Sydney Girls High School

MATHEMATICS EXTENSION

Year 11 Yearly

Assessment Task 3

2015

Time Allowed: 60 minutes + 5 minutes reading time

Total: 56 marks

Instructions:

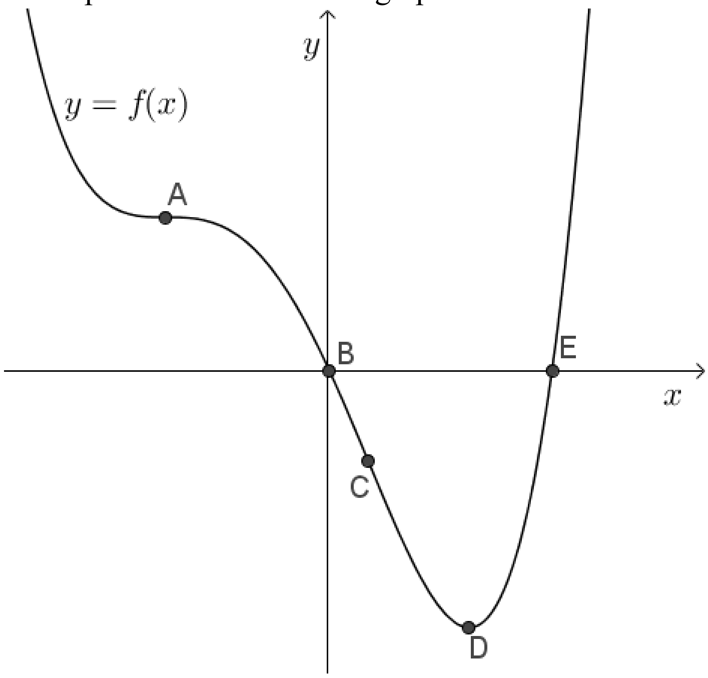
- There are FOUR (4) questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- All answer must be written to the simplest exact form.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your student number at the top of each question and clearly number each question.

Student 's Name:__________________**Teacher's Name:**__________________

Question 1 (14 marks)

Start Each Question on a New Sheet of Paper. **Marks**

(a) For the following graph of $y = f(x)$ the points A, B, C, D and E correspond to important features of the graph.



At which points A, B, C, D or E is

(i) $f'(x) = 0$ **1**

(ii) $f''(x) > 0$ **2**

(b) Write $3x^2 - x - 5$ in the form $Ax(x + 2) + Bx + C$. **2**

(c) Solve the equation $x^6 - 6x^3 - 16 = 0$. **2**

(d) For the equation $y = 2x^3 - 9x^2 - 24x + 40$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **2**

(ii) Find the stationary points and determine their natures. **3**

(iii) Sketch the graph of $y = 2x^3 - 9x^2 - 24x + 40$ showing the stationary points and y-intercept. **2**

End of Question 1

Question 2 (14 marks)

Start Each Question on a New Sheet of Paper. **Marks**

(a) Each student in a music class of 24 studies either the piano or the saxophone or both. It is known that 21 study the piano and 9 study the saxophone. Find the probability that a student selected at random studies both instruments. **1**

(b) Given the parabola $y = \frac{1}{2}x^2 - 2x - 2$.

(i) Rewrite in the form $(x - h)^2 = 4a(y - k)$. **2**

(ii) Write down the **3**

(α) axis of symmetry

(β) focus

(γ) directrix

(c) Given $P(x) = x^3 - 9x^2 + 6x + 56$

(i) Evaluate $P(-2)$ **1**

(ii) Hence solve $P(x) = 0$ **3**

(d) Julia has two pairs of identical purple socks and three pairs of identical green socks. Her socks are randomly mixed in her drawer. She takes two individual socks at random from the drawer in the dark.

(i) Draw a tree diagram to represent this situation. **2**

(ii) Find the probability that she obtains a matching pair. **2**

End of Question 2

Question 3 (14 marks)

Start Each Question on a New Sheet of Paper. **Marks**

(a) Find y as a function of x if $\frac{dy}{dx} = 6x^2 - 4$ and $y = 2$ when $x = 1$. **2**

(b) Solve the inequality $\frac{x^2-3x+3}{x-2} \leq 3$ **3**

(c) For the function $f(x) = x^3 - 3x + 1$.

(i) Show that $f(x)$ has a zero for x between 0 and 1. **1**

(ii) Take $x = 0$ as your first approximation and use two applications of Newton's method to find a better approximation to 3 decimal places. **3**

(iii) Explain why you could not use $x = 1$ as your first approximation. **1**

(d) A and B are points (–1, 7) and (5, –2) respectively; P divides AB internally in the ratio k: 1.

(i) Write down the coordinates of P in terms of k. **2**

(ii) If P lies on the line $5x - 4y = 1$, find the ratio of AP:PB. **2**

End of Question 3

Question 4 (14 marks)

Start Each Question on a New Sheet of Paper. **Marks**

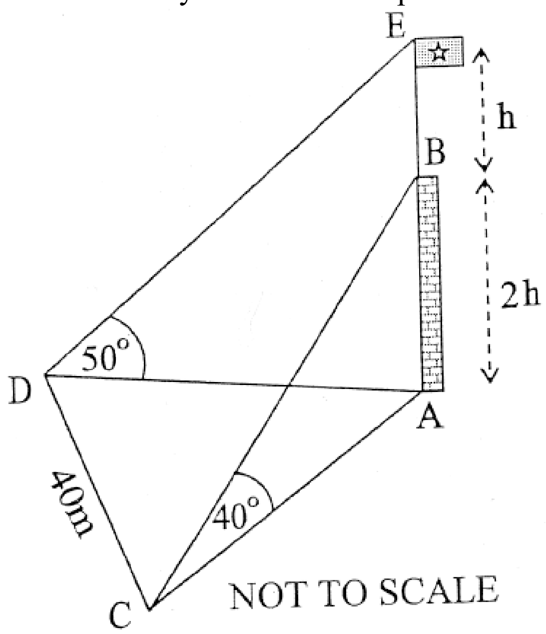
(a) Find the equation of the locus of the point $P(x, y)$ such that it is twice the distance from $y = 3$ as it is from $x = -1$. **2**

(b) The roots of the equation $2x^3 - 5x^2 + 8x + 2 = 0$ are α, β and γ.
Find the value of:

(i) $\alpha + \beta + \gamma - \alpha\beta\gamma$ **2**

(ii) $\alpha(\beta^2 + \gamma^2) + \beta(\alpha^2 + \gamma^2) + \gamma(\alpha^2 + \beta^2)$ **2**

(c) A building AB of height $2h$ metres has a flag pole of height h metres on top of it. From a point C, south west of the building, the angle of elevation of the top of the building is 40°. From a point D, due west of the building, the angle of elevation of the top of the flagpole is $50°$. The points C and D are on the same level as A and they are 40 metres apart.



(i) Find expressions for AC and AD in terms of h. **2**

(ii) Find the height of the building, that is AB, to three decimal places. **3**

(d) For the polynomial $P(x) = x^3 - 3x^2 - 9x + k$ find the values of k for which $P(x)$ has only one zero. **3**

End of Question 4

End of Exam

2015 YEAR 11 YEARLY EXTENSION

Question 1

$f'(x) = 0$ at the stationary pts and horizontal inflexion pt.

a) i) D, A ✓

ii) D and E ✓✓

$f''(x) > 0$ if $f'(x)$ is rising

b) $3x^2 - x - 5 \equiv Ax(x+2) + Bx + C$

$\equiv Ax^2 + 2Ax + Bx + C$

$\equiv Ax^2 + (2A + B)x + C$

Equating the coefficients of both sides

$A = 3$

$C = -5$

$2A + B = -1$

$B = -2A - 1$

$B = -7$

$\therefore \begin{cases} A = 3 \\ B = -7 \\ C = -5 \end{cases}$

c) $x^6 - 6x^3 - 16 = 0$

Let $U = x^3$

$U^2 - 6U - 16 = 0$

$(U - 8)(U + 2) = 0$

$\left[\begin{array}{l} U = 8 \\ U = -2 \end{array}\right. \quad \therefore \quad \left[\begin{array}{l} x^3 = 8 \\ x^3 = -2 \end{array}\right.$

$\left[\begin{array}{l} x = 2 \\ x = (-2)^{\frac{1}{3}} \end{array}\right.$

Note: $x^3 = -2$ is not undefined

$\boxed{Q1}$ d) $y = 2x^3 - 9x^2 - 24x + 40$

i) $\frac{dy}{dx} = 6x^2 - 18x - 24$ ✓

$\frac{d^2y}{dx^2} = 12x - 18$ ✓

ii) $y' = 0 \therefore 6(x^2 - 3x - 4) = 0$

stationary pts should include both x, y values.

$6(x-4)(x+1) = 0$

$\left[\begin{array}{l} x = 4 \rightarrow y = -72 \\ x = -1 \rightarrow y = 53 \end{array}\right.$ ✓

$f''(4) = 12 \times 4 - 18 = 30 > 0$

Testing for Max Min is important.

Min point $(4, -72)$ ✓

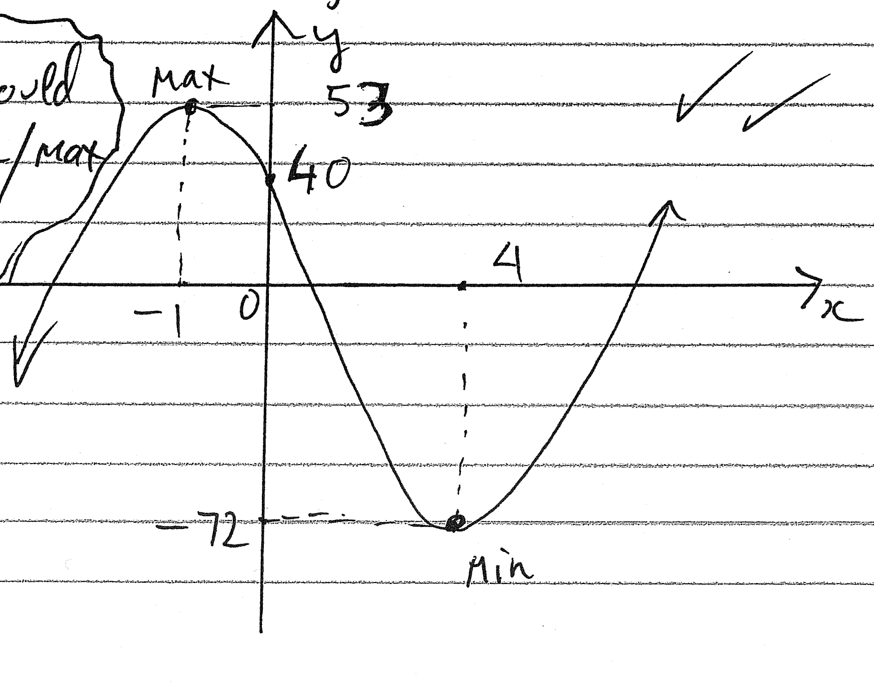
$f''(-1) = 12(-1) - 18$ ✓

$= -30 < 0$ Max $(-1, 53)$

iii) $y = 2x^3 - 9x^2 - 24x + 40$

when $x = 0 \rightarrow y = 40$

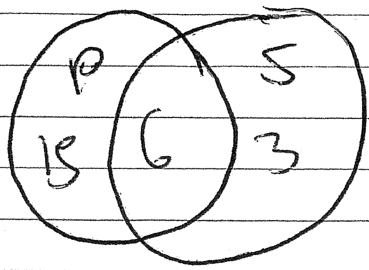
The graph should shows the Min/Max pts and the y-intercept.



✓ ✓

QUESTION 2.

(a)



$|P \cup S| = |P| + |S| + |P \cap S|$

$24 = 21 + 9 - |P \cap S|$

$|P \cap S| = 6.$

$P(P \cap S) = \frac{6}{24}$

$= \frac{1}{4}$

(b)(i) $y = \frac{1}{2}x^2 - 2x - 2.$

$x^2 - 4x - 4 = 2y$

$(x-2)^2 - 4 - 4 = 2y$

$(x-2)^2 = 2y + 8$

$(x-2)^2 = 2(y+4)$

$(x-2)^2 = 2\left(\frac{1}{2}\right)(y+4).$

This question was poorly done by many students. Errors either involved inattention to the sign of terms or an inability to compete the square correctly.

(ii)

(α) axis of symmetry $x = 2$

(β) focus $(2, -3\frac{1}{2}) = (2, -\frac{7}{2})$

(γ) directrix $y = -4\frac{1}{2}$ or $y = -\frac{9}{2}.$

(c) (i) $P(-2) = (-2)^3 - 9(-2)^2 + 6(-2) + 56$

$= 0.$

$\therefore (x+2)$ is a factor

(ii)

$$\begin{array}{r} x^2 - 11x + 28 \\ x+2 \enclose{longdiv}{x^3 - 9x^2 + 6x + 56} \\ \underline{x^3 + 2x^2 } \\ -11x^2 + 6x \\ \underline{-11x^2 - 22x } \\ 28x + 56 \\ \underline{28x + 56} \\ 0 \end{array}$$

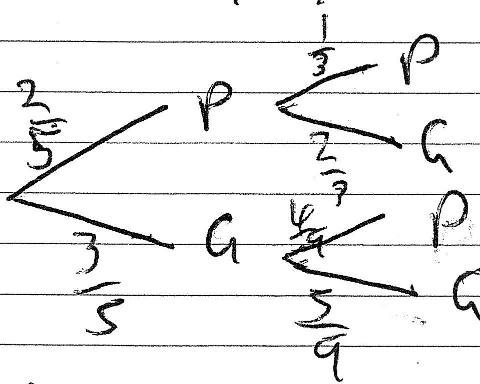
$(x+2)(x^2 - 11x + 28) = 0.$

$(x+2)(x-7)(x-4) = 0$

$x = -2, 7, 4.$

Many students lost a mark because they factorised but did not solve the equation.

(d) (i)



(ii) $P(PP) + P(GG) = \frac{2}{5} \times \frac{1}{3} + \frac{3}{5} \times \frac{5}{9}.$

$= \frac{7}{15}.$

Many trees were drawn but few were useful. The taking of the socks are dependent events. The first sock choosen affects the probabilities of the second choice.

3 (a) $y = 2x^3 - 4x + c$

$2 = 2\times 1^3 - 4\times 1 + c$

$c = 4$

$y = 2x^3 - 4x + 4$

Many students forgot to calculate c

(b) $(x-2)(x^2-3x+3) \le 3(x-2)^2$

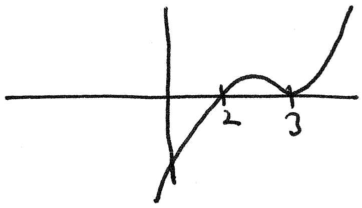
$(x-2)(x^2-3x+3) - 3(x-2)^2 \le 0$

$(x-2)\{x^2-3x+3-3(x-2)\} \le 0$

$(x-2)(x^2-3x+3-3x+6) \le 0$

$(x-2)(x^2-6x+9) \le 0$

$(x-2)(x-3)^2 \le 0$



x=2 makes the inequality undefined

$\therefore \quad x < 2$ or $x = 3$

(c) (i) $f(0) = 1$

$f(1) = -1$

(ii) $f'(x) = 3x^2 - 3$

$x_2 = 0 - \dfrac{f(0)}{f'(0)}$

$= 0 - \dfrac{1}{-3}$

$= \frac{1}{3}$

$x_3 = \frac{1}{3} - \dfrac{f(\frac{1}{3})}{f'(\frac{1}{3})}$

$= \frac{1}{3} - \dfrac{\frac{1}{27}}{-2\frac{2}{3}}$

$= 0.347|22222$

$\doteqdot 0.347$

Many students forgot to express their answer as a decimal

(iii) $f'(1) = 0$

(d) (i) $\left(\dfrac{5k - 1\times 1}{k+1}, \dfrac{-2k + 7\times 1}{k+1}\right)$

$= \left(\dfrac{5k-1}{k+1}, \dfrac{7-2k}{k+1}\right)$

(ii) $5\times\dfrac{(5k-1)}{k+1} - 4\times\dfrac{(7-2k)}{k+1} = 1$

$25k - 5 - 28 + 8k = k+1$

$33k - 33 = k+1$

$32k = 34$

$k = \dfrac{34}{32}$

$= \dfrac{17}{16}$

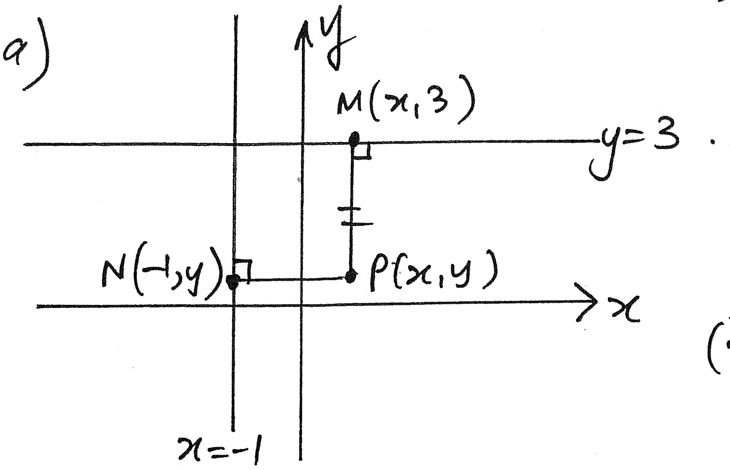
$\therefore AP : PB = \frac{17}{16} : 1$

$= 17 : 16$

Many arithmetical errors occurred in this part.

Question 4

a)



Let $P=(x,y)$, $M(x,3)$ and $N=(-1,y)$

$$PM = 2PN$$

$$PM^2 = 4PN^2$$

$$\underbrace{\cancel{(x-x)^2}}_{0} + (y-3)^2 = 4\left[(x+1)^2 + \underbrace{\cancel{(y-y)^2}}_{0}\right]$$

$$(y-3)^2 = 4(x+1)^2$$

$$\therefore y-3 = \pm(2(x+1))$$

$\therefore$ Locus is two lines:

$$y-3 = 2x+2 \qquad \text{or} \qquad y-3 = -(2x+2)$$

$$y = 2x+5 \qquad\qquad y-3 = -2x-2$$

$$y = -2x+1.$$

b)

$$\underset{+a}{2x^3} \underset{-b}{-5x^2} \underset{+c}{+8x} \underset{-d}{+2} = 0$$

i) $\alpha+\beta+\gamma-\alpha\beta\gamma$

$$= -\frac{b}{a} - \left(-\frac{d}{a}\right)$$

$$= \frac{5}{2} + \frac{2}{2}$$

$$= \frac{7}{2}.$$

4b) ii) $\alpha(\beta^2+\gamma^2)+\beta(\alpha^2+\gamma^2)+\gamma(\alpha^2+\beta^2)$

$= \alpha\beta^2+\alpha\gamma^2+\beta\alpha^2+\beta\gamma^2+\gamma\alpha^2+\gamma\beta^2$

$= \alpha^2\beta+\alpha^2\gamma + \beta^2\alpha+\beta^2\gamma + \gamma^2\alpha+\gamma^2\beta$

$= \alpha(\alpha\beta+\alpha\gamma)+\beta(\alpha\beta+\beta\gamma)+\gamma(\alpha\gamma+\beta\gamma)$

Now: $\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a} = 4.$

$\therefore \quad \alpha\beta+\alpha\gamma = 4-\beta\gamma.$

Similarly: $\alpha\beta+\beta\gamma = 4-\alpha\gamma$

and: $\alpha\gamma+\beta\gamma = 4-\alpha\beta$

$= \alpha(4-\beta\gamma)+\beta(4-\alpha\gamma)+\gamma(4-\alpha\beta)$

$= 4\alpha-\alpha\beta\gamma+4\beta-\alpha\beta\gamma+4\gamma-\alpha\beta\gamma$

$= 4(\alpha+\beta+\gamma)-3(\alpha\beta\gamma)$

$= 4\left(\frac{5}{2}\right)-3(-1)$

$= 10+3$

$= 13.$

OR (as recognised by some students:)

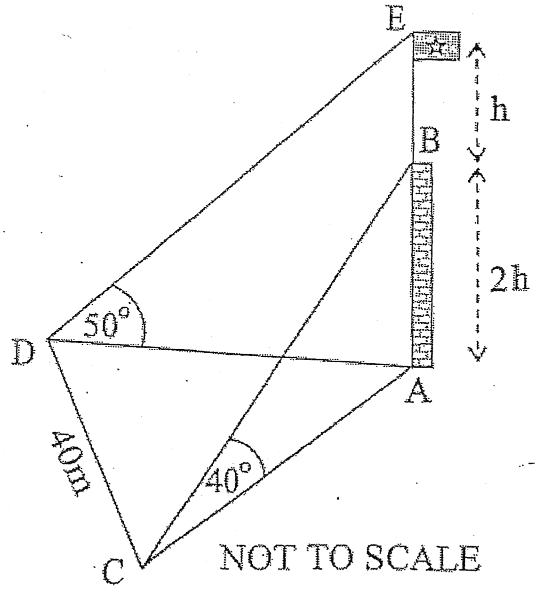
$\alpha(\beta^2+\gamma^2)+\beta(\alpha^2+\gamma^2)+\gamma(\alpha^2+\beta^2)$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma)-3(\alpha\beta\gamma)$

$= 4\left(\frac{5}{2}\right)-3(-1)$

$= 13.$

4 c)



i) $\tan 40° = \dfrac{2h}{AC}$

$\therefore AC = \dfrac{2h}{\tan 40°}$

[OR : $\tan 50° = \dfrac{AC}{2h}$

$\therefore AC = 2h \tan 50°$]

$\tan 50° = \dfrac{3h}{AD}$

$\therefore AD = \dfrac{3h}{\tan 50°}$

[OR : $\tan 40° = \dfrac{AD}{3h}$

$\therefore AD = 3h \tan 40°$]

ii) In ΔADC :

$$40^2 = AC^2 + AD^2 - 2AC.AD.\cos 45$$

$$1600 = (2h\tan 50°)^2 + (3h\tan 40°)^2 - \frac{2(2h\tan 50°)(3h\tan 40°)}{\sqrt{2}}$$

$$1600 = h^2(2\tan 50°)^2 + h^2(3\tan 40°)^2 - h^2(6\sqrt{2}\tan 50°\tan 40°)$$

$$\therefore h^2 = \frac{1600}{(2\tan 50°)^2 + (3\tan 40°)^2 - (6\sqrt{2}\tan 50°\tan 40°)}$$

Taking square root:

$h \doteqdot 21.28195892$

$\therefore \quad AB = 2h$

$= 42.564$ m.

* Many students thought that ΔADC was right-angled and this was incorrect.

d) $P(x) = x^3 - 3x^2 - 9x + k$

$P'(x) = 3x^2 - 6x - 9$

For turning points $P'(x) = 0$:

$0 = 3(x^2 - 2x - 3)$

$0 = (x-3)(x+1)$

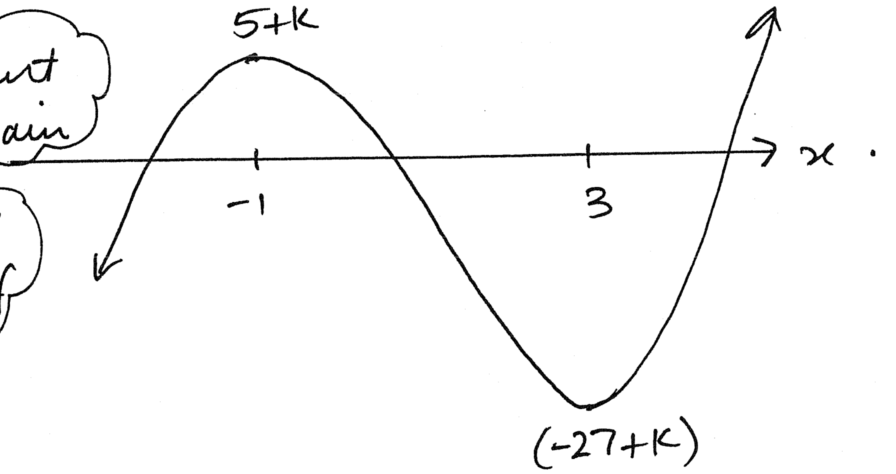
$P(3) = -27 + k \qquad P''(3) = 12 > 0 \therefore$ min. t.pt.

$P(-1) = 5 + k \qquad P''(-1) = -12 < 0 \therefore$ max. t.pt.

$\therefore$ Turning points are $(3, -27+k)$ and $(-1, 5+k)$

Possible sketch of $P(x)$:

* Many different attempts but main concept was misunderstood by majority of students.



* Only one zero means that $P(x)$ intersects the x-axis at only one point.
This can happen if:

$\underline{\underline{k < -5}}$ or $\underline{\underline{k > 27}}$ ✓