Sydney Girls High School

YEAR 11 MATHEMATICS EXTENSION 1

Assessment Task 3

2022

Reading time : 5 minutes

Writing time : 60 minutes

Total marks : 52

Topics: Permutations and Combinations, Inequalities, The Binomial Expansion and Pascal's Triangle, Graphical Relationships, Polynomials, Further Trigonometric Identities, Inverse Functions and Inverse Trigonometric Functions, Parametric Form of a Function or Relation

General Instructions:

- There are four (4) questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale, unless indicated otherwise.
- NESA approved calculators are allowed.
- Write using a black or blue pen.
- White-out or correction tape is NOT to be used.

Student Name:____________________________**Teacher Name:**____________________________

Question 1 (13 Marks)

(a) How many ways can 10 people be arranged in a straight line? **1**

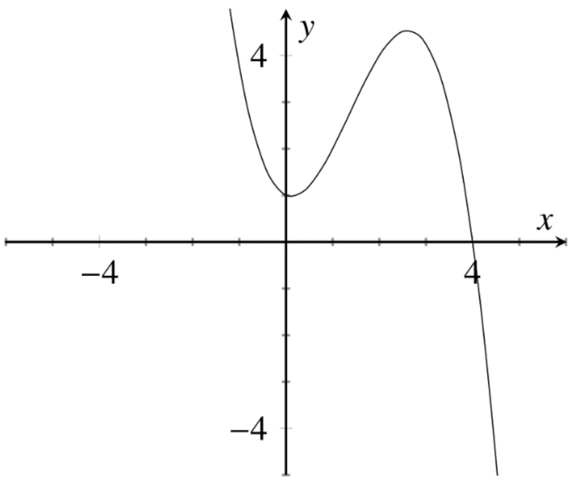
(b) Find the remainder when the polynomial $P(x) = x^3 + 5x - 1$ is divided by $x + 2$. **1**

(c) Solve the inequality $\frac{x}{x-1} \leq 2$. **3**

(d) Find the coefficient of x^5 in the expansion of $\left(1 + \frac{x}{2}\right)^{2022}$. Leave your answer in terms of $\binom{n}{k}$. **2**

(e) Evaluate $\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1)$. Express your answer in simplified exact form in terms of π. **2**

(f) The graph of $y = f(x)$ is shown below.



Sketch the graph of $y = |f(x)|$, showing all important features. **1**

(g) Let $P(x) = x^3 + ax^2 + bx + 8$ where a and b are real numbers. **3**

Find the values of a and b given that $(x - 2)^2$ is a factor of $P(x)$.

Question 2 (13 Marks)

(a) Solve $(n+2)! = 72(n!)$. **2**

(b)

(i) Prove that $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$. **2**

(ii) Hence, or otherwise, find the exact value of $\sin 75° - \cos 75°$. Leave your answer with a rational denominator. **2**

(c)

(i) Let $t = \tan\frac{\theta}{2}$. Express $\sin\theta$ and $\cos\theta$ in terms of t. **2**

(ii) Hence, or otherwise, prove that **2**

$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}.$$

(e) The ten members of a sporting team have the numbers 1 to 10 on the back of their jumpers. **1**

What is the minimum number of players that must be chosen to guarantee that there is at least one pair of players whose numbers add to 11?

(f) By using $(1+x)^{10} = (1+x)^5(1+x)^5$, show that **2**

$$\binom{10}{3} = \binom{5}{3}\binom{5}{0} + \binom{5}{2}\binom{5}{1} + \binom{5}{1}\binom{5}{2} + \binom{5}{0}\binom{5}{3}.$$

Question 3 (13 Marks)

(a) Sketch the graph of $y = 2\cos^{-1}\frac{x}{3}$, clearly labelling the axes intercepts and the endpoints. **2**

(b) An equation can be expressed in the parametric form **2**

$$\begin{cases} x = 2\cos\theta - 1 \\ y = 2 + 2\sin\theta \end{cases}$$

Express the equation in Cartesian form.

(c) The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β and γ.

(i) Show that $\beta + \gamma = 1 - \alpha$. **1**

(ii) Using part (i), and similar results, evaluate $\frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} + \frac{\alpha+\beta}{\gamma}$. **2**

(d) Find the exact value of $\sin\left(\sin^{-1}\left(\frac{12}{13}\right) - \cos^{-1}\left(\frac{2}{3}\right)\right)$. **3**

(e) Let α, β and γ be the angles of a triangle.

(i) Find an expression for $\alpha + \beta$ in terms of γ. **1**

(ii) It is known that $\cos\gamma = 2\sin\alpha\sin\beta$, where $\alpha > \beta$. **2**

By using the product-to-sum formula and part (i), or otherwise, show that 2β and γ are complementary.

Question 4 (13 Marks)

(a) Consider the function $f(x) = \frac{1}{2}(x-3)^2 + 4$.

(i) Sketch the parabola $y = f(x)$, showing clearly the coordinates of the vertex. Hence write down the largest domain containing the value $x = 2022$, for which the function has an inverse function $f^{-1}(x)$. **2**

(ii) For the domain found in part (i), find $f^{-1}(x)$. **2**

(iii) Let a be a real number NOT in the domain found in part (i). **2**

By considering the symmetry of the parabola, or otherwise, find $f^{-1}\big(f(a)\big)$.

(b) Four couples are to be seated around a circular table. All the couples are to sit together, except for Jeremy and Candace, in such a way that males and females alternate. **2**

How many seating arrangements are possible?

(c) If $\sin^{-1} x - \cos^{-1} x = k$, show that $x = \frac{1}{\sqrt{2}}\left(\sin\frac{k}{2} + \cos\frac{k}{2}\right)$. **2**

(d) Suppose that the roots of $Q(x) = Ax^3 + Bx^2 + Cx + D$ are α, α and γ, where A, B, C and $D \in \mathbb{R}$, and $A \neq 0$. **3**

Show that $B\alpha^2 + 2C\alpha + 3D = 0$.

End of paper

Question 1 (13 Marks)

(a) How many ways can 10 people be arranged in a straight line? **1**

$10! = 3\ 628\ 800$ ways.

(b) Find the remainder when the polynomial $P(x) = x^3 + 5x - 1$ is divided by $x + 2$. **1**

$x+2=0$

$x=-2$

remainder $= P(-2)$

$= (-2)^3 + 5(-2) - 1$

$= -8 - 10 - 1$

$= -19$

(c) Solve the inequality $\frac{x}{x-1} \leq 2$. **3**

$\frac{x}{x-1} \leq 2 \quad (x \neq 1)$

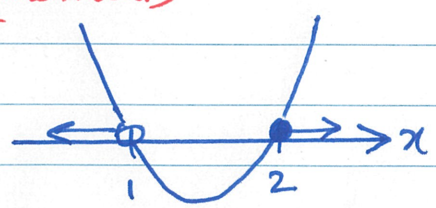
$x(x-1) \leq 2(x-1)^2$

$0 \leq 2(x-1)^2 - x(x-1)$

$0 \leq (x-1)[2(x-1) - x]$ (method)

$0 \leq (x-1)(x-2)$

$\therefore (x-1)(x-2) \geq 0$



$x < 1$ or $x \geq 2$

(d) Find the coefficient of x^5 in the expansion of $\left(1 + \frac{x}{2}\right)^{2022}$. Leave your answer in terms of $\binom{n}{k}$. **2**

$\left(1+\frac{x}{2}\right)^{2022} = {}^{2022}C_5 \ 1^{2017} \cdot \left(\frac{x}{2}\right)^5$

$= {}^{2022}C_5 \ \frac{x^5}{32}$

$\therefore$ coefficient $= \frac{1}{32} \ {}^{2022}C_5$.

(e) Evaluate $\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1)$. Express your answer in simplified exact form in terms of π. **2**

$\tan^{-1}\sqrt{3} - \tan^{-1}(-1)$

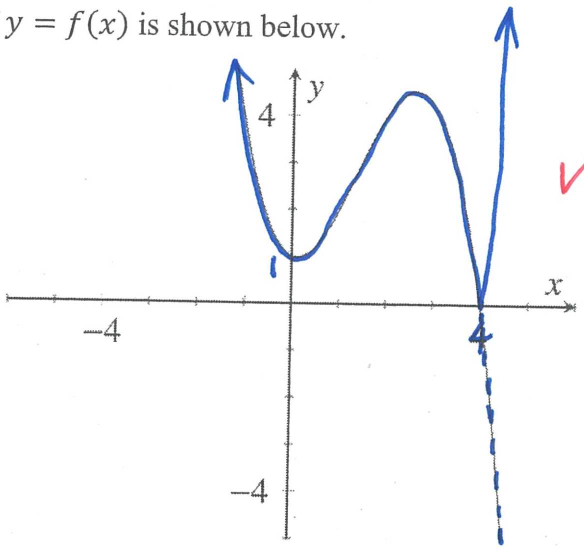
$= \frac{\pi}{3} - \left(-\frac{\pi}{4}\right)$

$= \frac{4\pi + 3\pi}{12}$

$= \frac{7\pi}{12}$.

* Students over complicated the answer to this question.

(f) The graph of $y = f(x)$ is shown below.



*Students had to reflect the curve properly for one mark.

Sketch the graph of $y = |f(x)|$, showing all important features. **1**

(g) Let $P(x) = x^3 + ax^2 + bx + 8$ where a and b are real numbers. **3**

Find the values of a and b given that $(x-2)^2$ is a factor of $P(x)$.

If $(x-2)^2$ is a factor of $P(x)$, then remaining factor must be $(x+2)$ since product of $(-2)^2 \times 2 = 8$

Hence $P(x) = (x-2)^2(x+2)$

expand: $= (x^2 - 4x + 4)(x+2)$

$= x^3 - 4x^2 + 4x + 2x^2 - 8x + 8$

$= x^3 - 2x^2 - 4x + 8$

$\equiv x^3 + ax^2 + bx + 8$ (method)

$\therefore a = -2$

$b = -4$

g) Alternate Solution:

* $(x-2)$ is a double factor of $P(x)$
hence $P(2) = P'(2) = 0$.

$$P(x) = x^3 + ax^2 + bx + 8$$
$$P'(x) = 3x^2 + 2ax + b$$

$P(2)$: $\quad 2^3 + a \times 2^2 + b(2) + 8 = 0$
$$4a + 2b + 16 = 0$$
$$2a + b + 8 = 0 \quad \ldots (1)$$

$P'(2)$: $\quad 3(2)^2 + 2a(2) + b = 0$
$$4a + b + 12 = 0 \quad \ldots (2)$$

$(2) - (1)$: $\quad 2a + 4 = 0$
$$2a = -4$$
$$\underline{a = -2}$$

From (1): $\quad 2(-2) + b + 8 = 0$
$$b + 4 = 0$$
$$\underline{b = -4}.$$

$\therefore P(x) = x^3 - 2x^2 - 4x + 8.$

Question 2

(a) $(n+2)! = 72(n!)$

$(n+2)(n+1)\,n! = 72\,(n!)$ ✓

$(n+2)(n+1) = 72$

$n^2 + 3n - 70 = 0$

$(n+10)(n-7) = 0$

$\therefore n = -10, 7$

$\therefore n = 7 \quad (\text{as } n \geq 0)$ ✓

Comment

Students needed to exclude $n = -10$ in their final line to receive full marks.

(b) (i) LHS $= \sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha$

$= \sin^2\alpha + \cos^2\alpha - 2\sin\alpha\cos\alpha$

$= 1 - \sin 2\alpha$

$=$ RHS

(ii) Let $\alpha = 75^\circ$ in (i)

$(\sin 75^\circ - \cos 75^\circ)^2 = 1 - \sin 150^\circ$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

$\therefore \sin 75^\circ - \cos 75^\circ = \frac{1}{\sqrt{2}}$ (since $\sin 75^\circ - \cos 75^\circ > 0$)

$= \frac{\sqrt{2}}{2}$

Comment

Students should write $\sin 75^\circ - \cos 75^\circ > 0$ to justify why only the positive square root is taken.

(c) (i) $\sin\theta = \dfrac{2t}{1+t^2}$ ✓

$\cos\theta = \dfrac{1-t^2}{1+t^2}$ ✓

(ii) $$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$$ ✓

$$= \frac{2t^2+2t}{2+2t}$$

$$= \frac{2t(t+1)}{2(1+t)}$$

$$= t$$ ✓

$$= \tan\frac{\theta}{2} = \text{RHS}$$

(d) Group the jumpers as such:

$\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$

Worst case scenario:
Choosing 5 players whose numbers do not add to 11.

But choosing 1 more would guarantee that there is at least one pair of players whose numbers add to 11.

$\therefore$ 6 is the minimum number of players required. ✓

(e) $(1+x)^{10} = (1+x)^5(1+x)^5$

<u>LHS</u> $(1+x)^{10} = \binom{10}{0} + \binom{10}{1}x + \binom{10}{2}x^2 + \binom{10}{3}x^3 + \cdots$

coeff of $x^3 = \binom{10}{3}$ ✓

<u>RHS</u> $(1+x)^5(1+x)^5 = \left(\binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5\right)$

$\left(\binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5\right)$

coeff of $x^3 = \binom{5}{0}\binom{5}{3} + \binom{5}{1}\binom{5}{2} + \binom{5}{2}\binom{5}{1} + \binom{5}{3}\binom{5}{0}$ ✓

Equating coeff of x^3:

$$\binom{10}{3} = \binom{5}{3}\binom{5}{0} + \binom{5}{2}\binom{5}{1} + \binom{5}{1}\binom{5}{2} + \binom{5}{0}\binom{5}{3}$$

<u>Comment</u>

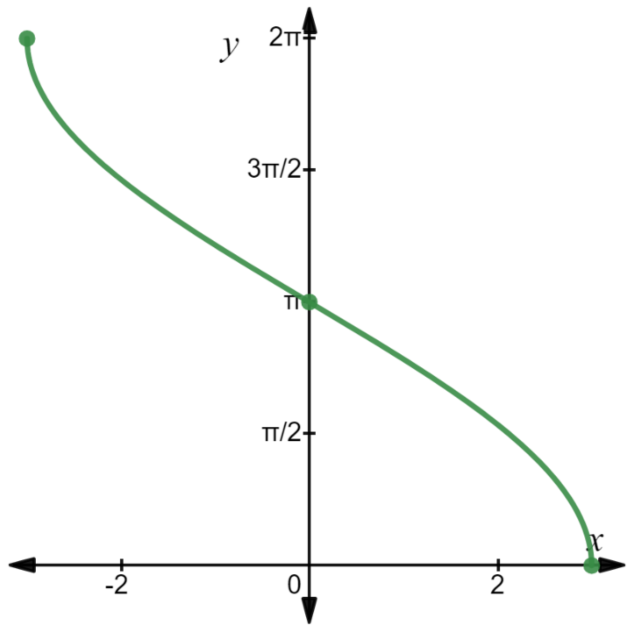
Proper working must be shown for full marks to be awarded.

Question 3 (13 marks)

(a)

$D: -1 \leq \dfrac{x}{3} \leq 1$ $\qquad$ $R: 0 \leq \cos^{-1}\left(\dfrac{x}{3}\right) \leq \pi$

$\therefore -3 \leq x \leq 3$ $\qquad$ $\therefore 0 \leq y \leq 2\pi$



2 marks

1st mark for one of the following:

- correct domain
- correct range
- correct shape

Comment:

Many students did not draw the shape correctly through the y-intercept. The graph is not horizontal as it passes through the point $(0, \pi)$. No marks were deducted in this instance but students should note the shape in the solution provided, and adjust their approach to sketching if needed.

(b)

$$\cos\theta = \frac{x+1}{2} \quad ; \quad \sin\theta = \frac{y-2}{2}$$

Since $\cos^2\theta + \sin^2\theta = 1$,

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$$

or equivalently, $(x+1)^2 + (y-2)^2 = 4$.

i.e. a circle with centre at $(-1,2)$ and radius of 2 units.

2 marks

1st mark for one of the following:

- correct expression for $\cos\theta$ or $\sin\theta$ in terms of x or y.
- Applying $\cos^2\theta + \sin^2\theta = 1$ correctly to a correct expression.

Comment:

A range of students successfully eliminated θ but did not express the final result in an ideal form. The Cartesian form should be expressed in a simplified form where possible and not left in terms of expressions such as $\sin\left(\cos^{-1}\left(\frac{x+1}{2}\right)\right)$.

Question 3 (continued)

(c) (i)

$$\text{Sum of roots} = -\frac{b}{a}$$

$$\alpha+\beta+\gamma = -\frac{(-1)}{1}$$

$$\alpha+\beta+\gamma = 1$$

$$\therefore \beta+\gamma = 1-\alpha$$

1 mark

Comment:
Many students did **not** provide a very convincing solution. Working for questions that require you to SHOW, should be explicit and include appropriate detail to justify clearly why the result is true.
Stating the result that is to be shown, is not indicating why it is true.

(ii) Similar to the result from (i)

$\alpha+\beta = 1-\gamma$ and $\alpha+\gamma = 1-\beta$.

$$\frac{\beta+\gamma}{\alpha}+\frac{\gamma+\alpha}{\beta}+\frac{\alpha+\beta}{\gamma} = \frac{1-\alpha}{\alpha}+\frac{1-\beta}{\beta}+\frac{1-\gamma}{\gamma}$$

$$= \frac{1}{\alpha}-\frac{\alpha}{\alpha}+\frac{1}{\beta}-\frac{\beta}{\beta}+\frac{1}{\gamma}-\frac{\gamma}{\gamma}$$

$$= \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-3$$

$$= \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}-3$$

$$= \frac{\frac{c}{a}}{-\frac{d}{a}}-3$$

$$= \frac{c}{-d}-3$$

$$= \frac{-5}{-2}-3$$

$$\frac{\beta+\gamma}{\alpha}+\frac{\gamma+\alpha}{\beta}+\frac{\alpha+\beta}{\gamma} = -\frac{1}{2}$$

2 marks

1st mark for:

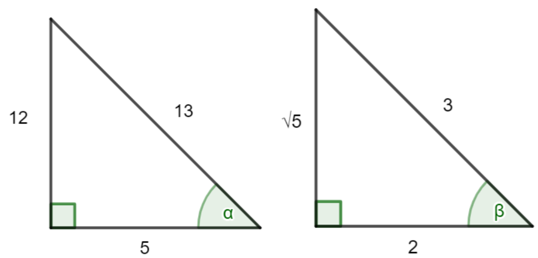
- correctly merging the three fractions into one fraction **and** using the part (i) result (and similar results) appropriately as directed by the phrasing of the question.

Comment:
Many students did not pay attention to the question in terms of using the result from part (i). A range of errors were made when calculating the result.

Question 3 (continued)

(d) Let $\alpha = \sin^{-1}\left(\frac{12}{13}\right)$ and $\beta = \cos^{-1}\left(\frac{2}{3}\right)$.

i.e. $\sin\alpha = \frac{12}{13}$ and $\cos\beta = \frac{2}{3}$.



$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$= \frac{12}{13} \times \frac{2}{3} - \frac{\sqrt{5}}{3} \times \frac{5}{13}$$

$$\therefore \sin\left(\sin^{-1}\left(\frac{12}{13}\right) - \cos^{-1}\left(\frac{2}{3}\right)\right) = \frac{24 - 5\sqrt{5}}{39}$$

3 marks

1st mark for one of the following OR

2 marks for **both** of the following:

- identifying either of the appropriate triangles needed with all three lengths shown.
- attempting to use the correct expansion for $\sin(\alpha - \beta)$.

Comment:

Mostly well done though there were a surprising number of students who did not connect the question to the expansion of $\sin(\alpha - \beta)$.

Question 3 (continued)

(e) (i)

$$\alpha + \beta + \gamma = \pi \ (\angle \text{ sum of } \Delta)$$

$$\therefore \alpha + \beta = \pi - \gamma$$

1 mark

Comment:

A surprising number of students did not read (nor answer) this elementary question properly.

All students should have gotten the mark for this part.

(ii)

$$\begin{aligned}\cos\gamma &= 2\sin\alpha\sin\beta \\ &= 2 \times \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \\ &= \cos(\alpha-\beta) - \cos(\pi-\gamma) \\ &= \cos(\alpha-\beta) - (-\cos\gamma) \ * \\ \cos\gamma &= \cos(\alpha-\beta) + \cos\gamma \\ 0 &= \cos(\alpha-\beta)\end{aligned}$$

$$\alpha - \beta = \frac{\pi}{2} \ \text{ since } \ \alpha > \beta$$

$$\alpha - \frac{\pi}{2} = \beta$$

$$\begin{aligned}\text{Hence, } 2\beta + \gamma &= \beta + \left(\alpha - \frac{\pi}{2}\right) + \gamma \\ &= \alpha + \beta + \gamma - \frac{\pi}{2} \\ &= \pi - \frac{\pi}{2}\end{aligned}$$

$$\therefore 2\beta + \gamma = \frac{\pi}{2}$$

i. e. 2β and γ are complementary.

2 marks

1st mark for:

- progressing to the step marked * in the solution on the left, by using the product-sum formula and the quadrant 2 relationship. $\cos(\pi - \theta) = -\cos\theta$.

Comment:

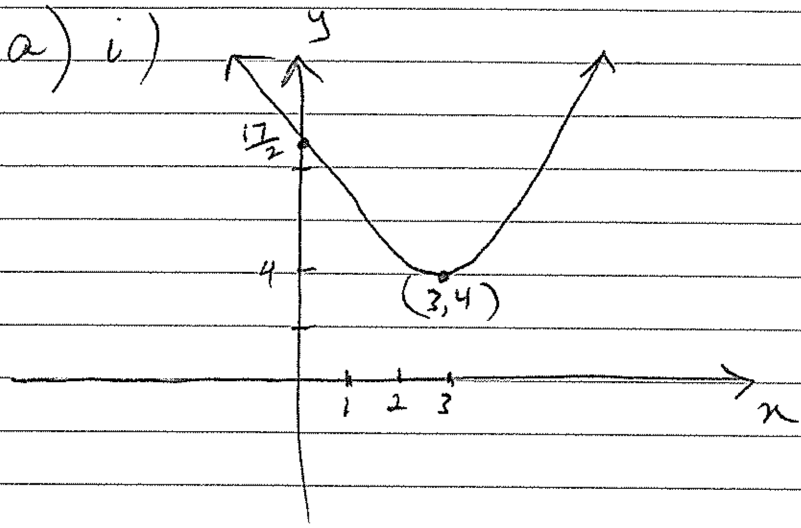
A good number of students managed to make appropriate use of the product-sum formula in conjunction with the result from part (i).

Only a small number of students managed to successfully construct a complete solution to the question.

WARNING: A number of students did not use the correct pronumerals supplied in the question. You should not use a, b, y instead of α, β, γ. Pay attention to these details and **do not** modify the symbols to suit your preference.

Q4. Yr 11 Task 3 2022

a) i)



Domain : $x \geq 3$ ✓

ii) $x = \frac{1}{2}(y-3)^2 + 4$

$x - 4 = \frac{1}{2}(y-3)^2$ ✓

$2x - 8 = (y-3)^2$

$y = 3 + \sqrt{2x-8}$ ✓

iii) $f^{-1}(f(a)) = 3 + \sqrt{2(\frac{1}{2}(a-3)^2 + 4) - 8}$

$= 3 + \sqrt{2(a-3)^2 + 8 - 8}$

$= 3 + \sqrt{(a-3)^2}$

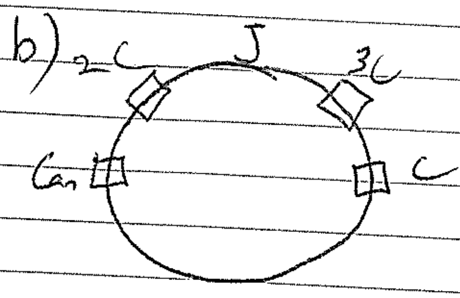
$= 3 + |a-3|$ 2

$= 3 + 3 - a$

$= 6 - a$

* This question was done poorly.
many student didn't know what to do.

b)



$$3 \times 2 \times 2$$
$$= 12$$

c)
$$\sin^{-1} x - \left(\frac{\pi}{2} - \sin^{-1} x\right) = k$$
$$2\sin^{-1} x - \frac{\pi}{2} = k$$
$$2\sin^{-1} x = k + \frac{\pi}{2}$$
$$\sin^{-1} x = \frac{1}{2}\left(k + \frac{\pi}{2}\right)$$
$$x = \sin\left(\frac{k}{2} + \frac{\pi}{4}\right)$$
$$= \sin\left(\frac{1}{2}k + \frac{\pi}{4}\right)$$
$$= \sin\frac{1}{2}k\cos\frac{\pi}{4} + \cos\frac{1}{2}k\sin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}\sin\frac{1}{2}k + \frac{1}{\sqrt{2}}\cos\frac{1}{2}k$$
$$= \frac{1}{\sqrt{2}}\left(\sin\frac{k}{2} + \cos\frac{k}{2}\right)$$

Only a handful of students were able to do this

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

had to be used.

d) $Q'(x) = 3Ax^2 + 2Bx + C$

$$2\alpha + \gamma = \frac{-B}{A}$$

$$\alpha^2 + 2\alpha\gamma = \frac{C}{A} \longrightarrow \alpha^2 = \frac{C}{A} - 2\alpha\gamma$$

$$\alpha^2\gamma = \frac{-D}{A} \longrightarrow \gamma = \frac{-D}{\alpha^2 A}$$

$$Q'(\alpha) = 0$$

$$3A\alpha^2 + 2B\alpha + C = 0$$

$$3A\left(\frac{C}{A} - 2\alpha\gamma\right) + 2B\alpha + C = 0$$

$$3C - 6A\alpha\gamma + 2B\alpha + C = 0$$

$$3C - 6\cancel{A}\alpha\left(\frac{-D}{\alpha^2 \cancel{A}}\right) + 2B\alpha + C = 0$$

$$3C + \frac{6D}{\alpha} + 2B\alpha + C = 0$$

$$3C\alpha + 6D + 2B\alpha^2 + C\alpha = 0$$

$$2B\alpha^2 + 4C\alpha + 6D = 0$$

$$B\alpha^2 + 2C\alpha + 3D = 0$$

This was the best method for this question

OR

$$A\alpha^3 + B\alpha^2 + C\alpha + D = 0 \qquad \times 3$$

$$3A\alpha^2 + 2B\alpha + C = 0 \qquad \times \alpha$$

$$3A\alpha^3 + 3B\alpha^2 + 3C\alpha + 3D = 0 \qquad (1)$$
$$3A\alpha^2 + 2B\alpha^2 + C\alpha = 0 \qquad (2)$$

$(1) - (2)$

$$B\alpha^2 + 2C\alpha + 3D = 0$$

This was the fastest method.