**Student Name:** ______________________________

**Teacher:** Ms Mandile / Mrs Underwood (circle)

Danebank

An Anglican School for Girls

# Year 12 Extension 1

## Wednesday 18th August 2021

## Task 4 – Trial

| General Instructions | |
|---|---|
| • Write using a black pen<br>• Graphs may be drawn in pencil<br>• No liquid paper or whiteout tape | |
| **Reading Time** | 5 minutes |
| **Working Time** | 120 minutes |

| Structure of the Paper | | |
|---|---|---|
| **Section I**<br>pages 2 – 5 | **10 marks** | Allow about 15 minutes for this section |
| **Section II**<br>pages 6 – 9 | **60 marks** | Allow about 1 hour 45 minutes for this section |
| **Total** | **70 marks** | **Weighting 30%** |

| Special Instructions |
|---|
| • NESA approved Calculator may be used.<br>• Attempt all questions<br>• This examination must not be removed from the Examination Room.<br>• **Use the multiple choice answer sheet for section I**<br>• In Section II show relevant mathematical reasoning and/or calculations.<br>• The mark allocation for each question provides a guide for the expected length of response.<br>• Full marks may not be awarded for correct answers without working.<br>• In section II answer each question in a separate writing booklet. Extra writing booklets are available if you need them. Number your answers clearly. |

**Section I**
**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1 – 10.

1. Peter draws a vector from the origin to the point $A(-2, 5)$. He then draws a vector $-3\underset{\sim}{\mathbf{i}} + 2\underset{\sim}{\mathbf{j}}$ from $A$, ending at point $B$.

   How far is point $B$ from the origin?

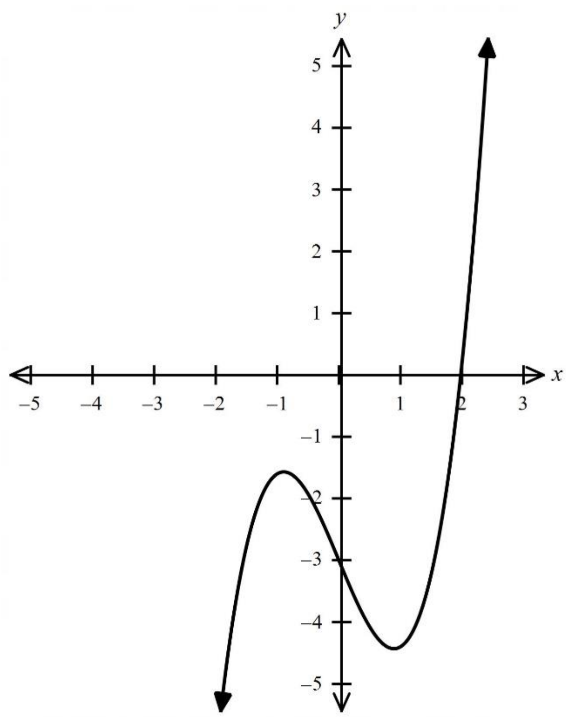
   A. $\sqrt{10}$ units

   B. $\sqrt{74}$ units

   C. $\sqrt{13}$ units

   D. $\sqrt{34}$ units

2. Consider the graph of $y = f(x)$ shown here:




   Which one of the following would have 2 more roots than $f(x)$?

   A. $y = -2 \times f(x)$

   B. $y = f(x) + 3$

   C. $y = f^{-1}(x)$

   D. $y = f(x + 3)$

3. The volume of the solid formed when the area enclosed between the curve $y = x^3 - 1$, the $y$-axis, and the lines $y = 1$ and $y = 2$ is rotated about the $y$-axis is given by:

A. $\pi \int_1^2 (x^3 - 1)dy$

B. $\int_1^2 (\sqrt[3]{y+1})\,dy$

C. $\pi \int_1^2 (\sqrt[3]{y} + 1)^2 dy$

D. $\pi \int_1^2 \sqrt[3]{(y+1)^2} dy$

4. Which one of the following differential equations is $y = 5xe^{2x}$ a solution?

A. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

B. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$

C. $\frac{d^2y}{dx^2} - 4y = 20e^{2x}$

D. $\frac{d^2y}{dx^2} - 4y = e^{2x}$

5. The vectors $4\underset{\sim}{i} - \underset{\sim}{j}$ and $3\underset{\sim}{i} + x\underset{\sim}{j}$ are perpendicular. What is the value of $x$?

A. $-12$

B. $-1$

C. 12

D. 13

6. Which of the following is an anti-derivative for $-\dfrac{1}{\sqrt{9 - x^2}}$ ?

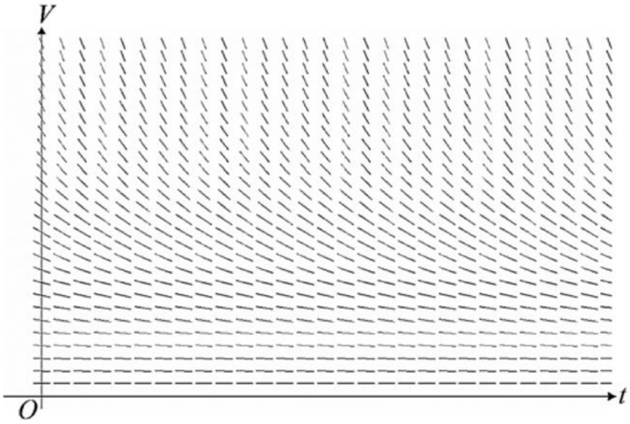
A. $y = \tan^{-1}\left(\frac{x}{3}\right) + C$

B. $y = \sin^{-1} 3x + C$

C. $y = \cos^{-1}\left(\frac{x}{3}\right) + C$

D. $y = \cos^{-1} 3x + C$

7. A direction field for the volume of water, $V$ megalitres, in a reservoir $t$ years after 2021 is shown below.




According to this model, for $k > 0$ what is the value of $\dfrac{dV}{dt}$?

A. $-kV^2$

B. $-\frac{k}{V}$

C. $\frac{k}{V}$

D. $kV^2$

8. A company has a board of twelve directors. Six of these were selected at random to be candidates in the election for the positions of President, Vice-President and Treasurer.

In how many ways can these three senior positions be filled?

A. ${}^{12}\mathbf{P}_3$

B. $\dfrac{12!}{3!}$

C. $\dfrac{{}^{12}\mathbf{P}_6}{3!}$

D. $\dfrac{{}^{12}\mathbf{P}_6}{3!3!}$

9. A particle is initially positioned at $x = -2$. Its motion has velocity of $v = \dfrac{1}{t+e}$.

Which of the following will be true when the particle reaches the origin?

A. $t = 2, v = \dfrac{1}{2+e}$

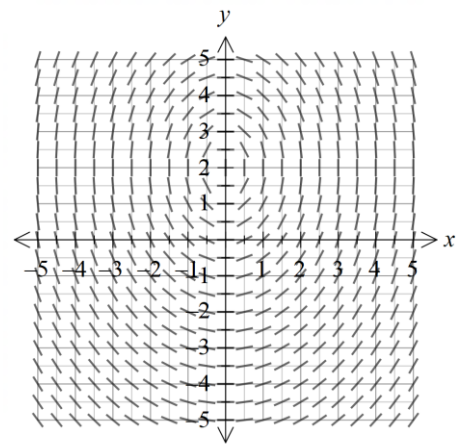
B. $t = e, \dot{x} = \dfrac{1}{2e}$

C. $t = e^2, v = \dfrac{1}{e^2+e}$

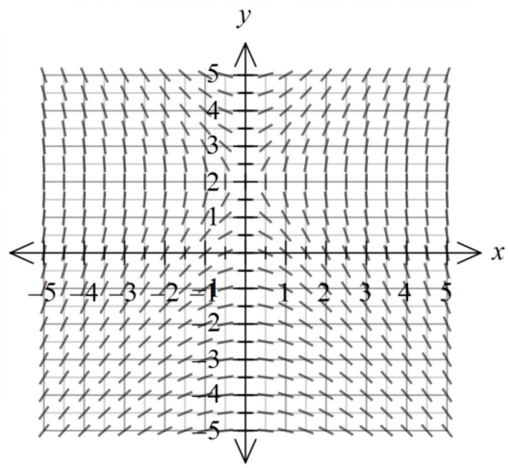
D. $t = e^3 - e, v = e^{-3}$

10. Which of the following best represents the direction field for the differential equation $\dfrac{dy}{dx} = \dfrac{-2x}{y-2}$ ?

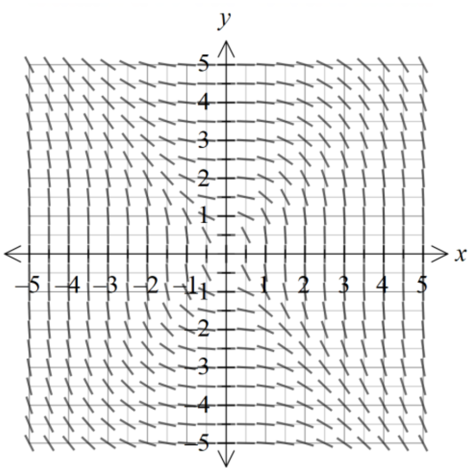
A.



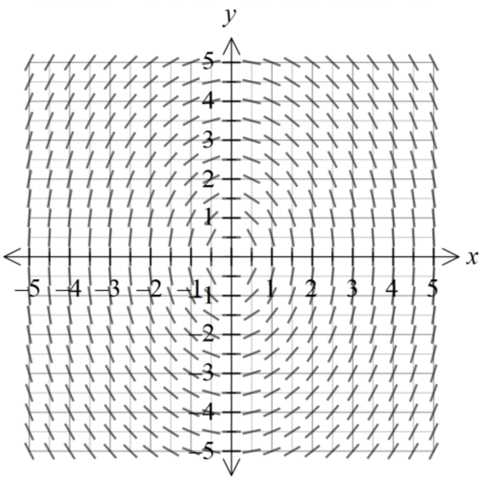
B.



C.



D.



**Section II**
**60 marks**
**Attempt Questions 11-14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Start a new Writing Booklet

(a) Given $P(x) = x^3 + 11x^2 - 56x + 60$ and $P'(2) = 0$

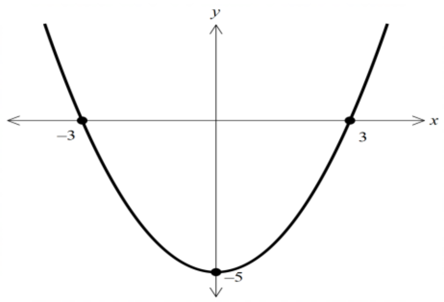
(i) Fully factorise the polynomial $P(x)$ **2**

(ii) Hence, solve $P(x) \leq 0$ **1**

(b) For what values of $x$ is $\dfrac{x}{x-1} \geq -2$? **3**

(c) Find the angle between the vectors $\underset{\sim}{\mathbf{a}} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\underset{\sim}{\mathbf{b}} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, correct to the nearest degree. **2**

(d) The diagram shows the graph of $y = f(x)$.




(i) Sketch $y = f^{-1}(x)$ after making an appropriate restriction on the domain of $f(x)$. **1**

(ii) Sketch $y = \dfrac{1}{f^{-1}(x)}$. **2**

(e) By expressing $\sin x - \sqrt{3}\cos x$ in the form $A\sin(x - \alpha)$, solve $\sin x - \sqrt{3}\cos x = -\sqrt{2}$, for $0 \leq x \leq 2\pi$. **4**

**End of Question 11**

**Question 12** (15 marks) Start a new Writing Booklet

(a) Use the principle of mathematical induction to show that for all integers $n \geq 1$, **3**

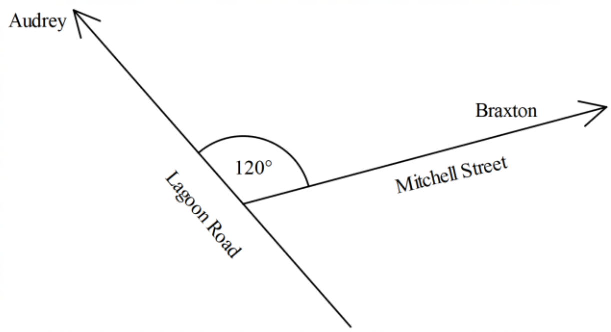
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2 .$$

(b) Use the substitution $x = u^2 - 1$, find the exact value of **3**

$$\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$$

(c) Use the pigeonhole principle to determine how many integers must be selected from the numbers 5, 6, 7, 8, 9 and 10 so that at least 2 of the numbers will have a difference of 3? **2**

(d) Two straight roads, Lagoon Road and Mitchell Street, intersect at an angle of $120°$.
Audrey leaves the intersection at 9:00 am and travels on Lagoon Road at a constant speed of 50 kilometres per hour.
Braxton leaves the intersection at 10:00 am, travelling along Mitchell Street at a constant speed of 60 kilometres per hour.
The diagram below shows the direction of Audrey and Braxton's travels.




Taking $t = 0$ at the time of Braxton's departure from the intersection:

(i) Show that the distance between Audrey and Braxton at time, t hours can be expressed as $D = 10\sqrt{91t^2 + 80t + 25}$ kilometres. **2**

(ii) Find the rate of change in the distance between Audrey and Braxton at 12:00 noon, correct to the nearest km/h. **2**

(e) It is known that $\sin x = \frac{1}{3}$, where $\frac{\pi}{2} < x < \pi$. Find the exact value of $\cot 2x$? **3**

**End of Question 12**

**Question 13** (15 marks) Start a new Writing Booklet

(a) Water is poured into a container at a rate of 8 $cm^3s^{-1}$ and the volume of the water in the container is given by: **3**

$$V = \frac{3}{2}(h^2 + 8h) \quad \text{where h cm is the depth of the water.}$$

Find the rate of change of the depth of the water when $V = 72$ $cm^3$.

(b) Using the substation $u = x - 1$,

(i) show that: **1**

$$\int \frac{x}{x^2 - 2x + 2}\,dx = \int \frac{u}{u^2 + 1}\,du + \int \frac{1}{u^2 + 1}\,du$$

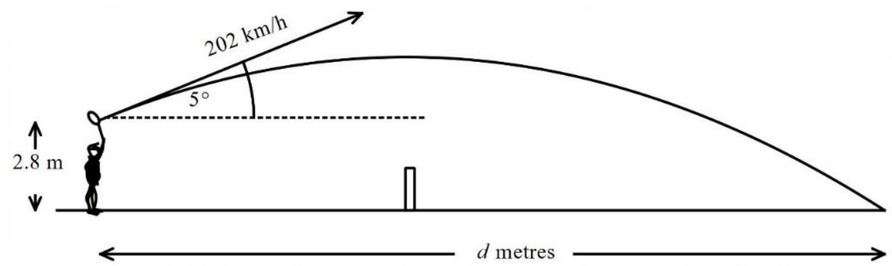
(ii) Hence or otherwise find **2**

$$\int \frac{x}{x^2 - 2x + 2}\,dx$$

(c) Mick hits a tennis ball from a height 2.8 metres above the ground. The angle of elevation as the racket hits the ball is 5° and the velocity is 202 km/h.

202 km/h is approximately 56.1 m/s.

The ball lands at a horizontal distance of $d$ metres from Mick's feet.




**3**

(i) Using $g = 9.8$ $ms^{-2}$ as acceleration due to gravity, show that the vector displacement of the tennis ball is:

$$\underset{\sim}{s}(t) \approx 55.9t\,\underset{\sim}{\boldsymbol{i}} + (4.9t - 4.9t^2 + 2.8)\,\underset{\sim}{\boldsymbol{j}}.$$

.

**2**

(ii) If $d$ is greater than 26.2, the ball will be called "out".

Find the value of $d$ and determine if the ball will be called "out".

**Question 13 continues on page 9**

**Question 13 (continued)**

(d) A piece of hot metal is placed in a room with a surrounding air temperature of 15˚C and allowed to cool. It loses heat according to Newton's law of cooling: **4**

$$\frac{dT}{dt} = -k(T - A)$$

where $T$ is the temperature of the metal in degrees Celsius at time $t$ minutes, $A$ is the surrounding air temperature and $k$ is a positive constant.
After 5 minutes the temperature of the metal is 75˚C, and after a further 3 minutes it is 45˚C.
Solve the differential equation, expressing your answer in the form $T = f(t)$ and giving all constants in exact form.

**End of Question 13**

**Question 14** (15 marks) Start a new Writing Booklet

(a) Consider the differential equation: $\dfrac{dy}{dx} = \dfrac{3x^2+4}{2y}$ **3**

Find a solution to the equation in the form $y = f(x)$, given that $f(2) = 1$.

(b) (i) Differentiate $x \cos^{-1} x$ with respect to $x$. **1**

(ii) Hence find $\int \cos^{-1} x \, dx$ **2**

(c) Given the function $f(x) = \sin^2 x + 2\cos x$

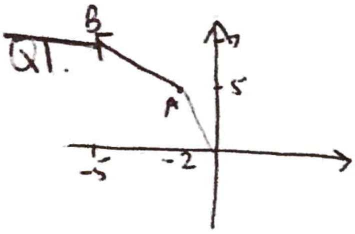
(i) Determine if the function $y = f(x)$ is odd, even or neither. **2**

(ii) Show that the $x$-intercepts of $f(x) = \sin^2 x + 2\cos x$ in the domain $-\pi \leq x \leq \pi$ are $x = \pm\cos^{-1}(1 - \sqrt{2})$ **3**

(iii) Hence or otherwise show that the exact area enclosed by the curve, $-\cos^{-1}(1 - \sqrt{2}) \leq x \leq \cos^{-1}(1 - \sqrt{2})$ and the $x$-axis is **4**

$$\cos^{-1}(1 - \sqrt{2}) + (3 + \sqrt{2})\sqrt{2\sqrt{2} - 2}$$

**End of Paper**

Q1. 


$OB = \sqrt{(5^2 + 7^2)}$
$= \sqrt{74}$

B

B

Q2 $y = f(x) + 3$

D

Q3 $Vol = \pi \int x^2 \, dy$

$y = x^3 - 1$
$y + 1 = x^3$
$x = \sqrt[3]{(y+1)}$
$x^2 = (\sqrt[3]{(y+1)})^2$

Q4 $y = 5x . e^{2x}$

$u = 5x$ $\quad v = e^{2x}$
$u' = 5$ $\quad v' = 2e^{2x}$

$\frac{dy}{dx} = 5e^{2x} + 10xe^{2x}$

$\frac{d^2y}{dx^2} = 10e^{2x} + 2(5e^{2x} + 10xe^{2x})$
$= 20e^{2x} + 20xe^{2x}$

- not A or B

$20e^{2x} + 20xe^{2x} - 4(5xe^{2x}) = 20e^{2x}$ $\therefore$ C

Q5 If perpendicular then dot product = 0

$4 \times 3 - x = 0$
$x = 12$

C

C

Q6.

All gradients are negative since $k > 0$
then either A or B.

As $V$ increases, $\frac{dV}{dt}$ gets steeper $\therefore$ A

Q8. $^{12}C_6 \times {}^6P_3 = \frac{12!}{6!\,\cancel{6!}} \times \frac{\cancel{6!}}{3!}$

$= \frac{^{12}P_6}{3!}$

C

Q9. $\frac{dx}{dt} = \frac{1}{t+e}$

$x = \int \frac{1}{t+e}\,dt$

$x = \ln|t+e| + c$

when $t = 0$ $x = -2$

$-2 = \ln e + c$

$c = -3$

$\therefore x = \ln(t+e) - 3$

when $x = 0$

$\ln(t+e) = 3$

$t+e = e^3$

$t = e^3 - e$

$\therefore$ D

Q10. $\frac{dy}{dx} = \frac{-2x}{y-2}$

at $(0,0)$ $\frac{dy}{dx} = 0$ $\therefore$ A or B

at $(-2,-2)$ $\frac{dy}{dx} = -\frac{2x-2}{-2-2}$

$= -1$

$\therefore$ A

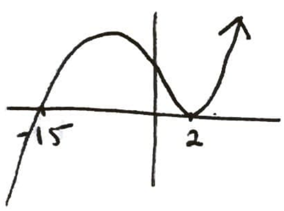
1.

a) i) $P(x) = x^3 + 11x^2 - 56x + 60$

$P'(2) = 0 \quad \therefore \quad (x-2)^2$ is a factor

$$\begin{array}{r} x + 15 \\ x^2 - 4x + 4 \overline{) x^3 + 11x^2 - 56x + 60} \\ \underline{x^3 - 4x^2 + 4x} \\ 15x^2 - 60x + 60 \\ 15x^2 - 60x + 60 \end{array}$$

$\therefore P(x) = (x-2)^2(x+15)$

ii) 


$P(x) \leqslant 0$

$\therefore x \leqslant 15, \; x = 2$

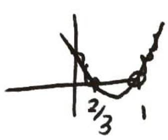
b) $\dfrac{x}{x-1} \geqslant -2 \qquad x \neq 1$

$(x-1)^{\cancel{2}} \times \dfrac{x}{\cancel{x-1}} \geqslant -2(x-1)^2$

$x(x-1) + 2(x-1)^2 \geqslant 0$

$(x-1)(x + 2x - 2) \geqslant 0$

$(x-1)(3x-2) \geqslant 0$




$\therefore x > 1, \; x \leq 2/3$

c) $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$
$= x_1 x_1 + y_1 y_1$

$|\underset{\sim}{a}| = \sqrt{(9+25)}$
$= \sqrt{34}$

$|\underset{\sim}{b}| = \sqrt{(1+16}$
$= \sqrt{17}$

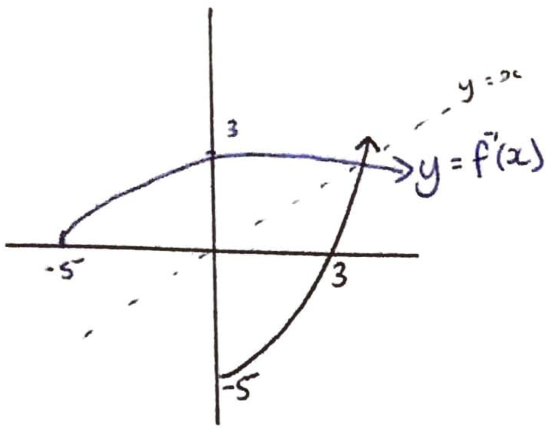
$\therefore \sqrt{34} \cdot \sqrt{17} \cos\theta = -3 - 20$

$\cos\theta = -0.9566$

$\theta = 163^\circ$

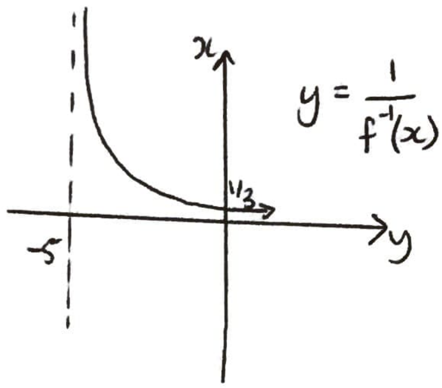
)

i)




or other half.

ii)




e)

$\sin x - \sqrt{3}\cos x$

$A = \sqrt{1^2 + \sqrt{3}^2}$

$= 2$

$\tan\alpha = \frac{\sqrt{3}}{1}$

$\alpha = \frac{\pi}{3}$

$\begin{Bmatrix} \cos\alpha = 1 \\ \sin\alpha = \sqrt{3} \end{Bmatrix}$

$\therefore \sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$

$2\sin\left(x - \frac{\pi}{3}\right) = -\sqrt{2}$

$0 \leq x \leq 2\pi$

$\sin\left(x - \frac{\pi}{3}\right) = \frac{-\sqrt{2}}{2}$

$-\frac{\pi}{3} \leq x - \frac{\pi}{3} \leq \frac{5\pi}{3}$

$= -\frac{1}{\sqrt{2}}$

related acute $\left(x - \frac{\pi}{3}\right) = \frac{\pi}{4}$

$x - \frac{\pi}{3} = \pi + \frac{\pi}{4}, \ -\frac{\pi}{4}$

$x = \frac{\pi}{12}, \ \frac{19\pi}{12}$

a) test $n=1$

$LHS = 1^3$ $\quad$ $RHS = \frac{1}{4}^2(1+1)^2$

$= 1$

$= 1$

$\therefore$ true for $n=1$

Assume true for $n=k$

$$S_k = \frac{k^2}{4}(k+1)^2$$

Prove true for $n=k+1$ ie show $S_{k+1} = \frac{(k+1)^2(k+2)^2}{4}$

Now $S_{k+1} = S_k + T_{k+1}$ using our assumption

$$= \frac{k^2}{4}(k+1)^2 + (k+1)^3$$

$$= (k+1)^2\left(\frac{k^2}{4} + k+1\right)$$

$$= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right)$$

$$= \frac{(k+1)^2}{4}\left(k^2+4k+4\right)$$

$$= \frac{(k+1)^2}{4}(k+2)^2$$

$\therefore$ true for $n=k+1$

By the principle of mathematical induction the above statement is proved to be true.

$x = u^2 - 1$ $\qquad \frac{dx}{du} = 2u$

$dx = 2u\,du$

when $x = 0$ $\qquad x = 1$

$u^2 - 1 = 0$ $\qquad u^2 - 1 = 1$

$u = 1$ $\qquad u = \sqrt{2}$

$$\int_0^1 \frac{x}{\sqrt{x+1}}\,dx = \int_1^{\sqrt{2}} \frac{u^2-1}{\sqrt{u^2-1+1}}\,2u.du$$

$$= \int_1^{\sqrt{2}} \frac{u^2-1}{\cancel{u}} . 2\cancel{u}\,du$$

$$= 2\int_1^{\sqrt{2}} (u^2-1)\,du$$

$$= 2\left[\frac{u^3}{3} - u\right]_1^{\sqrt{2}}$$

$$= 2\left\{\left(\frac{2\sqrt{2}}{3} - \sqrt{2}\right) - \left(\frac{1}{3} - 1\right)\right\}$$

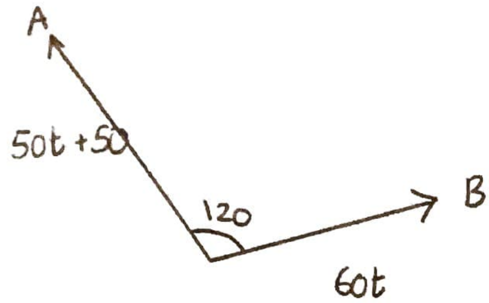
$$= 2\left\{\frac{2\sqrt{2}}{3} - \frac{3\sqrt{2}}{3} + \frac{2}{3}\right\}$$

$$= \frac{2(2 - \sqrt{2})}{3}$$

c) If we select 5, 6, 7 then there is no difference of 3

$\therefore$ we need 4 digits to be selected.

i)




$$D^2 = \left(50(t+1)\right)^2 + (60t)^2 - 2\times 50(t+1)\times 60t\cos 120$$

$$= 2500(t^2+2t+1) + 3600t^2 + 3000t(t+1)$$

$$= 2500t^2 + 5000t + 2500 + 3600t^2 + 3000t^2 + 3000t$$

$$= 9100t^2 + 8000t + 2500$$

$$= 100(91t^2 + 80t + 25)$$

$$D = \sqrt{100(91t^2+80t+25)}$$

$$= 10\sqrt{91t^2+80t+25}$$

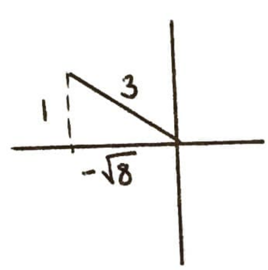
ii) $\frac{dD}{dt} = 10\times\frac{1}{2}(182t+80)(91t^2+80t+25)^{-1/2}$

when $t=2$

$\frac{dD}{dt} \doteqdot 94.747$

$= 95$ Km/h

e) $\sin x = \frac{1}{3}$ $\quad \frac{\pi}{2} < x < \pi$




$$\tan x = -\frac{1}{\sqrt{8}}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\cot 2x = \frac{1-\tan^2 x}{2\tan x} = \frac{1-\left(-\frac{1}{\sqrt{8}}\right)^2}{2\times\frac{1}{\sqrt{8}}} = \frac{-7\sqrt{8}}{16}$$

$$= \frac{-7\times 2\sqrt{2}}{\cancel{16}\ 8} = \frac{-7\sqrt{2}}{8}$$

a) $\frac{dV}{dt} = 8 \text{ cm}^3 s^{-1}$

$V = \frac{3}{2}(h^2+8h)$

$\frac{dV}{dh} = \frac{3}{2}(2h+8)$

$= 3h+12$

$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV}$

$\frac{dh}{dV} = \frac{1}{3h+12}$ ✓

$=$

When $V=72$

$\frac{3}{2}(h^2+8h)=72$

$h^2+8h-48=0$

$(h+12)(h-4)=0$

$h=-12, 4$ ✓

invalid sol$^n$

$\therefore \frac{dh}{dt} = 8 \times \frac{1}{3\times 4+12}$ ✓

$= \frac{1}{3} \text{ cms}^{-1}$

3

b) i) $u=x-1$

$\frac{du}{dx}=1$

$du=dx$

$\int \frac{x}{x^2-2x+2}\,dx = \int \frac{u+1}{u^2+1}\,du$

$= \int \frac{u}{u^2+1}\,du + \int \frac{1}{u^2+1}\,du$

1

ii) $\int \frac{x}{x^2-2x+2}\,dx = \frac{1}{2}\int \frac{2u}{u^2+1}\,du + \int \frac{1}{u^2+1}\,du$

$= \frac{1}{2}\ln|u^2+1| + \tan^{-1}(u) + c$ ✓

$= \frac{1}{2}\ln|(x-1)^2+1| + \tan^{-1}(x-1) + c$

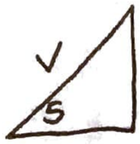
$= \frac{1}{2}\ln|x^2-2x+2| + \tan^{-1}(x-1) + c$ ✓

2

$V = 202$ Km/h

i) $= \frac{202 \times 1000}{3600}$

$= 56^{1}/_{9}\ ms^{-1}$




$\dot{y} = 56\tfrac{1}{9}\sin 5$
$\doteq 4.9$

$\dot{x} = 56\tfrac{1}{9}\cos 5$
$\doteq 55.9$

$\underset{\sim}{a}(t) = \ddot{x}\underset{\sim}{i} + \ddot{y}\underset{\sim}{y}$

$= 0\underset{\sim}{i} - 9.8\underset{\sim}{j}$ ✓

$\underset{\sim}{v}(t) = \int 0\underset{\sim}{i} - 9.8\underset{\sim}{j}\ dt$

$= c\underset{\sim}{i} + (-9.8t + c_1)\underset{\sim}{j}$

when $t = 0$

$c = 55.9 \qquad c_1 = 4.9$

$\underset{\sim}{v}(t) = 55.9\underset{\sim}{i} + (-9.8t + 4.9)\underset{\sim}{j}$ ✓

$\underset{\sim}{s}(t) = \int 55.9\underset{\sim}{i} + (-9.8t + 4.9)\underset{\sim}{j}\ dt$

$= (55.9t + c_2)\underset{\sim}{i} + (-4.9t^2 + 4.9t + c_3)\underset{\sim}{j}$

when $t = 0$

$x = 0 \qquad \therefore c_2 = 0$
$y = 2.8 \qquad c_3 = 2.8$

$\underset{\sim}{s}(t) = 55.9t\underset{\sim}{i} + (-4.9t^2 + 4.9t + 2.8)\underset{\sim}{j}$ ✓

3

ii) Ball hits the ground when vertical displacement is zero

$-4.9t^2 + 4.9t + 2.8 = 0$

$t = \frac{-4.9 \pm \sqrt{4.9^2 - 4(-4.9)(2.8)}}{2 \times -4.9}$

$= -0.4, 1.41 \qquad t > 0 \therefore t = 1.41$ /

$\therefore d = 55.9 \times 1.41$
$\doteq 78.6m$ ✓
$\therefore$ ball is out

2

9
13.

$$\frac{dT}{dt} = -K(T-15)$$

when $t=5$ $T=75$

$t=8$ $T=45$

$$\frac{dt}{dT} = -\frac{1}{K(T-15)}$$

$$t = -\frac{1}{K}\int \frac{1}{T-15}\, dT$$

$$-Kt = \ln|T-15| + c$$ ✓

$$= \ln|T-15| + \ln e^{c}$$

$$= \ln e^{c}(T-15)$$

$$\therefore e^{-Kt} = e^{c}(T-15)$$

$$= B(T-15) \quad \text{where } B = e^{c}$$

$$\therefore e^{-5K} = 60B \quad \ldots (1)$$

$$e^{-8K} = 30B \quad \ldots (2)$$

$(1) \div (2)$

$$e^{3K} = 2$$

$$3K = \ln 2$$

$$K = \frac{\ln 2}{3}$$ ✓

$$\therefore \frac{1}{B} e^{-t \cdot \frac{\ln 2}{3}} = T - 15$$

$$T = 15 + \frac{1}{B} e^{-t \cdot \frac{\ln 2}{3}}$$

ontinued

$$B = \frac{e^{-\frac{\ln 2}{3} \cdot t}}{T-15} \qquad \text{when } t=5 \; T=75$$

$$B = \frac{e^{-5\frac{\ln 2}{3}}}{60}$$

$$= \frac{e^{\ln 2^{-5/3}}}{60}$$

$$= \frac{2^{-5/3}}{60}$$

$$= \frac{1}{120\sqrt[3]{4}}$$ ✓

$$\therefore T = 15 + 120\sqrt[3]{4}\, e^{-t \cdot \frac{\ln 2}{3}}$$ ✓

Q14.

a) $$\frac{dy}{dx} = \frac{3x^2+4}{2y}$$

$$\int 2y\,dy = \int (3x^2+4)\,dx$$ ✓

$$y^2 = x^3 + 4x + C$$

$f(2) = 1$

$$1^2 = 2^3 + 4\times 2 + C$$

$$C = -15$$ ✓

$$\therefore y^2 = x^3 + 4x - 15$$ ✓

$$y = \pm\sqrt{x^3+4x-15}$$

but $f(2)=1$ $\therefore$ $y = \sqrt{x^3+4x-15}$

$$\frac{d}{dx}(x\cos^{-1}x)$$

$u = x \quad v = \cos^{-1}x$

$u' = 1 \quad v' = -\frac{1}{\sqrt{1-x^2}}$

$$= 1 \times \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$$

$$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$$

1

ii) $\therefore \int \frac{d}{dx}(x\cos^{-1}x)\,dx = \int \cos^{-1}x\,dx + \int \frac{-x}{\sqrt{1-x^2}}\,dx$

$$x\cos^{-1}x = \int \cos^{-1}x\,dx + \frac{1}{2}\int -2x(1-x^2)^{-1/2}\,dx \checkmark$$

$$\int \cos^{-1}x\,dx = x\cos^{-1}x - \frac{1}{2}\frac{(1-x^2)^{1/2}}{\cancel{1/2}} + c$$

$$= x\cos^{-1}x - \sqrt{1-x^2} + c \checkmark$$

2

c) i) $f(x) = \sin^2 x + 2\cos x$

$f(-x) = \sin^2(-x) + 2\cos(-x)$

$= \sin^2 x \checkmark + 2\cos x \checkmark$

2

$\therefore f(x) = f(-x)$ so the function is even.

ii) $\sin^2 x + 2\cos x = 0$

$1 - \cos^2 x + 2\cos x = 0$

$\cos^2 x - 2\cos x - 1 = 0$

$$\cos x = \frac{2 \pm \sqrt{(2^2 - 4\times1\times-1)}}{2}$$

continued

$\cos x = 1 \pm \sqrt{2}$ ✓

but $\cos x \neq 1+\sqrt{2}$ $\quad \therefore \cos x = 1-\sqrt{2}$

$$x = \cos^{-1}(1-\sqrt{2}) \quad ✓$$

but since $f(x)$ is even, it is symmetrical about $y$-axis

$\therefore$ $x$ intercepts are $x = \pm \cos^{-1}(1-\sqrt{2})$ ✓ 3

(iii)

$$\text{Area} = \int_{-\cos^{-1}(1-\sqrt{2})}^{\cos^{-1}(1+\sqrt{2})} (\sin^2 x + 2\cos x)\,dx$$

$$= 2\int_0^{\cos^{-1}(1+\sqrt{2})} \sin^2 x + 2\cos x \,dx \quad ✓$$

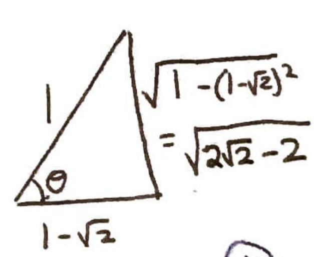
$$= 2\int_0^{\cos^{-1}(1+\sqrt{2})} \left(\tfrac{1}{2}(1-\cos 2x) + 2\cos x\right) dx$$

$$= 2\left[\tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) + 2\sin x\right]_0^{\cos^{-1}(1-\sqrt{2})}$$

$$= \left[x - \tfrac{1}{2}\sin 2x + 4\sin x\right]_0^{\cos^{-1}(1-\sqrt{2})} \quad ✓$$

$$= \left[x - \sin x\cos x + 4\sin x\right]_0^{\cos^{-1}(1-\sqrt{2})}$$

$$= \cos^{-1}(1-\sqrt{2}) - \sin(\cos^{-1}(1-\sqrt{2})).\cos(\cos^{-1}(1-\sqrt{2})) + 4\sin(\cos^{-1}(1-\sqrt{2})) \quad ✓$$




$$= \cos^{-1}(1-\sqrt{2}) - \sqrt{2\sqrt{2}-2}\,.\,(1-\sqrt{2}) + 4\sqrt{2\sqrt{2}-2}$$

$$= \cos^{-1}(1-\sqrt{2}) + \sqrt{2\sqrt{2}-2}\,(\sqrt{2}-1+4)$$

$$= \cos^{-1}(1-\sqrt{2}) + \sqrt{2\sqrt{2}-2}\,(3+\sqrt{2}) \quad ✓$$

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