)^2-4(y-2)(y+2)\\ &= 9-12y+4y^2-4y^2+16\\ &= 25-12y\end{aligned}$$

The domain of C is defined for when $\Delta \ge 0$

$$\therefore 25-12y \ge 0$$
$$\therefore y \le 2\tfrac{1}{12}$$

Alternative solution on the next page

(e) (continued)

$$y = \frac{2x^2 - 3x - 2}{(x-1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(x-1)^2(4x-3) - (2x^2 - 3x - 2) \times 2(x-1)}{(x-1)^4}$$

$$= \frac{(x-1)\left[(x-1)(4x-3) - 2(2x^2 - 3x - 2)\right]}{(x-1)^4}$$

$$= \frac{(x-1)\left(4x^2 - 7x + 3 - 4x^2 + 6x + 4\right)}{(x-1)^4}$$

$$= \frac{4x^2 - 7x + 3 - 4x^2 + 6x + 4}{(x-1)^3}$$

$$= \frac{7-x}{(x-1)^3}$$

$\therefore$ a stationary point at $(7,\ 2\tfrac{1}{12})$

What type of stationary point?

Note: For $x > 2$, $(x-1)^3 > 0$ and so one only needs to consider the sign of the numerator

x	6	7	8
$\frac{dy}{dx}^{*}$	5	0	–1
	/	–	\

$\therefore (7,\ 2\tfrac{1}{12})$ is a maximum turning point.

Being the only turning point, it is a global (absolute) maximum turning point.
$\therefore y \le 2\tfrac{1}{12}$

Comment

It is clear by inspection that $y \to 2$ as $x \to \infty$ and $x \to -\infty$, but too many students tried to go down this path.

Students who chose the calculus path needed to help themselves out by factorising wherever possible rather than the BFI approach of expanding.
When justifying the nature of the turning point, one can't use a value of x on the other side of the asymptote i.e. students needed to choose a value of x such that $x > 1$.

16) a) i) $v = 1.5 + 0.4t$

$a = \frac{dv}{dt} = 0.4 \ m/s^2$

ii) For $0 \le t \le 5$

$a > 0$ and $v(0) > 0$

$\therefore$ will not come to rest in first 5 seconds

For $t > 5$

$\frac{100}{t^2} - 0.1t = 0$

$100 - 0.1t^3 = 0$

$0.1t^3 = 100$

$t^3 = 1000$

$t = 10 \ s$

iii) $d = \left|\int_0^5 (1.5 + 0.4t)\,dt\right| + \left|\int_5^{10} (100t^{-2} - 0.1t)\,dt\right| + \left|\int_{10}^{12} (100t^{-2} - 0.1t)\,dt\right|$

$= \left|\left[1.5t + 0.2t^2\right]_0^5\right| + \left|\left[-100t^{-1} - 0.05t^2\right]_5^{10}\right| + \left|\left[-100t^{-1} - 0.05t^2\right]_{10}^{12}\right|$

$= \left|1.5(5) + 0.2(5)^2 - (0)\right| + \left|\frac{-100}{(10)} - 0.05(10)^2 - \left(\frac{-100}{(5)} - 0.05(5)^2\right)\right|$

$+ \left|\frac{-100}{(12)} - 0.05(12)^2 - \left(\frac{-100}{(10)} - 0.05(10)^2\right)\right|$

$= \left|\frac{25}{2}\right| + \left|\frac{25}{4}\right| + \left|-\frac{8}{15}\right|$

$= \frac{1157}{60} \ m$

$\approx 19.3 \ m$

COMMENT:

Part (i) & (ii) were done well. In Part (iii) many students forgot to take into account the answer from part (ii) which is where the particle could change direction.

Remember a definite integral gives us the change in displacement.
In the solutions absolute values have been used so that direction of travel does not need to be determined.
Errors were common in the substitution of values also.

b) $\int_0^2 3^{x+1}\,dx$

$$= \int_0^2 e^{\ln 3^{x+1}}\,dx$$

$$= \int_0^2 e^{(x+1)\ln 3}\,dx$$

$$= \left[\frac{1}{\ln 3} e^{(x+1)\ln 3}\right]_0^2$$

$$= \left[\frac{e^{\ln 3^{x+1}}}{\ln 3}\right]_0^2$$

$$= \left[\frac{3^{x+1}}{\ln 3}\right]_0^2$$

$$= \frac{3^{(2)+1}}{\ln 3} - \frac{3^{(0)+1}}{\ln 3}$$

$$= \frac{24}{\ln 3}$$

COMMENT:
It was shocking how few students got this question correct.
We need the base to be e!
Note: $\boxed{a^{\log_a b} = b}$
This allows us to write 3^{x+1} in the form e^{ax+b}.

c) a) considering areas

$$(n-(n-1)) \times \frac{1}{n} < \int_{n-1}^{n} \frac{1}{x}\, dx < (n-(n-1)) \times \frac{1}{n-1}$$

$$\frac{1}{n} < \left[\ln x\right]_{n-1}^{n} < \frac{1}{n-1}$$

$$\frac{1}{n} < \ln n - \ln(n-1) < \frac{1}{n-1}$$

$$\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$$

$$e^{\frac{1}{n}} < e^{\ln\left(\frac{n}{n-1}\right)} < e^{\frac{1}{n-1}}$$ since $f(x)=e^x$ is an increasing function

$$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$$

$$\frac{1}{e^{\frac{1}{n}}} > \frac{n-1}{n} > \frac{1}{e^{\frac{1}{n-1}}}$$ since $f(x)=\frac{1}{x}$ is a decreasing function for $x>0$

$$e^{-\frac{1}{n}} > 1-\frac{1}{n} > e^{-\frac{1}{n-1}}$$

$$\left(e^{-\frac{1}{n}}\right)^n > \left(1-\frac{1}{n}\right)^n > \left(e^{-\frac{1}{n-1}}\right)^n$$ since $f(x)=x^n$ is an increasing function for $x>0$.

$$e^{-1} > \left(1-\frac{1}{n}\right)^n > e^{-\frac{n}{n-1}}$$

$$\therefore\ e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$$

b) $$e^{-\frac{n}{n-1}} \leq \left(1-\frac{1}{n}\right)^n \leq e^{-1}$$

$$\lim_{n\to\infty} e^{-\frac{n}{n-1}} \leq \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n \leq \lim_{n\to\infty} e^{-1}$$

$$\lim_{n\to\infty} e^{-\frac{1}{1-\frac{1}{n}}} \leq \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n \leq e^{-1}$$

$$e^{-\frac{1}{1-0}} \leq \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n \leq e^{-1}$$

$$e^{-1} \le \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n \le e^{-1}$$

$$\therefore \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1}$$

by the sandwich theorem

or squeeze theorem.

COMMENT:

It is important not to skip steps in a show that question.

It may be helpful to consider increasing & decreasing functions with inequalities.

ie If $a > b$

$$\begin{cases} f(a) > f(b), \text{ if } f(x) \text{ is an increasing function} \\ f(a) < f(b), \text{ if } f(x) \text{ is a decreasing function.} \end{cases}$$

Few students completed the proof adequately. Students were able to gain some marks for either establishing an inequality or making a logical progression towards the result.

It is a requirement of a hence question to use the previous result(s)