Sydney Girls High School

Mathematics Extension 1

Year 11

Assessment Task 3

2017

Reading time : 5 minutes

Writing time : 60 minutes

Total marks : 60

Topics: Other inequalities, Internal/External Division of an Interval, 3 Dimensional Trigonometry, the Quadratic Polynomial, Locus, Further Trigonometry (excluding Integration), Geometrical application of Differentiation, Series and Series Applications

Instructions:

- There are FOUR (4) questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write using a black or blue pen.

Student 's Name:____________________**Teacher's Name:**_____________________

Question 1 **15 Marks**

(a) What is the 20th term of the sequence defined by $T_n = (n-1)(3+n)$? **1**

(b) Given the equation of the parabola $(x-3)^2 = 16(y+2)$, find:

(i) the coordinates of the vertex . **1**

(ii) the coordinates of the focus. **1**

(iii) the equation of the axis of symmetry. **1**

(iv) the equation of the directrix. **1**

(c) The quadratic equation $3x^2 - 2x - 5 = 0$ has roots α and β. Evaluate :

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta^2 + \beta\alpha^2$ **1**

(iii) $\alpha^2 + \beta^2$ **2**

(d) Find the coordinates of the point that divides the interval joining the points (0,4) and (2,7) externally in the ratio 2 : 3. **3**

(e) Solve $\dfrac{2}{x+1} > 4$. **3**

Question 2 (Start a new page) **15 Marks**

(a) Find the obtuse angle between the lines $y = 2x - 1$ and $3x - 2y + 1 = 0$.
Give your answer correct to the nearest degree. **3**

(b) Find the limiting sum of the series $8 - 4 + 2 - 1 + \ldots\ldots$ **2**

(c) Find A, B and C such that

$$3x^2 - x + 3 = A(x-1)(x+2) + B(x-2) + C\ .$$ **3**

(d) Find the primitive function of $4x - \frac{1}{x^2}$. **2**

(e) Given $2\sin^2\theta = 3\cos\theta$, find all possible values of θ where $0^\circ \le \theta \le 360^\circ$. **3**

(f) Given that $\tan A = 2\tan B + \cot B$, prove the following identity:

$$2\tan(A - B) = \cot B$$ **2**

Question 3 (Start a new page) **15 Marks**

(a) A and B are the points (4, 2) and (1, 5) respectively. Find the equation of the locus of a point $Q(x, y)$ which moves such that QA is perpendicular to QB. **2**

(b)

(i) Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0^\circ \le \alpha \le 90^\circ$. **2**

(ii) Hence, or otherwise, solve the equation $\sin x + \sqrt{3}\cos x = \sqrt{2}$ for $0^\circ \le x \le 360^\circ$. **2**

(c) For what values of k does the quadratic equation $kx^2 - 4kx - k + 5 = 0$ have real, distinct roots ? **2**

(d) Consider the curve $y = x^3 - x^2 - 5x + 1$

(i) Find the coordinates of any turning points and find their nature. **3**

(ii) Find any points of inflexion. **1**

(iii) Sketch the curve. You do not need to find the x-intercepts. **2**

(iv) For what values of x is the curve both decreasing and concave up ? **1**

Question 4 (Start a new page) **15 marks**

(a) An open cylindrical can, without the top lid, has an external surface area of $20\pi \text{ cm}^2$.

(i) Let r be the radius and h be its height in centimetres. Show that **1**

$$h = \frac{20 - r^2}{2r}.$$

(ii) Hence, show that the volume of the can is given by **1**

$$V = 10\pi r - \frac{1}{2}\pi r^3.$$

(iii) Show that the maximum volume is obtained when the height of the can is equal to its radius. **3**

(b) A surveyor observes two towers A and B. Tower A is due north of the surveyor with a height of 80 m, and Tower B is on a bearing of θ (where $\theta < 90^\circ$) from the surveyor with a height of 120 m. The angles of elevation of the tops of the towers A and B are 40° and 36° respectively from the surveyor. The towers are 150 m apart on a horizontal plane. Calculate the value of θ to the nearest minute. **3**

(c) Mathew borrowed \$100000 to buy an apartment, which he is going to rent. He plans to repay \$500 every fortnight from his salary. On every 4th week he will repay an extra \$900 from the rent he receives. Interest on the loan is 6.5% p.a. reducing fortnightly.

Let A_n is the amount owing on the loan after n fortnights and R=1.0025

(i) Show that $A_2 = 100000R^2 - 500(R+1) - 900$ **1**

(ii) Show that **2**

$$A_4 = 100000R^4 - 500(R^3 + R^2 + R + 1) - 900(R^2 + 1)$$

(iii) Hence show that $A_{100} = 100000R^{100} - 500\dfrac{R^{100} - 1}{R - 1} - 900 \times \dfrac{R^{100} - 1}{R^2 - 1}$ **2**

(iv) What percentage of the money paid in first 100 repayments will be interest? **2**

END OF PAPER

Q1

a) $T_{20} = (20-1)(3+20)$
$= 437$

b) $(x-3)^2 = 16(y+2)$

i) $V(3,-2)$

ii) $4a = 16$
$a = 4$
Focus point $S(x_0, y_0 + a)$
$S(3, 2)$

iii) $x = 3$

iv) Directrix: $y = -6$

c) $3x^2 - 2x - 5 = 0$

i) $\alpha + \beta = \frac{2}{3}$

ii) $\alpha\beta = -5/3$
$\alpha\beta^2 + \beta\alpha^2 = \alpha\beta(\alpha+\beta)$
$= -\frac{5}{3}\left(\frac{2}{3}\right)$
$= -\frac{10}{9}$

iii) $(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$
$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= \frac{4}{9} + \frac{10}{3} = 34/9$

$\boxed{Q1}$

d) $A(0,4)$ $B(2,7)$. External Division ratio $2:3$

Let $P(x,y)$ be the point that divides the interval.

$$x = \frac{0(-3) + 2(2)}{2 + -3} = -4$$

$$y = \frac{4(-3) + 7(2)}{2 + -3} = -2$$

$P(-4, -2)$

* A number of students could not recognise the difference between Internal and External Ratio Division.

e) $\dfrac{2}{x+1} > 4$

$$2(x+1) > 4(x+1)^2$$

$$4(x+1)^2 - 2(x+1) < 0$$

$$2(x+1)\left[2(x+1) - 1\right] < 0$$

$$2(x+1)(2x+1) < 0$$

$$-1 < x < -\tfrac{1}{2}$$

2 (a) $m_1 = 2$, $m_2 = \frac{3}{2}$

$$\tan\theta = \frac{2 - \frac{3}{2}}{1 + 2 \times \frac{3}{2}}$$

Many students used the wrong formula.

$\theta = 7.125016349$
$= 7°$

obtuse $\angle = 180° - 7°$
$= 173°$

Many students forgot what an obtuse $\angle$ is.

(b) $$S_\infty = \frac{8}{1 - -\frac{1}{2}}$$

$$= \frac{16}{3}$$

(c) $3x^2 - x + 3 = A(x^2 + x - 2) + Bx - 2B + C$
$= Ax^2 + (A + B)x - 2B + C - 2A$

$A = 3$, $A + B = -1$ $-2B + C - 2A = 3$
$3 + B = -1$ $8 + C - 6 = 3$
$B = -4$ $C = 1$

(d) $4x - x^{-2}$

Primitive function

$$\frac{4x^2}{2} - \frac{x^{-1}}{-1} + c = 2x^2 + \frac{1}{x} + c$$

(e) $2(1 - \cos^2\theta) - 3\cos\theta = 0$
$2 - 2\cos^2\theta - 3\cos\theta = 0$
$2\cos^2\theta + 3\cos\theta - 2 = 0$
$2\cos^2\theta + 4\cos\theta - \cos\theta - 2 = 0$
$2\cos\theta(\cos\theta + 2) - 1(\cos\theta + 2) = 0$
$(2\cos\theta - 1)(\cos\theta + 2) = 0$
$\cos\theta = \frac{1}{2}$ $\cos\theta = -2$

$\therefore \theta = 60°, 300°$

Students who used the t method were generally unable to complete the process correctly.

(f) $2\tan(A - B)$

$$= 2\,\frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= 2\,\frac{2\tan B + \cot B - \tan B}{1 + \tan A \tan B}$$

$$= 2 \cdot \frac{\tan B + \cot B}{1 + (2\tan B + \cot B)\tan B}$$

$$= 2\,\frac{\tan B + \cot B}{1 + 2\tan^2 B + 1}$$

$$= 2\,\frac{(\tan^2 B + 1)\cot B}{2 + 2\tan^2 B}$$

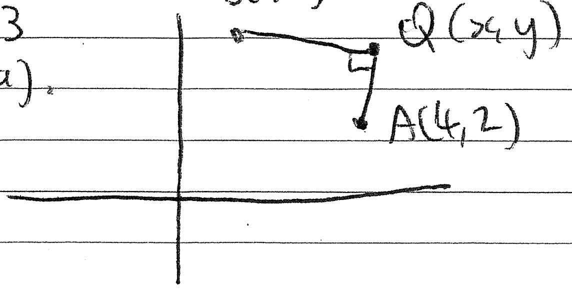
$$= \frac{(1 + \tan^2 B)\cot B}{1 + \tan^2 B}$$

$$= \cot B$$

2017 YEAR 11 EXT TASK 3.

Q3

(a).



If $QA \perp QB$ then $m_{QA} \times m_{QB} = -1$

$m_{QA} = \dfrac{y-2}{x-4}$ $\quad$ $m_{QB} = \dfrac{y-5}{x-1}$

$$\frac{y-2}{x-4} \times \frac{y-5}{x-1} = -1$$

$$(y-2)(y-5) = -(x-4)(x-1)$$

$$y^2 - 7y + 10 = -x^2 + 5x - 4.$$

$$x^2 - 5x + y^2 - 7y + 14 = 0.$$

(b) (i) $R\sin(x+\alpha) = R\cos\alpha \sin x + R\sin\alpha \cos x.$

$R\cos\alpha = 1$ $\quad$ $R\sin\alpha = \sqrt{3}.$

$R^2 = 1+3$ $\quad$ $\tan\alpha = \sqrt{3}$

$R = 2$ $\quad$ $\alpha = 60^\circ$

$$\sin x + \sqrt{3}\cos x \equiv 2\sin(x+60^\circ).$$

(b)(ii). $\sin x + \sqrt{3}\cos x = \sqrt{2}$

$2\sin(x+60°) = \sqrt{2}$

$\sin(x+60°) = \frac{1}{\sqrt{2}}$.

When you lose a solution at the lower end of the domain you gain one at the upper end.

$x+60° = 45°, 135°, 405°$

$x = -15°, 75°, 345°$

So $x = 75°, 345°$ since $0° \leq x \leq 360°$

(c) $\Delta > 0$

For distinct roots the discriminant can not equal zero.

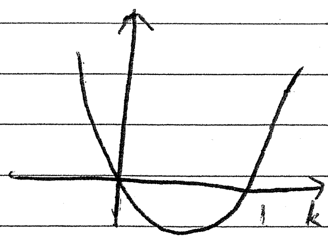
$16k^2 - 4k(-k+5) > 0$

$16k^2 + 4k^2 - 20k > 0$.

$20k^2 - 20k > 0$

$k^2 - k > 0$.

$k(k-1) > 0$



$k < 0, k > 1$.

(d)(i) $y = x^3 - x^2 - 5x + 1$

$\frac{dy}{dx} = 3x^2 - 2x - 5$.

$\frac{d^2y}{dx^2} = 6x - 2$.

Stat Pts $\frac{dy}{dx} = 0$

$$3x^2 - 2x - 5 = 0$$

$$\frac{(3x-5)(3x+3)}{3} = 0$$

$$(3x-5)(x+1) = 0$$

$$x = \frac{5}{3}, -1.$$

Nature

at $x = \frac{5}{3}$

$$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8 > 0 \quad \text{minima}$$

at $x = -1$

$$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8 < 0 \quad \text{maxima.}$$

Points

at $x = \frac{5}{3}$ $\quad y = -\frac{148}{27} \approx -5.5.$

at $x = -1$ $\quad y = 4.$

Turning Points $\left(\frac{5}{3}, -\frac{148}{27}\right)$ minima.

$(-1, 4)$ maxima.

(ii) Inflexion point

$$\frac{d^2y}{dx^2} = 0.$$

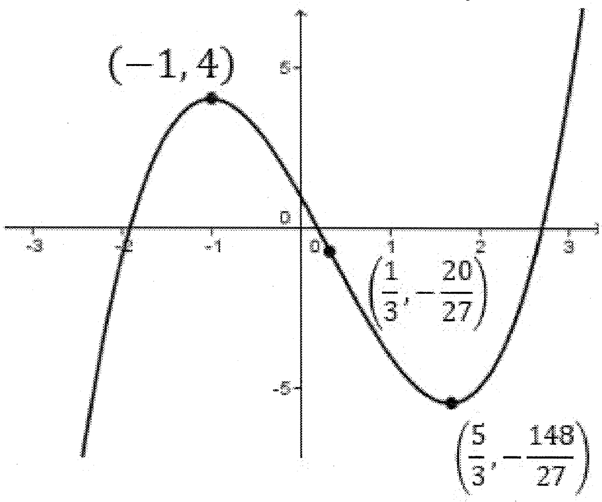
$$6x-2=0.$$

$$x = \frac{1}{3}.$$

$$y = -\frac{20}{27} \approx -0.7$$

Inflexion $\left(\frac{1}{3}, -\frac{20}{27}\right)$.

(iii)



(iv) $\frac{1}{3} < x < \frac{5}{3}$.

Many students included the end points with less than or equal to.

11 Ext 1 2017 Task 3

4) a)

i) $SA = \pi r^2 + 2\pi r h$

$20\pi = \pi r^2 + 2\pi r h$

$20 = r^2 + 2rh$

$2rh = 20 - r^2$

$h = \frac{20 - r^2}{2r}$

ii) $V = \pi r^2 h$

$= \pi (r\cancel{r})\left(\frac{20 - r^2}{2\cancel{r}}\right)$

$= \pi\left(\frac{20r - r^3}{2}\right)$

$= \pi\left(10r - \frac{r^3}{2}\right)$

$= 10\pi r - \frac{\pi r^3}{2}$

iii) $V' = 10\pi - \frac{3\pi r^2}{2}$

$V' = 0$

$20\cancel{\pi} = 3\cancel{\pi} r^2$

$r^2 = \frac{20}{3}$

$r = \sqrt{\frac{20}{3}}$

$V'' = -3\pi r$

< 0

$\therefore$ max

$r = \sqrt{\frac{20}{3}}$

* If V'' not shown no marks was awarded

$h = \frac{20 - \frac{20}{3}}{2\left(\sqrt{\frac{20}{3}}\right)}$

$= \frac{\frac{40}{3}}{\frac{2\sqrt{20}}{\sqrt{3}}}$

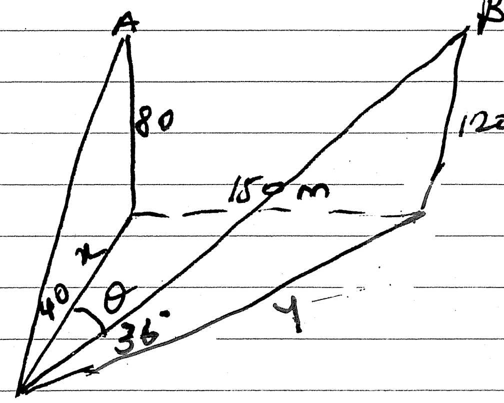
$= \frac{20\sqrt{3}}{3\sqrt{20}} \times \frac{\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{20}}{\sqrt{20}}$

$= \frac{\cancel{20} \times \cancel{3}\sqrt{20}}{\cancel{3}\sqrt{3} \times \cancel{20}}$

$= \frac{\sqrt{20}}{\sqrt{3}} = r$

Many students couldn't show this step if you didn't rationalise you couldn't get the mark.

b)



$\tan 40 = \frac{80}{x}$

$x = \frac{80}{\tan 40}$

$\tan 36 = \frac{120}{y}$

$y = \frac{120}{\tan 36}$

$\cos\theta = \frac{\left(\frac{120}{\tan 36}\right)^2 + \left(\frac{80}{\tan 40}\right)^2 - 150^2}{2 \times \left(\frac{120}{\tan 36}\right)\left(\frac{80}{\tan 40}\right)}$

$\theta = 63°52'$

c)

i) $A_1 = 100000(R) - 500$

$A_2 = (100000R - 500)R - 500 - 900$

$= 100000R^2 - 500R - 500 - 900$

$= 100000R^2 - 500(R+1) - 900$

* Many students couldn't show these steps properly. so no marks were awarded if steps not shown.

ii) $A_3 = (100000R^2 - 500(R+1) - 900)R - 500$

$= 100000R^3 - 500(R^2+R) - 900R - 500$

$= 100000R^3 - 500(R^2+R+1) - 900R$

$A_4 = (100000R^3 - 500(R^2+R+1) - 900R)R - 500 - 900$

$= 100000R^4 - 500(R^3+R^2+R) - 900R^2 - 500 - 900$

$= 100000R^4 - 500(R^3+R^2+R+1) - 900(R^2+1)$

kih you didn't show $(R^2)^{50}$ no mark.

iii) $A_{100} = 100000R^{100} - 500(R^{99}+R^{98}+\dots+1) - 900(R^{98}+R^{96}+\dots 1)$

$= 100000R^{100} - 500\left(\frac{1(R^{100}-1)}{R-1}\right) - 900\left(\frac{1((R^2)^{50}-1)}{R^2-1}\right)$

$= 100000R^{100} - 500\frac{(R^{100}-1)}{R-1} - 900\left(\frac{R^{100}-1}{R^2-1}\right)$

iv) $A_{100} = \$20648.77$ paid $\$79351.23$

Actually paid $= 500 \times 100 + 50 \times 900 = 95000$

$I\% = \frac{15648.77}{95000} \times 100\% = 16.47\%$

* Many students had problem with this question