Sydney Girls High School

MATHEMATICS EXTENSION

Year 11

Assessment Task 3

2020

Reading time : 5 minutes

Writing time : 60 minutes

Total marks : 60

Topics: Permutations and Combinations, Inequalities, The Binomial Expansion and Pascal's Triangle, Graphical Relationships, Polynomials, Further Trigonometric Identities, Inverse functions and Inverse Trigonometric Functions, Parametric Form of a Function or Relation, Related Rates of Change, Exponential Growth and Decay

General Instructions:

- There are four (4) questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale, unless indicated otherwise.
- Write using a black or blue pen.

Student Name:_______________ Teacher Name:_________________

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Question 1 **15 marks**

(a) Let $P(x) = x^3 - 3x^2 - x + 3$.

(i) Show that $(x - 1)$ is a factor of $P(x)$. **1**

(ii) Hence, or otherwise, factorise $P(x)$ completely. **2**

(b) (i) Sketch the graph of $y = (x + 4)(x - 3)^2$. **2**

(ii) Hence, or otherwise, solve the inequality $(x + 4)(x - 3)^2 \geq 0$. **1**

(c) Find the coefficient of x^3 in the expansion of $(x + 2)^5$. **2**

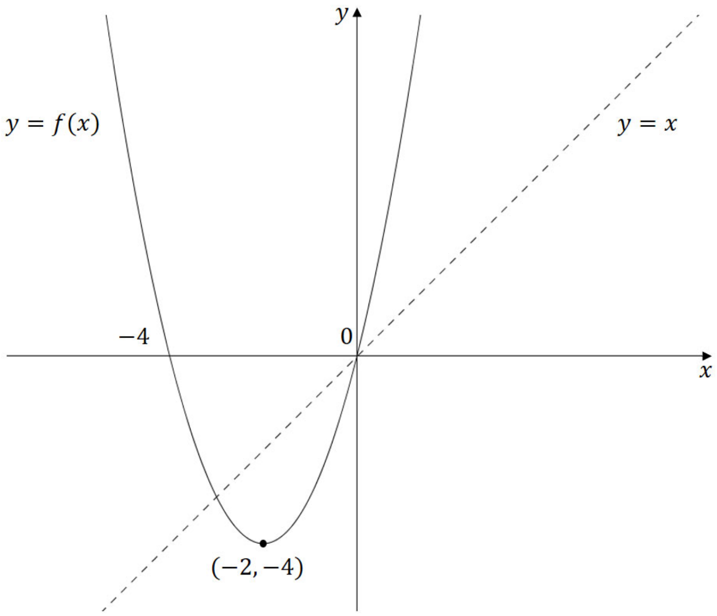
(d) Simplify $\sin(\alpha + \beta) - \sin(\alpha - \beta)$. **2**

Question 1 continues on page 4

Question 1 (continued)

(e) Let $f(x) = (x + 2)^2 - 4$.

The graph of $y = f(x)$ is shown below.



(i) State the largest possible domain of $f(x)$ including zero such that an inverse function $f^{-1}(x)$ exists. **1**

(ii) Find the equation of $f^{-1}(x)$. **2**

(iii) Sketch $y = f^{-1}(x)$. **2**

End of Question 1

Question 2 **15 marks**

(a) Find the domain and range of $y = 2\cos^{-1} 4x$. **2**

(b) Find the Cartesian equation of the curve defined by the parametric equations **2**

$$x = 2t \quad \text{and} \quad y = t^2 - 1.$$

(c) If $P(x) = 2x^3 - 9x^2 + 27$ has a double zero at $x = a$, find the value of a. **3**

(d) The temperature T of a cup of coffee is governed by Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_r)$$

where T_r is the temperature of the room and k is a positive constant.

(i) Show that $T = T_r + Be^{-kt}$ is a solution to this differential equation, where t is the time in minutes. **1**

(ii) The coffee is initially 58° C and the room temperature is 22° C. Five minutes later, the temperature of the coffee is 50° C. Show that **3**

$$T = 22 + 36e^{-kt}, \text{ where } k = \frac{1}{5}\ln\frac{9}{7}.$$

(e) Find the exact value of $\sin\left(\tan^{-1}\frac{2}{5}\right)$. **2**

Question 2 continues on page 6

Question 2 (continued)

(f) Five boys and six girls are to be seated in a row.

Find the number of ways this can be done if:

(i) there are no restrictions; **1**

(ii) the boys all sit together. **1**

End of Question 2

Question 3 **15 marks**

(a) Given $t = \tan\frac{\theta}{2}$, prove that **2**

$$\frac{\cos\theta}{\sin\theta + 1} = \frac{1-t}{1+t}.$$

(b) Let $P(x) = (x+4)(x-4)Q(x) + ax + b$. **2**

When $P(x)$ is divided by $x - 4$ the remainder is -8.

Furthermore, $P(x)$ has a factor of $x + 4$.

Find the values of a and b.

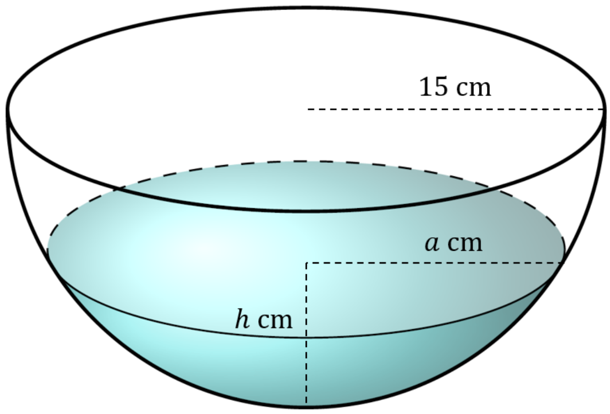
(c) (i) Sketch the graphs of $y = \sin^{-1}(x)$ and $y = -x$ on the same set of axes. **2**

(ii) Hence, sketch the graph of $y = \sin^{-1}(x) - x$, including coordinates of key points. **2**

Question 3 continues on page 8

Question 3 (continued)

(d) A cracked hemispherical bowl of radius 15 cm is leaking water. The surface of the water forms a circle of radius a cm. The depth of the water, h cm, is decreasing at a constant rate of 5 cm/min.



(i) Show that the surface area of the water is given by $A = \pi(30h - h^2)$ cm^2. **2**

(ii) Find the exact rate at which the surface area is decreasing when the depth is 3 cm. **2**

(e) Let A, B and C be the three angles of a triangle.

(i) Prove that $\tan(A + B) = -\tan C$. **1**

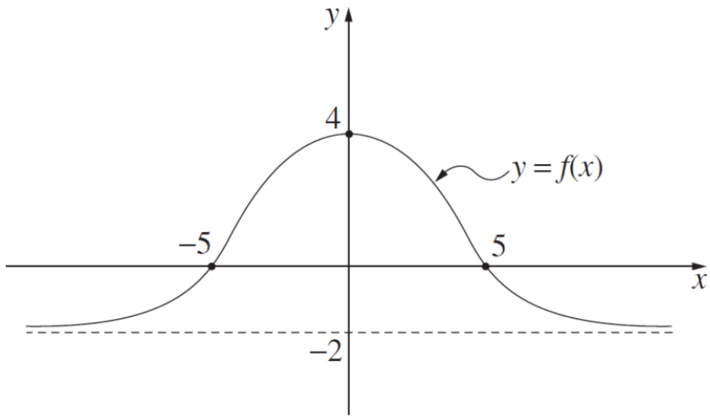
(ii) Hence, or otherwise, prove that **2**

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

End of Question 3

Question 4 **15 marks**

(a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the following:

(i) $y = \frac{1}{f(x)}$ **2**

(ii) $y = \left(f(x)\right)^2$ **2**

(iii) $|y| = f(x)$ **2**

(b) The cubic equation $x^3 + px^2 + qx + r = 0$ has roots $\alpha, \frac{1}{\alpha}$ and β.

(i) Show that $\alpha + \frac{1}{\alpha} = r - p$. **2**

(ii) Hence, show that $q = 1 - r^2 + rp$. **2**

Question 4 continues on page 10

Question 4 (continued)

(c) Suppose that x is a real number and that m and n are positive integers.

(i) Express $\sin mx \sin \frac{x}{2}$ as a difference of cosine functions. **1**

(ii) Hence, show that **2**

$$\sin x + \sin 2x + \sin 3x + \cdots + \sin nx = \frac{\cos \frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x}{2 \sin \frac{x}{2}}.$$

(d) Suppose that x is a real number and that n is a positive integer. **2**

Show that

$$\cos^n 2x = \binom{n}{0}\cos^{2n} x - \binom{n}{1}\cos^{2n-2} x \sin^2 x + \binom{n}{2}\cos^{2n-4} x \sin^4 x - \cdots + (-1)^n \binom{n}{n}\sin^{2n} x$$

End of task

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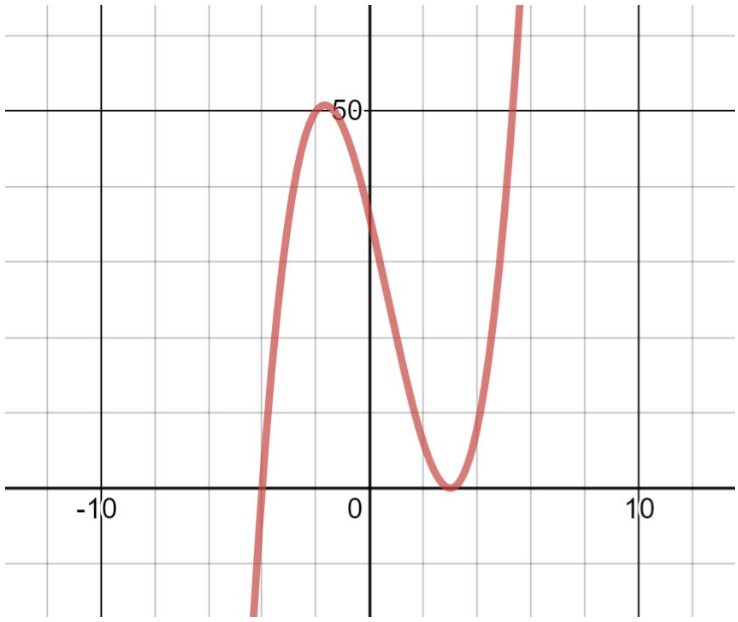
a) $P(x) = x^3 - 3x^2 - x + 3$ i) $P(1) = 1^3 - 3 \times 1^2 - 1 + 2$ $P(1) = 0$, $\therefore$ $x - 1$ is a factor of $P(x)$. ii) $\mathbf{x^2 - 2x - 3}$ $\mathbf{x - 1} \overline{) \, \mathbf{x^3 - 3\,x^2 - x + 3}}$ $\mathbf{x^3 - x^2}$ $\mathbf{-2x^2 - x}$ $\mathbf{-2x^2 + 2x}$ $\mathbf{-3x + 3}$ $\mathbf{-3x + 3}$ 0 0 $\therefore P(x) = (x - 1)(x^2 - 2x - 3)$ $P(x) = (x - 1)(x + 1)(x - 3)$	*Some student did not factorize fully*
b) i) $y = (x + 4)(x - 3)^2$  ii) $x \geq -4$	*Student use $-4 \leq x \leq 3$ and $x \geq 3$ which is not the best answers.*
c) $\binom{5}{3} \times 2^2 = 40$	*Student wasted the time to expand .*
d) $\sin(\alpha + \beta) - \sin(\alpha - \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta - (\sin\alpha\cos\beta - \cos\alpha\sin\beta)$ $= 2\cos\alpha\sin\beta$	

<table>
<tr>
<td>
e) i) $x \geq -2$

ii) $y = (x+2)^2 - 4$

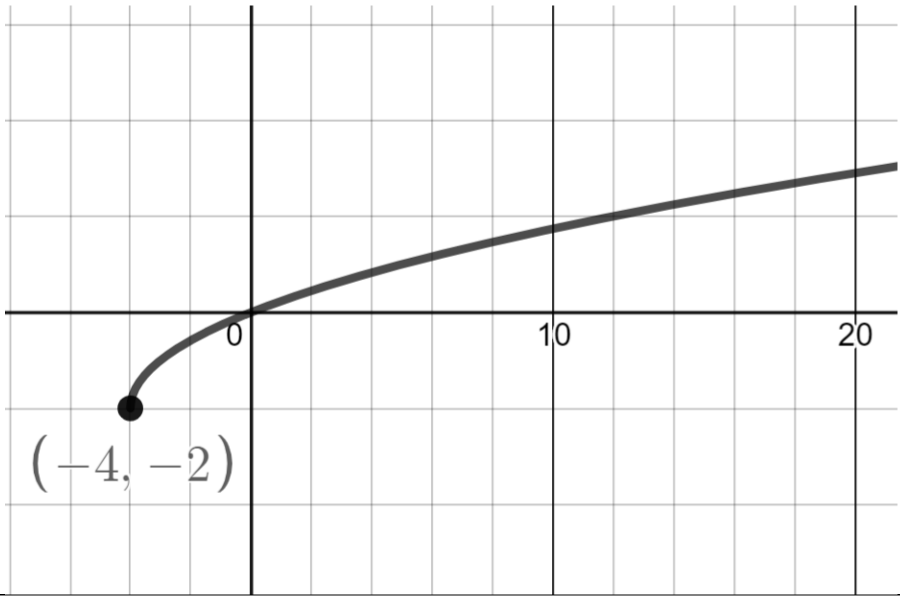
$x = (y+2)^2 - 4$

$x + 4 = (y+2)^2$

$y + 4 = \pm\sqrt{x+4}$

We reject $-\sqrt{x+4} - 2$ as $y \geq -2$

$\therefore\ y = \sqrt{x+4} - 2$

iii)


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<td>
Sum student did not reject
$-\sqrt{x+4} - 2$
as $y \geq -2$.
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Ext 1 Yr 11 Task 3 2020

Q2)

a) $-1 \leq 4x \leq 1$

$-\frac{1}{4} \leq x \leq \frac{1}{4}$ ✓

$0 \leq \frac{y}{2} \leq \pi$

$0 \leq y \leq 2\pi$ ✓

b) $t = \frac{x}{2}$ ✓

$y = \left(\frac{x}{2}\right)^2 - 1$ ✓

$y = \frac{x^2}{4} - 1$

c) $P'(x) = 6x^2 - 18x$ ✓

$= 6x(x-3)$ ✓

$x = 0$ or $x = 3$

double root $x = 3$ ✓

some students didn't show why the double root is 3 ∴ lost mark

d) i) $\frac{dT}{dt} = -kBe^{-kt}$

$T - T_r = Be^{-kt}$

$\therefore \frac{dT}{dt} = -k(T - T_r)$ ✓

ii) $58 = 22 + Be^{0}$

$B = 58 - 22$ ✓

$= 36$

If students didn't show why $B = 36$ they lost mark

$T = 22 + 36e^{-kt}$

$50 = 22 + 36e^{-5k}$

$\frac{28}{36} = e^{-5k}$

$\frac{7}{9} = e^{-5k}$

$\ln \frac{7}{9} = -5k$

$-\ln\left(\frac{7}{9}\right) = 5k$

$\ln\left(\frac{7}{9}\right)^{-1} = 5k$

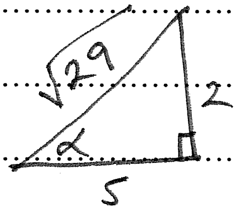
$\ln\left(\frac{9}{7}\right) = 5k$

$k = \frac{1}{5} \ln \frac{9}{7}$

some students didn't show properly why the value of k is $\frac{1}{5} \ln \frac{9}{7}$

e.) $\sin\left(\tan^{-1} \frac{2}{5}\right)$ Let $\alpha = \tan^{-1} \frac{2}{5}$

$\tan \alpha = \frac{2}{5}$



$\sin\left(\tan^{-1} \frac{2}{5}\right) = \frac{2}{\sqrt{29}}$

f) i) 11!

ii) 5! 7!

Question 3

(a) $t = \tan\frac{\theta}{2}$

$$LHS = \frac{\cos\theta}{\sin\theta + 1}$$

$$= \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2} + 1}$$

$$= \frac{1-t^2}{2t + 1 + t^2}$$

$$= \frac{1-t^2}{t^2 + 2t + 1}$$

$$= \frac{(1-t)(1+t)}{(t+1)^2}$$

$$= \frac{1-t}{1+t}$$

$\therefore LHS = RHS$

This question was well done by those students who knew the t-formulae. Unfortunately, there were a number of students who did not know the formulae.

(b) $P(x) = (x+4)(x-4)Q(x) + ax + b$

$P(4) = -8,\ P(-4) = 0$

$\therefore -8 = 4a + b$ ①

$0 = -4a + b$ ②

① + ② $2b = -8$

$b = -4$

① − ② $8a = -8$

$a = -1$

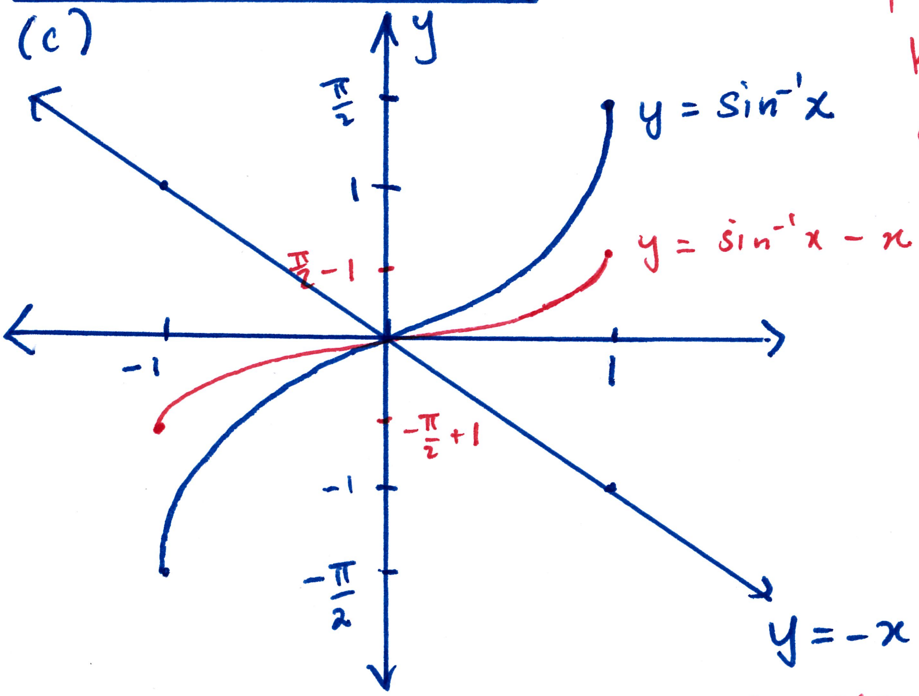
$\therefore a = -1,\ b = -4$

Well done by most students. Common errors included:

- Working with $P(4) = +8$.
- Working with $4x + b = -8$ instead of $4a + b = -8$.

Q3 (continued)

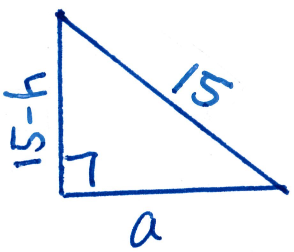
(c)



(i) Many graphs were poorly constructed. Key points were not always shown. There were problems with scale. Some students had $y=-x$ incorrectly passing through $(1, -\frac{\pi}{2})$.

(ii) Not well done, in general. Many unusually constructed graphs. Many students mistakenly identified key y-values at $\frac{\pi}{2}+1$ and $-\frac{\pi}{2}-1$.

(d)(i)



$a^2 = 15^2 - (15-h)^2$

$= 225 - (225 - 30h + h^2)$

$\therefore a^2 = 30h - h^2$

Surface area $A = \pi r^2$

$= \pi a^2$

$\therefore A = \pi(30h - h^2)$

(i) Challenging - only a small handful of students could see how to prove the result.

(ii) $\frac{dA}{dt} = ?$ when $h = 3$ $\quad \frac{dh}{dt} = -5$

$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

$= \pi(30-2h) \times -5$

$= \pi(30-2\times 3) \times -5$

$= -120\pi$

$\therefore$ Area is decreasing at $120\pi\ cm^2/min$.

(ii) Many students did not include the minus sign in $\frac{dh}{dt}$. Since h is decreasing, $\frac{dh}{dt}$ should be negative. A concluding sentence is highly encouraged.

Q3 (continued)

(e) (i) $A+B+C = 180°$ ($\angle$ sum of Δ)

$\therefore A+B = 180° - C$

$\tan(A+B) = \tan(180° - C)$

$\therefore \tan(A+B) = -\tan C$

(ii) From (i) $\tan(A+B) = -\tan C$

$\therefore \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\tan A + \tan B = -\tan C(1 - \tan A \tan B)$

$= -\tan C + \tan A \tan B \tan C$

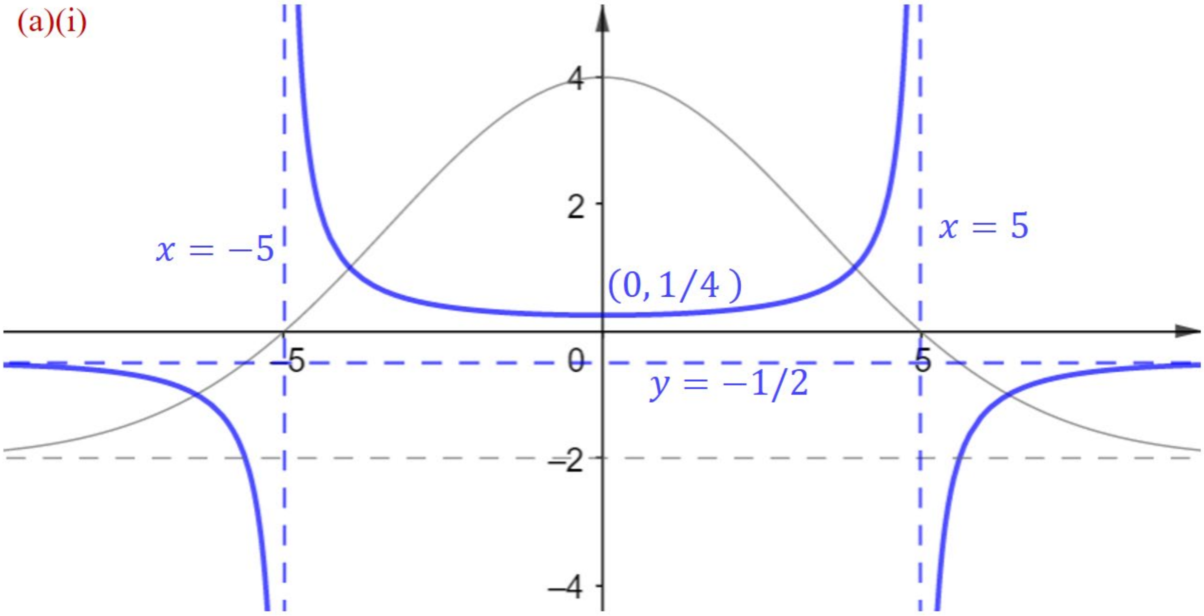
$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$

Part (e) was poorly attempted by many and in Some cases not attempted.

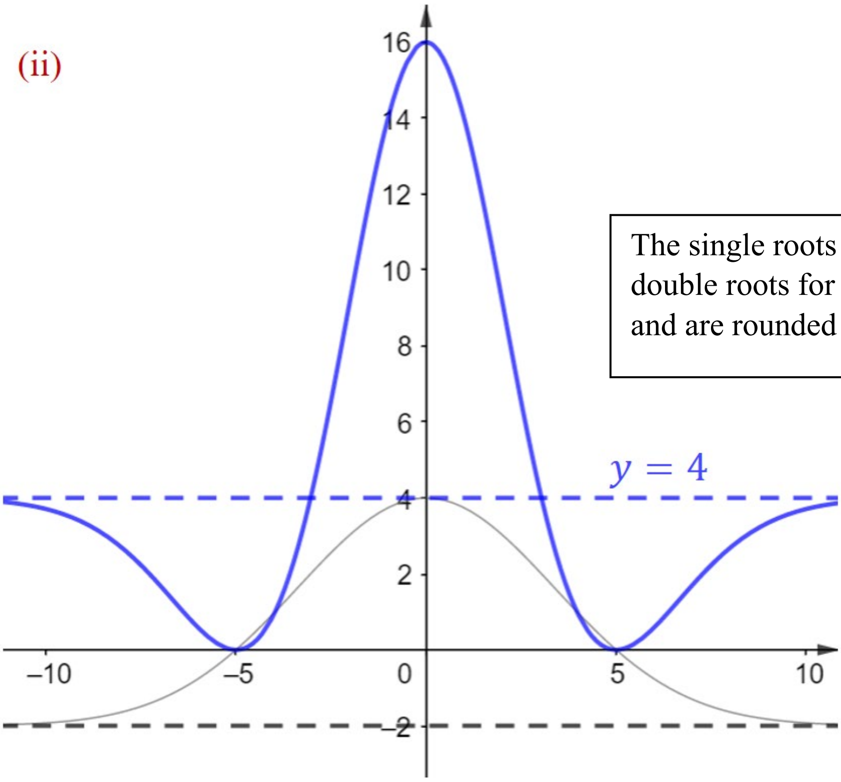
In some solutions, the logic was not clearly presented and some solutions were rather lengthy.

Question 4

(a)(i)

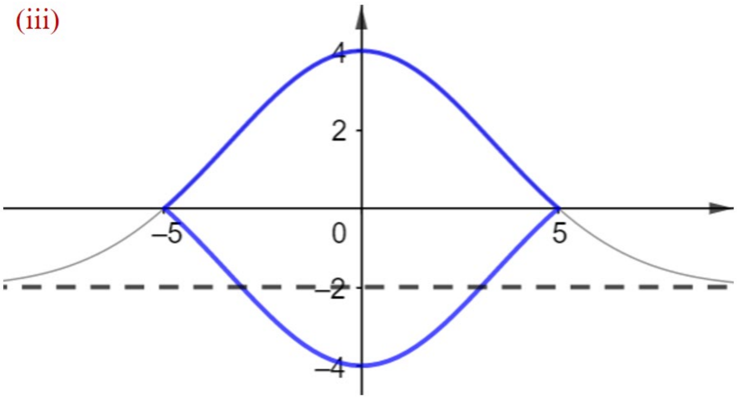


(ii)



The single roots at -5 and 5 become double roots for the squared function and are rounded on the x-axis.

(iii)



(b)(i)

$$\alpha + \frac{1}{\alpha} + \beta = -p \quad (1)$$

$$\alpha\frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha}\beta = q \Rightarrow 1 + \alpha\beta + \frac{\beta}{\alpha} = q \quad (2)$$

$$\alpha\frac{1}{\alpha}\beta = -r \Rightarrow \beta = -r \quad (3)$$

Substitute (3) into (1)

$$\alpha + \frac{1}{\alpha} - r = -p$$

$$\alpha + \frac{1}{\alpha} = r - p$$

(ii) Factorise (2) and substitute (3)

$$1 + \beta\left(\alpha + \frac{1}{\alpha}\right) = q$$

$$1 - r\left(\alpha + \frac{1}{\alpha}\right) = q$$

Using part (i)

$$1 - r(r - p) = q$$

$$1 - r^2 + rp = q$$

(c)(i) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$\sin mx \sin\frac{x}{2} = \frac{1}{2}\cos\left(mx - \frac{x}{2}\right) - \frac{1}{2}\cos\left(mx + \frac{x}{2}\right)$$

$$= \frac{1}{2}\cos\left(m - \frac{1}{2}\right)x - \frac{1}{2}\cos\left(m + \frac{1}{2}\right)x$$

(ii) $\sin x \sin\frac{x}{2} + \sin 2x \sin\frac{x}{2} + \sin 3x \sin\frac{x}{2} + \cdots + \sin(n-1)x \sin\frac{x}{2} + \sin nx \sin\frac{x}{2}$

$$= \frac{1}{2}\cos\left(1-\frac{1}{2}\right)x - \frac{1}{2}\cos\left(1+\frac{1}{2}\right)x + \frac{1}{2}\cos\left(2-\frac{1}{2}\right)x - \frac{1}{2}\cos\left(2+\frac{1}{2}\right)x + \frac{1}{2}\cos\left(3-\frac{1}{2}\right)x - \frac{1}{2}\cos\left(3+\frac{1}{2}\right)x + \cdots$$

$$+\frac{1}{2}\cos\left((n-1)-\frac{1}{2}\right)x - \frac{1}{2}\cos\left((n-1)+\frac{1}{2}\right)x + \frac{1}{2}\cos\left(n-\frac{1}{2}\right)x - \frac{1}{2}\cos\left(n+\frac{1}{2}\right)x$$

$$= \frac{1}{2}\cos\frac{1}{2}x - \cancel{\frac{1}{2}\cos\frac{3}{2}x} + \cancel{\frac{1}{2}\cos\frac{3}{2}x} - \cancel{\frac{1}{2}\cos\frac{5}{2}x} + \cancel{\frac{1}{2}\cos\frac{5}{2}x} - \cancel{\frac{1}{2}\cos\frac{7}{2}x} + \cdots$$

$$+\cancel{\frac{1}{2}\cos\left(n-\frac{3}{2}\right)x} - \cancel{\frac{1}{2}\cos\left(n-\frac{1}{2}\right)x} + \cancel{\frac{1}{2}\cos\left(n-\frac{1}{2}\right)x} - \frac{1}{2}\cos\left(n+\frac{1}{2}\right)x$$

$$= \frac{1}{2}\cos\frac{1}{2}x - \frac{1}{2}\cos\left(n+\frac{1}{2}\right)x \qquad \text{Telescoping series}$$

Thus

$$\sin x \sin\frac{x}{2} + \sin 2x \sin\frac{x}{2} + \sin 3x \sin\frac{x}{2} + \cdots + \sin nx \sin\frac{x}{2} = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2}$$

That is

$$\sin x + \sin 2x + \sin 3x + \cdots + \sin nx = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

(d) $\cos^n 2x = (\cos^2 x - \sin^2 x)^n$

$$= \sum_{r=0}^{n}\binom{n}{r}(\cos^2 x)^{n-r}(-1)^r(\sin^2 x)^r$$

$$= \binom{n}{0}(\cos^2 x)^n(-1)^0(\sin^2 x)^0 + \binom{n}{1}(\cos^2 x)^{n-1}(-1)^1(\sin^2 x)^1 + \binom{n}{2}(\cos^2 x)^{n-2}(-1)^2(\sin^2 x)^2 + \cdots$$

$$+\binom{n}{n}(\cos^2 x)^{n-n}(-1)^n(\sin^2 x)^n$$

$$= \binom{n}{0}\cos^{2n} x - \binom{n}{1}\cos^{2n-2} x \sin^2 x + \binom{n}{2}\cos^{2n-4} x \sin^4 x - \cdots + \binom{n}{n}(-1)^n \sin^{2n} x$$

Students lost a mark if they did not explicitly show how the $(-1)^n$ comes about in the last term.