Sydney Girls High School

MATHEMATICS EXTENSION 1

2021 (Online)YEAR 11 ASSESSMENT TASK 3

Time Allowed: 60 minutes + 5 minutes reading time

Total: 40 marks

Topics: Permutations and combinations; The binomial expansion and Pascal's triangle; Inequalities; Graphical relationships; Polynomials; Inverse functions (Part topic); Further trigonometric identities.

General Instructions:

- There are FOUR (4) Questions which are of equal value.
- Attempt all questions.
- Show all necessary working.
- Marks may be deducted for badly arranged work or incomplete working.
- An answer without any working will be awarded a maximum of 1 mark.
- Start each Question on a New Page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your Name clearly on the first page

Question 1 **(10 Marks)**

(a) The number plates for scooters appear as 3 letters followed by 2 numbers such as MAX21. How many different number plates are possible? 1

(b) Simplify $2\sin 3\theta \cos 3\theta$. 1

(c) Consider the polynomial $P(x) = x^3 + ax + b$ where a and b are real numbers. $P(x)$ has a factor of $(x-2)$ and when $P(x)$ is divided by $(x+1)$ the remainder is 6.

(i) Show that $2a + b = -8$ and $-a + b = 7$. 2

(ii) Find the values of a and b. 2

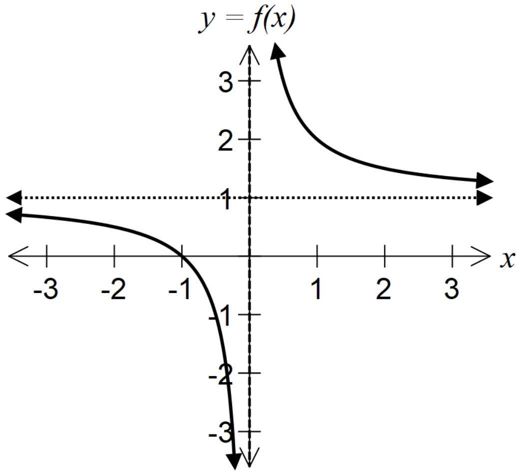
(d) Solve $\dfrac{|1-4x|}{2} \geq x$. 2

(e) Find the exact value of $\sin 255^\circ \cos 15^\circ$, in simplest form. 2

End of Question 1

Question 2 Start writing on a New Page (10 Marks)

(a) Consider the function $y = f(x)$, which is graphed below:



Sketch on separate diagrams the graphs of:

(i) $y = |f(x)|$. 1

(ii) $y = 2 - f(x)$. 2

(b) Given α, β and γ are the roots of $2x^3 + mx^2 + x + 3 = 0$:

(i) Find the value of m given the sum of the roots is 5. 1

(ii) Evaluate $(\alpha - 1)(\beta - 1)(\gamma - 1)$. 2

(c) Consider the function $f(x) = x^2 - 6x$ and the largest possible domain that contains $x = 0$, such that $f(x)$ is a one-to-one function. Find $f^{-1}(x)$. 2

(d) 4 Maths books are chosen from 6 different Maths books and 3 English books are chosen from 5 different English books.

(i) How many different ways can 7 books be selected? 1

(ii) Given that the 7 books have been selected, how many ways can they be arranged so that the Maths books stay together and the English books stay together? 1

End of Question 2

Question 3 **Start writing on a New Page** **(10 Marks)**

(a) Consider the binomial expansion: $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 \ldots + \binom{n}{n}x^n$. 1

Find the value of: $\binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \ldots + \binom{100}{100}$.

(b) Four couples, each consisting of a male and a female, sit around a circular table. In how many different ways is it possible to seat the four couples so that males and females sit in alternating positions and nobody sits next to his or her partner? 2

(c)

(i) Use the t-formulae to prove that $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$. 2

(ii) Hence find the exact value of $\cot 67\frac{1}{2}^{\circ}$, in simplest surd form. 2

(d)

(i) On the same set of axes, sketch the graphs of $y = |2-x|$ and $y = x(2-x)$. 2

(ii) Use the diagram, or otherwise, to solve $|2-x| \geq x(2-x)$ for all real x 1

End of Question 3

Question 4 Start writing on a New Page (10 Marks)

(a) What is the smallest group of people that can be assembled where it is guaranteed that at least two people have the same first and last initials? **1**
For example, Clark Kent and Calvin Klein both have the same initials.

(b) Consider the function $g(x) = 2x + 4$ for all real x. **1**
A second function $f(x)$ is defined as: $f[g(x)] = x$.
Find the function $f(x)$.

(c)

(i) Sketch the graph of $P(x) = x(2x+3)^3(1-x)^4$, clearly indicating all intercepts with the axes. **2**

(ii) Hence, solve $x(2x+3)^3(1-x)^4 \geq 0$. **1**

(d) Prove that: $\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right)=2\sec 2\theta$ **3**

(e) What is the coefficient of x^5 in the expansion of $\left(1-3x+2x^3\right)\left(1-2x\right)^6$. **2**

End of paper

2021 (Online) Year 11 Extension 1, Task 3, SOLUTIONS

Question One

(a) $26 \times 26 \times 26 \times 10 \times 10 = 26^3 \times 100$

$= \underline{\underline{1\,757\,600}}$. ✓

(b) $2\sin 3\theta \cos 3\theta = \underline{\underline{\sin 6\theta}}$ ✓

(c) i) $P(x) = x^3 + ax + b$

$P(2) = 0$

$\therefore P(2) = 8 + 2a + b$ ✓

$\therefore 8 + 2a + b = 0$

$2a + b = -8 \quad \cdots (1)$

$P(-1) = 6$

$P(-1) = -1 - a + b$ ✓

$-1 - a + b = 6$

$-a + b = 7 \quad \cdots (2)$

ii) Solve (1) and (2) simultaneously

$$\begin{aligned} 2a + b &= -8 \\ -a + b &= 7 \\ \hline 3a \quad &= -15 \\ \underline{\underline{a}} &\underline{\underline{= -5}} \end{aligned}$$ ✓

From (2):

$-(-5) + b = 7$

$5 + b = 7$

$\underline{\underline{b = 2}}$ ✓

(d) $\dfrac{|1-4x|}{2} \geq x$

$|1-4x| \geq 2x$

$\Rightarrow$ Note: $1 - 4x \geq 0$

$1 \geq 4x$

$x \leq \frac{1}{4}$.

Two cases:

$1 - 4x \geq 2x$

$1 \geq 6x$

$\frac{1}{6} \geq x$ ✓

or

$-(1-4x) \geq 2x$

$-1 + 4x \geq 2x$

$2x \geq 1$

$x \geq \frac{1}{2}$ ✓

$\underline{\underline{x \leq \frac{1}{6} \text{ or } x \geq \frac{1}{2}}}$

(e) $\sin 255^\circ \cos 15^\circ$

$= \frac{1}{2}\left(\sin(255^\circ + 15^\circ) + \sin(255^\circ - 15^\circ)\right)$

$= \frac{1}{2}\left(\sin 270^\circ + \sin 240^\circ\right)$ ✓ (progress).

$= \frac{1}{2}\left(-1 + \left(-\frac{\sqrt{3}}{2}\right)\right)$

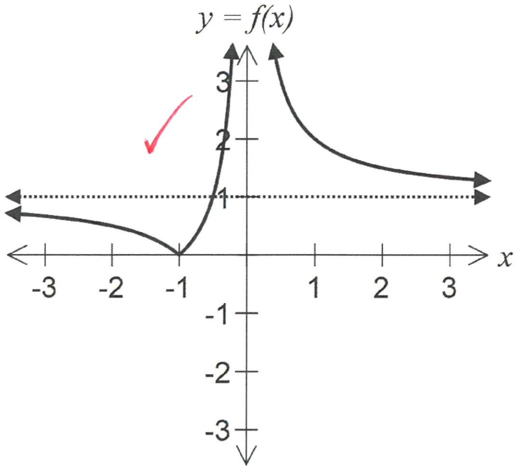
$= -\frac{1}{2} - \frac{\sqrt{3}}{4}$ ✓ (mark awarded here)

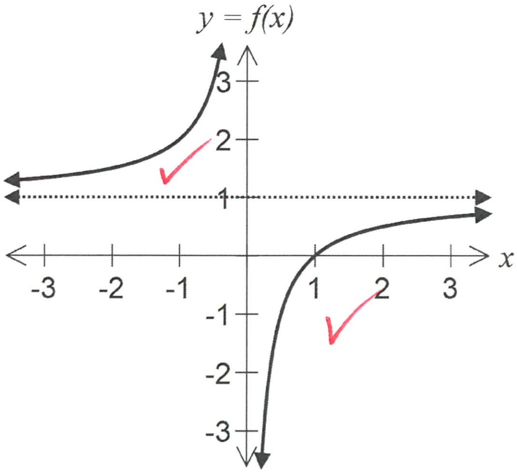
$= \frac{-2-\sqrt{3}}{4}$

(2)

Question Two:

(a)





(b) $2x^3 + mx^2 + x + 3 = 0.$

i) $\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha + \beta + \gamma = \frac{-m}{2}$

$5 = \frac{-m}{2}$

$\therefore \underline{\underline{m = -10}}.$ ✓ (correct answer)

ii) $(\alpha - 1)(\beta - 1)(\gamma - 1)$

$= (\alpha\beta - \alpha - \beta + 1)(\gamma - 1)$

$= \alpha\beta\gamma - \alpha\gamma - \beta\gamma + \gamma - \alpha\beta + \alpha + \beta - 1$

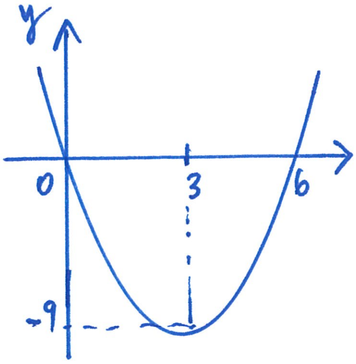
$= \alpha\beta\gamma - (\alpha\gamma + \beta\gamma + \alpha\beta) + (\alpha + \beta + \gamma) - 1$ ✓ (progress)

$= -\frac{d}{a} - \left(\frac{c}{a}\right) + 5 - 1$

$= -\frac{3}{2} - \frac{1}{2} + 5 - 1$

$= \underline{\underline{2}}.$ ✓ (correct answer)

(c) $f(x) = x^2 - 6x$
$= x(x-6)$



Domain containing $x=0$ is: $x \leq 3$.

$f^{-1}(x)$: $x = y^2 - 6y$

$$\left(\frac{-6}{2}\right)^2 + x = y^2 - 6y + \left(\frac{-6}{2}\right)^2$$

$$9 + x = (y-3)^2$$

(Finding inverse progress).

$$\therefore y - 3 = \pm\sqrt{9+x}$$

$$y = 3 \pm \sqrt{9+x}$$

$$\therefore \underline{\underline{f^{-1}(x) = 3 - \sqrt{9+x}, \text{ Range: } y \leq 3.}}$$

d) i) $^6C_4 \times {}^5C_3 = \underline{\underline{150 \text{ ways}}}$

ii) $4! \times 3! \times 2! = \underline{\underline{288 \text{ ways}}}$.

Question Three.

(a)

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

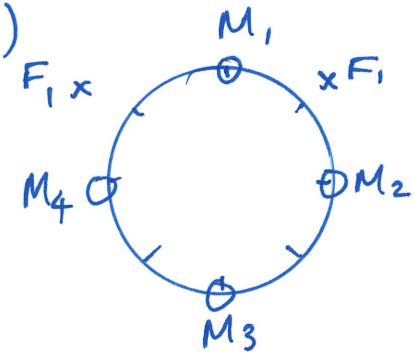
when $x=1$:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

when $n=100$

$$\therefore \quad \underline{\underline{2^{100}}} = \binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \binom{100}{3} + \cdots + \binom{100}{100} \quad ✓$$

(b)



$\begin{cases} M_1\ M_2\ M_3\ M_4 \\ F_1\ F_2\ F_3\ F_4 \end{cases}$

- arrange all males first in a circle $(4-1)! = 3!$ ways. ✓ (progress)
- F_1 can be seated in 2 ways.
- Once this occurs, there is only one way for each of F_2, F_3, F_4.

$\therefore$ total number of ways $= 3! \times 2$ ✓ (answer)

$= \underline{\underline{12 \text{ ways}}}.$

* bold answer 1 mark.

(c) P.T.O.

(c)i) Let $t = \tan\theta$.

$$\text{LHS} = \frac{\sin 2\theta}{1-\cos 2\theta}$$

$$= \frac{\frac{2t}{1+t^2}}{1-\left(\frac{1-t^2}{1+t^2}\right)} \quad \text{(substitution)}$$

$$= \frac{2t}{1+t^2} \div \frac{1+t^2-1+t^2}{1+t^2}$$

$$= \frac{2t}{\cancel{1+t^2}} \times \frac{\cancel{1+t^2}}{2t^2}$$

$$= \frac{2t}{2t^2}$$

$$= \frac{1}{t}$$

$$= \cot x$$

$$= \text{RHS}.$$

ii)

$$\cot 67\tfrac{1}{2}^\circ$$

$$= \frac{\sin 2\left(67\frac{1}{2}\right)^\circ}{1-\cos 2\left(67\frac{1}{2}\right)^\circ}$$

$$= \frac{\sin 135^\circ}{1-\cos 135^\circ}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}} \div \frac{\sqrt{2}+1}{\sqrt{2}}$$

$$= \frac{1}{\cancel{\sqrt{2}}} \times \frac{\cancel{\sqrt{2}}}{\sqrt{2}+1}$$

$$= \frac{1}{\sqrt{2}+1}$$

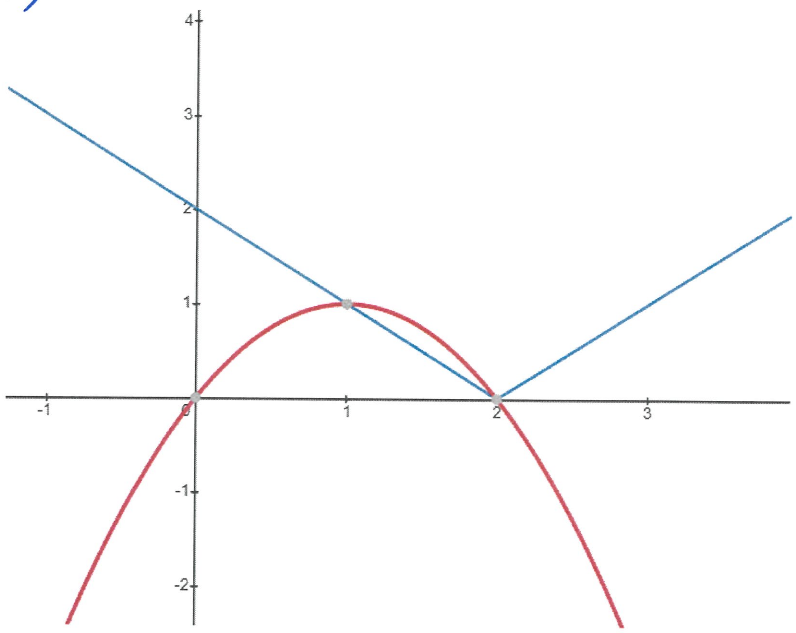
rationalise denominator

$$= \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$= \frac{\sqrt{2}-1}{2-1}$$

$$= \sqrt{2}-1.$$

d) i)



✓ correct parabola

✓ correct abs. value graph.

ii) Solve: $|2-x| \geq x(2-x)$

Solution by observation:

$x \leq 1$ or $x \geqslant 2$. ✓

(carry error if graphs in i) are incorrect)

Question Four

(a) As there are 2 initials, then $n = 2$.

The number of initial pairings $(k) = 26^2$
$= 676$.

Smallest group $= (n-1)k + 1$
$= 1 \times 676 + 1$
$= 677$ ✓ (correct answer)

$\therefore$ the group must have at least 677 people.

(b) $f(x)$ and $g(x)$ are mutually inverse functions, if $f[(g(x)] = x$,

then $f(x) = g^{-1}(x)$

$$f(x) = \frac{x-4}{2}$$ ✓ (correct answer)

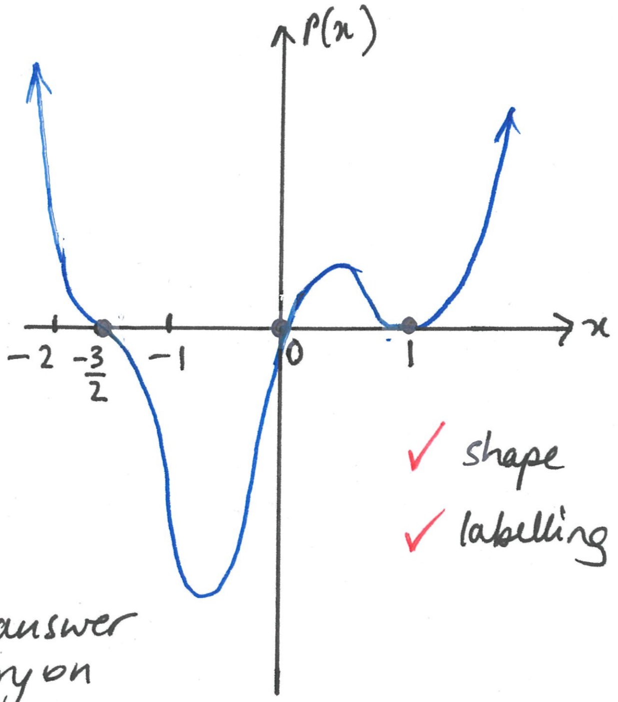
where $f[g(x)] = \dfrac{g(x)-4}{2} \Rightarrow \dfrac{2x+4-4}{2} \Rightarrow \dfrac{2x}{2} \Rightarrow x$.

(c) $P(x) = x(2x+3)^3(1-x)^4$

roots are: $0, -\frac{3}{2}, 1$

y-intercept $= 0$.

leading term $= 8x^8$ ↑↑



✓ shape
✓ labelling

ii) $x(2x+3)^3(1-x)^4 \geq 0$

$x \leq -\frac{3}{2}$. or $x \geq 0$

✓ (correct answer or carry on error from graph)

(d) $LHS = \tan\left(\frac{\pi}{4}+\theta\right) + \tan\left(\frac{\pi}{4}-\theta\right)$

$$= \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{1-\tan^2\theta}$$

$$= \frac{1+\cancel{2\tan\theta}+\tan^2\theta+1-\cancel{2\tan\theta}+\tan^2\theta}{1-\tan^2\theta}$$

$$= \frac{2+2\tan^2\theta}{1-\tan^2\theta}$$

$$= \frac{2\sec^2\theta}{1-\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= 2\sec^2\theta \div \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}$$

$$= \frac{2}{\cancel{\cos^2\theta}} \times \frac{\cancel{\cos^2\theta}}{\cos 2\theta}$$

$$= 2\sec 2\theta$$

$= RHS$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

(e) $(1-3x+2x^3)(1-2x)^6$

$= (1-3x+2x^3)(1-6(2x)$

$= (1-3x+2x^3)(1-6(2x)+15(2x)^2-20(2x)^3$
$+15(2x)^4-6(2x)^5+(2x)^6)$

Terms of x^5:

(progress)

$1\times -6(2x)^5 - 3x\times 15(2x)^4 + 2x^3\times 15(2x)^2$

$= -192x^5 - 720x^5 + 120x^5$

$= -792x^5$

$\therefore$ Coefficient of $x^5 = -792$. (correct answer).