Sydney Girls High School

MATHEMATICS EXTENSION 1

YEAR 11 ASSESSMENT TASK 3 2019

Time Allowed: 60 minutes + 5 minutes reading time

Total: 60 marks

Topics: Permutations and combinations; The binomial expansion and Pascal's triangle; Inequalities; Graphical relationships; Polynomials; Inverse functions; Parametric form of a function or relation; Further trigonometric identities.

General Instructions:

- There are FOUR (4) Questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- A correct answer without any working will be awarded a maximum of 1 mark.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.

Student Name :__________________ **Teacher Name :**__________________

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Question 1 (15 Marks)

(a) How many different arrangements of the letters of the word MATHS are possible? 1

(b) Simplify $\dfrac{\sin 2x}{1+\cos 2x}$. 2

(c) A curve has parametric equations $x = 2t$ and $y = 3 - t^2$. 2

Find the Cartesian equation of the curve.

(d) The polynomial $P(x)$ is given by $P(x) = 2x^3 + 7x^2 - 46x + 21$.

(i) Show that $P(x)$ is divisible by $(x-3)$. 1

(ii) Hence, fully factorise $P(x)$. 3

(e)

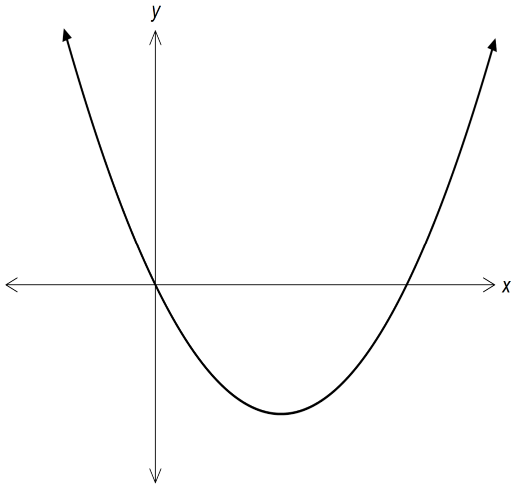
(i) State the domain and range of $y = 3\sin^{-1}(2x)$. 2

(ii) Hence, sketch $y = 3\sin^{-1}(2x)$. 1

(f) Solve the inequality $\dfrac{2x}{x-1} \geq 1$. 3

End of Question 1

(a)



The graph of the parabola $y = f(x)$ is given above. Sketch on separate number planes the graphs of:

(i) $y = |f(x)|$. 1

(ii) $y = \dfrac{1}{f(x)}$. 2

(b) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. 2

(c) A debating group consists of 12 students, 8 of whom are girls.
How many debating teams of three students can be formed if:

(i) There are no girls at all. 1

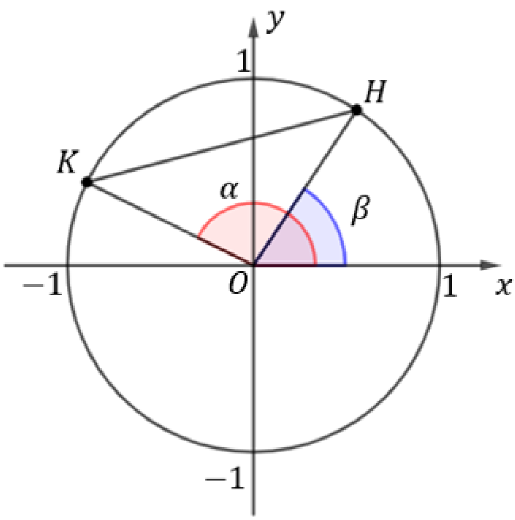
(ii) There is exactly one girl. 1

(iii) There is at least one girl. 2

Question 2 continues on the next page.

(d) The diagram below shows the circle $x^2 + y^2 = 1$, where the point H has coordinates $(\cos\beta, \sin\beta)$ and the point K has coordinates $(\cos\alpha, \sin\alpha)$.

The line OH makes an angle of β with the positive x-axis and the line OK makes an angle of α with the positive x-axis.



(i) Find an expression for the distance HK. 1

(ii) By considering the triangle KOH, show that $HK^2 = 2 - 2\cos(\alpha - \beta)$. 1

(iii) Hence show that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. 2

(iv) If $\alpha = \dfrac{\pi}{3}$ and $\beta = \dfrac{\pi}{4}$ find the exact value of $\cos\dfrac{\pi}{12}$. 2

End of Question 2

Question 3 — Start writing on a New Page — (15 Marks)

(a) Find the term independent of x in the expansion $\left(x^2 + \frac{2}{x}\right)^{15}$. **2**

(b) Find an expression for y in terms of x if for $x > 0$ and $y > 0$, $\tan^{-1} x = \tan^{-1} y + \frac{\pi}{4}$. **3**

(c)

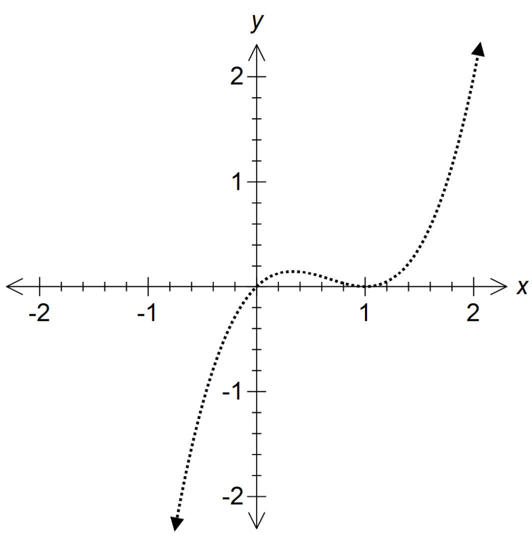
(i) Given the function $f(x) = 6x - x^2$, find the largest possible domain including zero, such that an inverse function $f^{-1}(x)$ exists. **1**

(ii) Find the equation of $f^{-1}(x)$. **3**

(d) Simplify $\dfrac{\cos(\theta+\alpha) - \cos(\theta-\alpha)}{\sin(\theta+\alpha) + \sin(\theta-\alpha)}$. **2**

(e) In the binomial expansion of $\left(1 + \frac{x}{k}\right)^n$, the coefficient of x^3 is twice the coefficient of x^2. Prove that $n = 6k + 2$. **2**

(f) The graph of the $y = f(x)$ is given below. Sketch the graph of $y^2 = f(x)$. **2**



End of Question 3

Question 4 **Start writing on a New Page** **(15 Marks)**

(a) The polynomial equation $x^3 + bx^2 + cx + d = 0$ has roots α, α^2 and α^3 for some real number $\alpha \neq 0$.

(i) Find in terms of b, c and d the value of $\dfrac{1}{\alpha} + \dfrac{1}{\alpha^2} + \dfrac{1}{\alpha^3}$. 2

(ii) Show that $b^3 d - c^3 = 0$. 2

(b) By substituting a suitable value for x in the expansion $(1+x)^n$, show that 2

$$1 + 2\binom{n}{1} + 4\binom{n}{2} + \ldots + 2^{n-1}\binom{n}{n-1} = 3^n - 2^n.$$

(c) Show that $\dfrac{\sin\theta + \sin(90^\circ - \theta) + 1}{\sin\theta - \sin(90^\circ - \theta) + 1} = \cot\dfrac{\theta}{2}$. 3

(d) A monic cubic polynomial when divided by $x^2 + 7$ leaves a remainder of $x + 12$ and when divided by x leaves a remainder of –6. 3

Find the polynomial in the form $ax^3 + bx^2 + cx + d$.

(e)

(i) State the domain of $y = x^2 \cos^{-1} x$. 1

(ii) Hence, sketch the graph of $y = x^2 \cos^{-1} x$. 2

End of paper

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2019

Yr 11 Ext. 1 Task # 3

Question 1

(a) $5! = 120$ $/1$

(b) $\dfrac{\sin 2x}{1+\cos 2x}$

$= \dfrac{2\sin x \cos x}{1 + 2\cos^2 x - 1}$

$= \dfrac{\sin x}{\cos x}$

$= \tan x$ $/2$

(c) $x = 2t$ ..... ①

$y = 3 - t^2$ ..... ②

From ①: $t = \frac{x}{2}$ ... ③

Sub ③ in ②:

$y = 3 - \left(\frac{x}{2}\right)^2$

$y = 3 - \frac{x^2}{4}$

$y = -\frac{x^2}{4} + 3$ $/2$

(d)(i)

$P(x) = 2x^3 + 7x^2 - 46x + 21$

$P(3) = 2(27) + 7(9) - 46(3) + 21$

$= 0$

∴ $(x-3)$ is a ~~factor~~

∴ $P(x)$ is divisible by $(x-3)$

$/1$

d)ii)

$$\begin{array}{r} 2x^2 + 13x - 7 \\ x-3 \overline{) 2x^3 + 7x^2 - 46x + 21} \\ 2x^3 - 6x^2 \\ \hline 13x^2 - 46x \\ 13x^2 - 39x \\ \hline -7x + 21 \\ -7x + 21 \\ \hline 0 \end{array}$$

$\therefore P(x) = (x-3)(2x^2 + 13x - 7)$

$= (x-3)(x+7)(2x-1)$

$/3$

e)i) $y = 3\sin^{-1} 2x$

$D: -1 \le 2x \le 1$

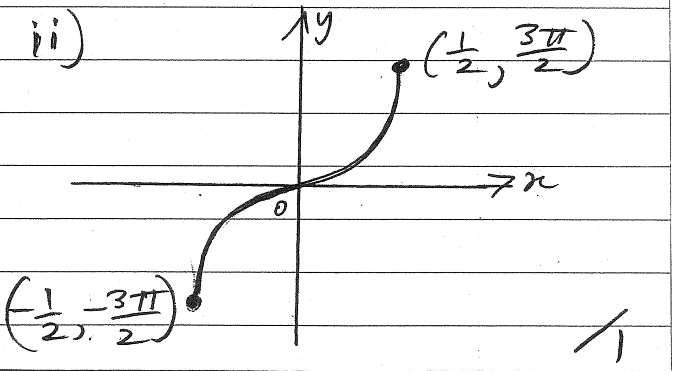
$-\frac{1}{2} \le x \le \frac{1}{2}$

$\frac{y}{3} = \sin^{-1} 2x$

$R: -\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ $/2$

ii)



$/1$

(f) $\frac{2x}{x-1} \times (x-1)^2 \ge 1 \times (x-1)^2$

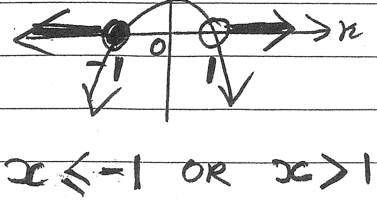
$2x(x-1) \ge (x-1)^2$

$(x-1)^2 - 2x(x-1) \le 0$

$(x-1)[(x-1) - 2x] \le 0$

$(x-1)(-x-1) \le 0$

$-(x-1)(x+1) \le 0$



$x \ne 1$

$x \le -1$ OR $x > 1$

$/3$

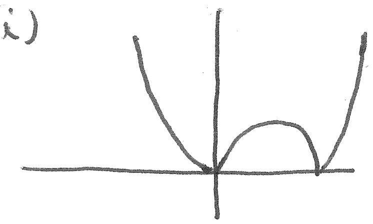
Notes for Q1

e) ii) Shape of curve needs to be more defined
(more "curvy")

f) - Test for Quadratic Inequality
needs to be more clearly shown

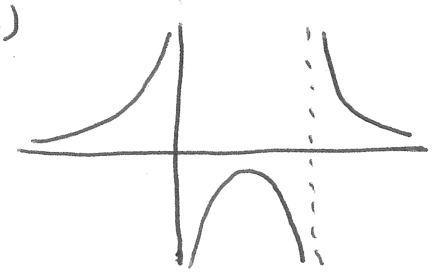
- Many students forgot the restriction $x \neq 1$

2 (a) (i)



This graph must have sharp corners on the x axis

(ii)



(b) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta \mp \sin 2\theta \sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta \mp 2\sin\theta\cos\theta\sin\theta$

$= \cos^3\theta - \sin^2\theta\cos\theta \mp 2\sin^2\theta\cos\theta$

$= \cos^3\theta - 3\sin^2\theta\cos\theta$

$= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ Q.E.D.

In show questions important steps must not be left out.

(c) (i) ${}^4C_3 = 4$

(ii) $8 \times {}^4C_2 = 48$

(iii) $12C_3 - 4 = 216$

(d) (i) $\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$

Co-ordinate geometry is required here.

(ii) $HK^2 = 1^2 + 1^2 - 2\times1\times1\times\cos(\alpha-\beta)$

$= 2 - 2\cos(\alpha-\beta)$

The cosine rule is required here.

(iii) $2 - 2\cos(\alpha-\beta) = \left(\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}\right)^2$

$= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin\beta^2$

$= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$

$\therefore \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Q.E.D.

(iv) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \dfrac{1+\sqrt{3}}{2\sqrt{2}}$

Q3 Ext 1 2019

a) $T_{k+1} = {}^{15}C_k \, (x^2)^{15-k} \cdot 2^k \, x^{-k}$

$x^{30-2k-k} = x^0$

$x^{30-3k} = x^0$

This question was done well.

$30 - 3k = 0$
$k = 10$

$T_{11} = {}^{15}C_{10} \, (x^2)^5 \cdot 2^{10} \cdot x^{-10}$

$= 3075072$

b) $\tan^{-1}x - \tan^{-1}y = \frac{\pi}{4}$

This question was done poorly so many students didn't know where to start.

Let $\alpha = \tan^{-1}x$ $\quad$ $B = \tan^{-1}y$

$\tan\alpha = x$ $\quad$ $\tan B = y$

$\tan(\alpha - B) = \tan\frac{\pi}{4}$

$$\frac{\tan\alpha - \tan B}{1 + \tan\alpha\tan B} = 1$$

$$\frac{x - y}{1 + xy} = 1$$

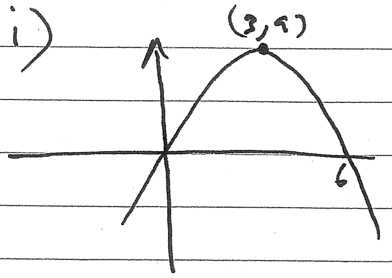
$x - y = 1 + xy$

$x - 1 = xy + y$

$x - 1 = y(x+1)$

$$y = \frac{x-1}{x+1}$$

c) i)



D: $x \leq 3$

* Many students didn't know the correct domain

ii) $x = 6y - y^2$

$y^2 - 6y + 9 = 9 - x$

$(y-3)^2 = 9 - x$

$y = 3 \pm \sqrt{9-x}$

* Many students didn't choose the correct branch

$x \leq 3 \quad \therefore \; y = 3 - \sqrt{9-x}$

d) $\dfrac{-\cancel{2}\,\cancel{\sin\theta}\sin\alpha}{\cancel{2}\,\cancel{\sin\theta}\cos\alpha}$

$= -\tan\alpha$

Some students used the wrong signs ∴ didn't get the correct answer

e) $2\,{}^nC_2\dfrac{x^2}{k^2} = {}^nC_3\dfrac{x^3}{k^3}$

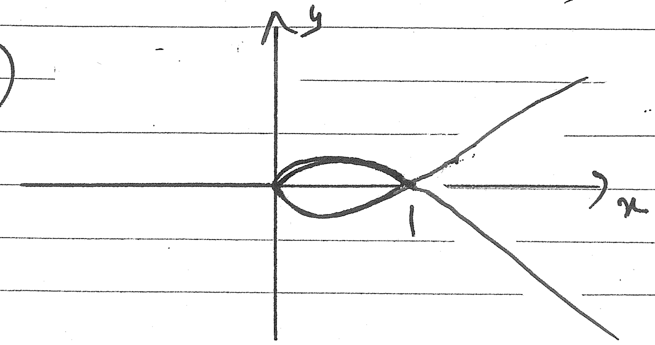
$\dfrac{\cancel{2}\cancel{n!}}{(n-2)!\,\cancel{2!}\,k^2} = \dfrac{\cancel{n!}}{(n-3)!\,3!\,k^3}$

* This question was done poorly

$6k^3(n-3)! = (n-2)!\,k^2$

$6k\cancel{(n-3)!} = (n-2)\cancel{(n-3)!} \quad \therefore\; n = 6k+2$

f)



* Many students couldn't draw the correct graph.

$\boxed{Q4}$

a) $x^3 + bx^2 + cx + d = 0$

$$\alpha + \alpha^2 + \alpha^3 = -b$$

$$\alpha(\alpha^2) + \alpha(\alpha^3) + \alpha^2\alpha^3 = c$$

$$\alpha(\alpha^2)(\alpha^3) = -d$$

i) $\frac{1}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3}$

$$= \frac{\alpha^2\alpha^3 + \alpha\alpha^3 + \alpha\alpha^2}{\alpha\alpha^2\alpha^3} = -\frac{c}{d}$$

ii) $b^3d - c^3 = 0$

$$b^3d = c^3$$

$$\left[-(\alpha + \alpha^2 + \alpha^3)\right]^3\left[-(\alpha^6)\right] = \left[\alpha^2(\alpha + \alpha^2 + \alpha^3)\right]^3$$

$$\alpha^6(\alpha + \alpha^2 + \alpha^3)^3 = \alpha^6(\alpha + \alpha^2 + \alpha^3)^3$$

This is true

$$\therefore b^3d = c^3$$

OR $b^3d - c^3 = 0$

b) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

Subs $x = 2$

$$3^n = {}^nC_0 + 2\,{}^nC_1 + 4\,{}^nC_2 + \dots 2^{n-2}\,{}^nC_{n-1} + 2^n$$

$$3^n - 2^n = \binom{n}{0}\ 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^{n-1}\binom{n}{n-1}$$

proven.

$\boxed{Q4}$

c) $$\frac{\sin\theta + \sin(90-\theta) + 1}{\sin\theta - \sin(90-\theta) + 1} = \cot\frac{\theta}{2}$$

$$LHS = \frac{\sin\theta + \cos\theta + 1}{\sin\theta - \cos\theta + 1}$$

$$= \frac{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1}{\frac{2t}{1+t^2} - \left(\frac{1-t^2}{1+t^2}\right) + 1}$$

$$= \frac{2t + 1 - t^2 + 1 + t^2}{2t - 1 + t^2 + 1 + t^2}$$

$$= \frac{2(1+t)}{2t(1+t)} = \frac{1}{t} = \frac{1}{\tan\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2} = RHS$$

Q4

d) $p(x) = (x^2+7)(ax+b) + x + 12$

$p(x) = ax^3 + bx^2 + 7ax + 7b + x + 12$

$p(x) = ax^3 + bx^2 + (7a+1)x + 7b + 12$

$p(x) = x^3 + bx^2 + 8x + 7b + 12$ (Monic poly)

$p(0) = -6$

$7b + 12 = -6$

$b = -\frac{18}{7}$

$\therefore \ p(x) = x^3 - \frac{18}{7}x^2 + 8x - 6$

e) $y = x^2 . \cos^{-1} x$

D: $-1 \leq x \leq 1$

R: $0 \leq y \leq \pi$