Student Number:

Teacher: Karagiannis / Blair (Circle)

or Student Name: ____________________

Mathematics Extension 1
Year 12 Examination Block 2022
Task 4 - Trial HSC Examination

General Instructions	
Write using black or blue pen Graphs can be drawn in pencil No liquid paper or correction tape are to be used.	
Reading Time	10 minutes
Working time	2 hours

Structure of the Paper		
Section 1	/10 Marks	Allow about 15 minutes for this section
Section 2	/ 60 Marks	Allow about 1 hour and 45 minutes for this section
Total	**/70 Marks**	**Weighting: 30 %**

Special Instructions
NESA approved Calculator may be used. A reference sheet is provided at the back of this paper. A separate multiple choice answer sheet is provided. Attempt all questions. This examination must not be removed from the Examination Room. In Questions 11-14, show relevant mathematical reasoning and/or calculations. Answer each question in a separate writing booklet.

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Attempt Questions 1-10

Answer questions on the multiple choice answer sheet.

1. Jack starts at the origin and walks along vector $2\underset{\sim}{i} + 3\underset{\sim}{j}$ and then turns and walks along vector $4\underset{\sim}{i} - 2\underset{\sim}{j}$. How far is Jack from the origin ?

(A) 5

(B) $\sqrt{11}$

(C) $\sqrt{37}$

(D) $\sqrt{61}$

2. What is the derivative of $\tan^{-1}(x^4)$ with respect to x?

(A) $\frac{1}{1+x^8}$

(B) $\frac{4x^3}{1+x^8}$

(C) $\frac{4x^3}{1+x^4}$

(D) $\frac{x^4}{1+x^8}$

3. Laura projects an arrow at an angle of 45° to the horizontal with an initial velocity of 50 ms^{-1}. What is the horizontal speed of the arrow?

(A) $\sqrt{2}\ \text{ms}^{-1}$

(B) $50\sqrt{2}\ \text{ms}^{-1}$

(C) $\frac{1}{\sqrt{2}}\ \text{ms}^{-1}$

(D) $\frac{50}{\sqrt{2}}\ \text{ms}^{-1}$

4. A curve is described by the parametric equations below.

   $x = \frac{t}{2}$ $\qquad$ $y = 3t^2$

   What is the cartesian equation of the curve?

   (A) $x = 12y^2$

   (B) $y = \frac{3x^2}{4}$

   (C) $y = \frac{2}{x}$

   (D) $y = 12x^2$

5. Correct to 3 significant figures, find the value of

   $$\int_1^{\sqrt{3}} \frac{dx}{\sqrt{3 - x^2}}$$

   (A) -1.05

   (B) -0.955

   (C) 0.955

   (D) 1.05

6. A Year 12 Biology class consists of 10 girls and 15 boys.

   In how many ways can a group of three boys and two girls be chosen from this class for a group to work on an investigation project?

   (A) 250

   (B) 900

   (C) 12 600

   (D) 20 475

7. What is the value of $\sin 2x$ given that $\sin x = 0.8$ and x is obtuse ?

(A) $-\frac{12}{25}$

(B) $-\frac{24}{25}$

(C) $\frac{12}{25}$

(D) $\frac{24}{25}$

8. What is the domain and range of the function $y = 6\cos^{-1}(3x)$?

(A) Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$, Range $[0, 6\pi]$

(B) Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$, Range $[0, 3\pi]$

(C) Domain $[0, 6\pi]$, Range $\left[-\frac{1}{3}, \frac{1}{3}\right]$

(D) Domain $[0, 3\pi]$, Range $\left[-\frac{1}{3}, \frac{1}{3}\right]$

9. A particle moves in a straight line. Its position at any time t is given by

$x = 3\cos 2t + 4\sin 2t$

What is the acceleration of the particle in terms of x?

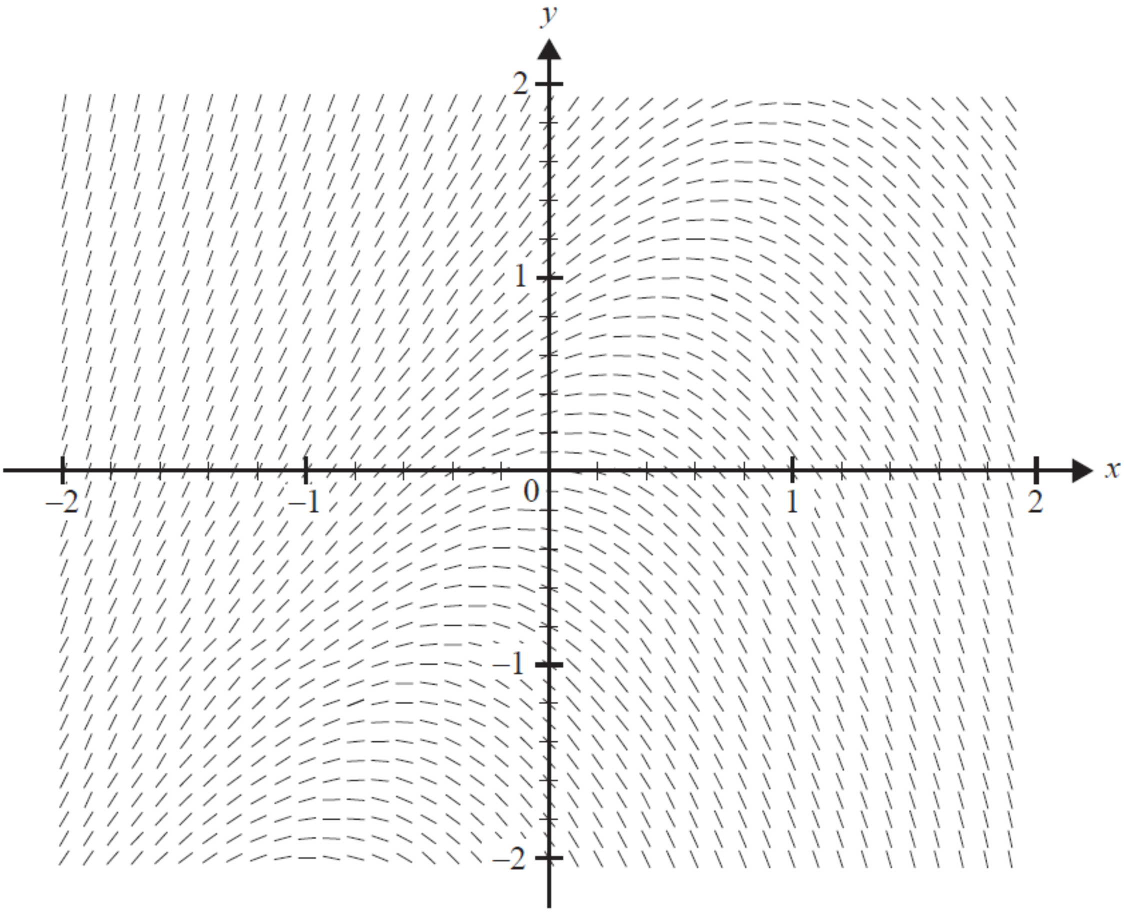
(A) $\ddot{x} = -3x$

(B) $\ddot{x} = -4x$

(C) $\ddot{x} = -16x^2$

(D) $\ddot{x} = -6\cos 2x + 8\sin 2x$

10.



The differential equation which has this slope field diagram is

(A) $\frac{dy}{dx} = y + 2x$

(B) $\frac{dy}{dx} = 2x - y$

(C) $\frac{dy}{dx} = 2y - x$

(D) $\frac{dy}{dx} = y - 2x$

END OF SECTION 1

SECTION 2 **60 marks**

Attempt Questions 11-14.

Allow about 1 hour and 45 minutes for this section

Answer each question in a new writing booklet.

Show all mathematical working / reasoning.

<u>Question 11</u> (15 marks) Start a new writing booklet

a) (i) Find the remainder when $x^3 - 2x^2 - 4x + 8$ is divided by $x - 2$. **1**

(ii) Hence or otherwise, find all the solutions to the equation **2**

$$x^3 - 2x^2 - 4x + 8 = 0$$

b) Using the substitution $u = 2x - 1$ find **3**

$$\int 2x\,(2x-1)^3 dx$$

c) A function is defined by $f(x) = 1 + \frac{3}{x-2}$ for $x > 2$

(i) Find the vertical and horizontal asymptotes for $y = f(x)$. **2**

(ii) Find the inverse function $f^{-1}(x)$. **1**

(iii) State the domain of $f^{-1}(x)$. **1**

d) Find the equation of the curve that passes through the point $\left(0, -\frac{\pi}{2}\right)$ which has a gradient function $f'(x) = -\frac{3}{\sqrt{16-9x^2}}$ **2**

e) Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all positive integers n. **3**

Question 12 (15 marks) Start a new writing booklet

a) Show that $y = Ae^{3x} + Be^{-2x}$ satisfies the differential equation $y'' - y' - 6y = 0$ for any real values of A and B. **2**

b) Find $\lim\limits_{x \to 0} \dfrac{\sin^2 4x}{4x^2}$ **1**

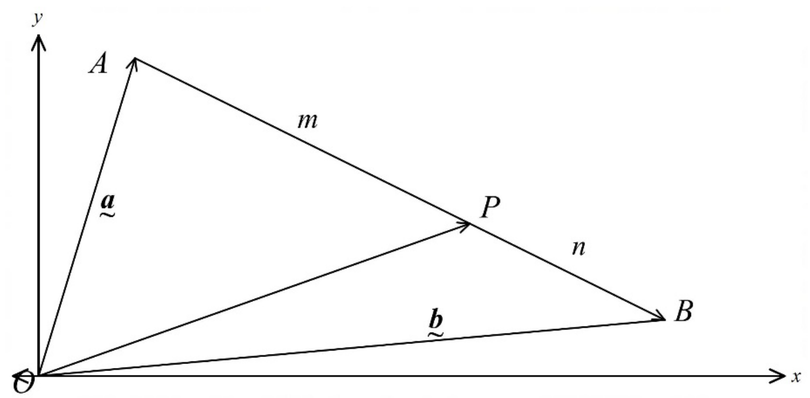
c) Find **1**

$$\int \cos x \sin^2 x \, dx$$

d) Solve the differential equation to find an equation for y in terms of x, given that $y(0) = 0$. **3**

$$\frac{dy}{dx} = e^x \cos^2 y.$$

e) The diagram below shows a vector $\overrightarrow{AB}$ and a point P which divides $\overrightarrow{AB}$ in a ratio $m : n$. **2**



If $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, prove that $\overrightarrow{OP} = \dfrac{n\underset{\sim}{a} + m\underset{\sim}{b}}{m + n}$.

f) Megan has taken a frozen quiche from the freezer with an initial temperature $T = 0°\text{C}$ and placed it directly in an oven set to 200°C. After ten minutes she checks the quiche and now it has a temperature of $T = 20°\text{C}$. The rate that the temperature of the quiche increases is proportional to the difference in temperature between the oven and the quiche.

(i) Write a differential equation to model this scenario and use it to derive the equation $T = 200 + Ae^{-kt}$. **2**

(ii) Find the value of k in exact form. **2**

(iii) The quiche is ready when the internal temperature reaches 60° C. How much longer after Megan checked the quiche will it be ready? Answer to the nearest minute. **2**

Question 13 (15 marks) Start a new writing booklet

a) Solve $\sqrt{3}\sin\theta - \cos\theta - 1 = 0$ for $[0, 2\pi]$. **3**

b) A circular metal plate is heated so that its diameter is increasing at a constant rate of 0.005 m/s. **2**

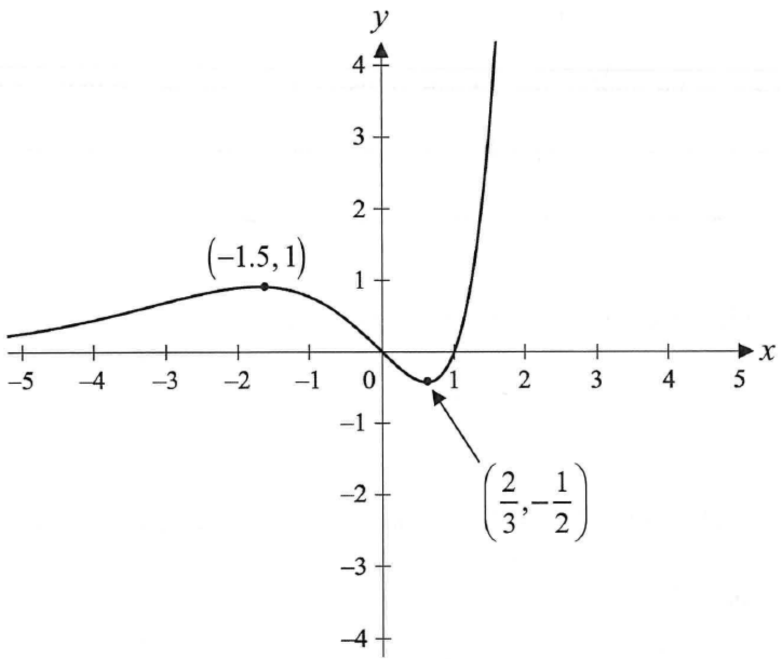
At what rate is the area of the circular surface of the plate increasing when its diameter is 6 metres? (Answer in m^2/s, correct to 2 significant figures.)

c) Consider the vectors given by $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$, where m is a constant. **3**

If the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 60°, find the exact value(s) of m.

d) Given $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$, find $proj_{\underset{\sim}{u}}\underset{\sim}{v}$ **2**

e)



Use the graph of $y = f(x)$ above to draw a separate, half page sketch of $y = \frac{1}{f(x)}$. **2**

f) Sara arranges $2n + 2$ different objects taken n at a time and Nikki arranges $2n$ different objects taken n at a time. **3**

If the number of objects from Sara and Nikki are in the ratio of 18 : 5, find the value of n.

Question 14 (15 marks) Start a new writing booklet

a) (i) Show that $\dfrac{\sec^2 x}{\tan x} = \dfrac{\operatorname{cosec} x}{\cos x}$ **1**

(ii) Use the substitution $u = \tan x$ to find the exact value of the integral **3**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\operatorname{cosec} x}{\cos x}\, dx$$

b) A region is bounded by the line $y = x + 1$ and a curve of the form $y = (x + 1)^n$, where n can be an even integer and $n \geq 2$.

This region is rotated about the x-axis to form a solid of revolution.

(i) Show the volume of the solid formed is $V_n = \pi\left(\frac{1}{3} - \frac{1}{2n+1}\right)$ **3**

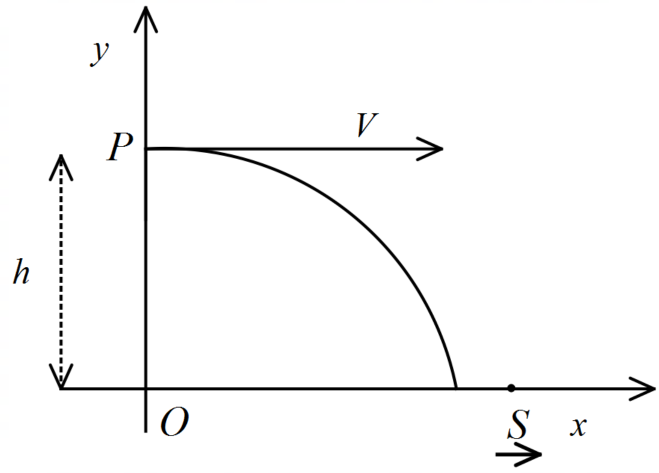
(ii) Find $\lim\limits_{n\to\infty} V_n$ **1**

(iii) Explain this result geometrically. **1**

Question 14 continues on the next page....

c) A particle is projected horizontally from a point P, h metres above O, with a velocity of V metres per second. The only acceleration that the particle undergoes is the acceleration due to gravity vertically in a downward direction. There is no horizontal acceleration.

Take $g = 10\ m/s^2$.



(i) Find the displacement vector $\underset{\sim}{s}$ of the particle at time t. **2**

(ii) A package of supplies is dropped from a plane to a disabled sailboat (S) in the ocean. The plane is travelling at a constant velocity of 252 km/h and is 150 m above sea level and directly due west of the disabled sailboat. **1**

How long will it take the package to hit the water?

(iii) A current is causing the sailboat to drift at a speed of 0.5 m/s in the same direction as the plane is travelling. Between what two values (distances) can the plane drop the package so that it lands at most 40 m from the disabled sailboat? **3**

END OF SECTION 2

END OF ASSESSMENT 😊

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Student Name:______________________________

Student Number:

SECTION 1 — Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample

$2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✖ C ○ D ○

correct

A ✖ B ✖ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word correct and drawing an arrow as follows: correct

---

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

2022 Year 12 Ext 1 HSC Trial - Task 4 - Solutions

Tuesday, 28 June 2022 8:30 AM

Multiple Choice Answer Summary

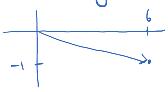
1. C. 2. B 3. D 4. D 5. C 6. D 7. B 8. A 9. B 10. D

3.20

Section I :

1. $2\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{i}-2\underset{\sim}{j}$

$= 6i - j$



$\therefore$ Distance

$d^2 = 6^2+1^2$

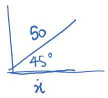
$d = \sqrt{37}$

(C)

2. $y = \tan^{-1}(\underbrace{x^4}_{f(x)})$

$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+(f(x))^2}$

$= \dfrac{4x^3}{1+(x^4)^2}$ (B)

3. 

$\dot{x} = V\cos\theta$

$= 50\cos 45°$

$= \dfrac{50}{\sqrt{2}}$ (D)

4. $x = \dfrac{t}{2}$

$t = 2x$

$y = 3t^2$

$= 3\times(2x)^2$

$y = 12x^2$ (D)

5. $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}}$

$= \left[\sin^{-1}\dfrac{x}{\sqrt{3}}\right]_1^{\sqrt{3}}$

Calc in Rad Mode

$= \sin^{-1}[1] - \sin^{-1}\left[\dfrac{1}{\sqrt{3}}\right]$

$= 0.955$ (C)

6. ${}^{15}C_3 \times {}^{10}C_2$

$= 20\,475$ (D)

7. $\sin 2x = 2\sin x\cos x$

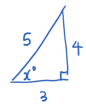
x in 2nd Quad

$\therefore \cos x = -\dfrac{3}{5}$

$\sin 2x = 2\times\dfrac{4}{5}\times -\dfrac{3}{5}$

$= \dfrac{-24}{25}$ (B)

$\sin x = 0.8 = \dfrac{4}{5}$



$\therefore \cos x = \dfrac{3}{5}$

8. $\dfrac{y}{6} = \cos^{-1}(3x)$

$0 \le \dfrac{y}{6} \le \pi$ $\quad -1 \le 3x \le 1$

$0 \le y \le 6\pi$ $\quad -\dfrac{1}{3} \le x \le \dfrac{1}{3}$

(A)

9. $x = 3\cos 2t + 4\sin 2t$

$\dot{x} = -6\sin 2t + 8\cos 2t$

$\ddot{x} = -12\cos 2t - 16\sin 2t$

$= -4(3\cos 2t + 4\sin 2t)$

$= -4x$ (B)

10. In 4th Quad $x>0$, $y<0$, $\dfrac{dy}{dx}<0$

this eliminates A + B.

Note: @ $(-1,-2)$ $\dfrac{dy}{dx}=0$ and

@ $(1,2)$ $\dfrac{dy}{dx}=0$.

$\therefore$ (D) is correct answer.

SECTION 2

QUESTION 11

a) i) $P(x) = x^3 - 2x^2 - 4x + 8$

$P(2) = 2^3 - 2\times2^2 - 4\times2 + 8$

$= 0$

$\therefore$

$$\begin{array}{r|l} & x^2 \qquad\quad -4 \\ \hline x-2 & x^3 - 2x^2 - 4x + 8 \\ & x^3 - 2x^2 \quad \downarrow \quad \downarrow \end{array}$$

$\therefore P(x) = (x-2)(x^2-4)$

$= (x-2)(x-2)(x+4)$

$\therefore$ solutions to $P(x) = 0$

are $x = 2$ or $x = -2$

$\therefore \quad x-2 \,\big)\, x^3 - 2x^2 - 4x + 8$

$\quad\quad\quad\;\; x^3 - 2x^2 \quad \downarrow \quad \downarrow$

$\quad\quad\quad\quad\quad 0 \quad -4x + 8$

$\quad\quad\quad\quad\quad\quad\; -4x + 8$

$\quad\quad\quad\quad\quad\quad\quad\quad 0$

are $\underline{x = 2 \text{ or } x = -2}$

b) $\frac{1}{2}\int \underbrace{2x}_{u+1} \underbrace{(2x-1)^3}_{u} \underbrace{2dx}_{du}$

Let $u = 2x - 1 \quad \therefore 2x = u + 1$

$\frac{du}{dx} = 2$

$du = 2\,dx.$

$= \frac{1}{2}\int (u+1)u^3 \, du$

$= \frac{1}{2}\int (u^4 + u^3)\, du$

$= \frac{1}{2}\left[\frac{u^5}{5} + \frac{u^4}{4}\right]$

$= \frac{1}{2}\left[\frac{(2x-1)^5}{5} + \frac{(2x-1)^4}{4}\right] + c$

$= \frac{(2x-1)^5}{10} + \frac{(2x-1)^4}{8} + c$

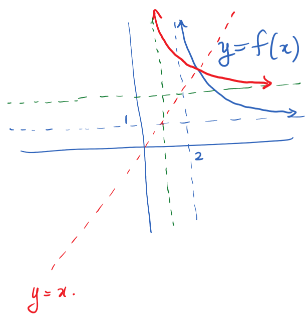
c) $f(x) = 1 + \frac{3}{x-2} \qquad x > 2.$

i) vertical asympt : $\underline{x = 2}$

horiz. asymptote ; $x \to \infty$,

$\frac{3}{x-2} \to 0 \quad \therefore \underline{y = 1}$

ii) $f: \quad y = 1 + \frac{3}{x-2}$



$f^{-1}: \quad x = 1 + \frac{3}{y-2} \qquad y > 2$

$x - 1 = \frac{3}{y-2}$

$y - 2 = \frac{3}{x-1}$

$y = 2 + \frac{3}{x-1}$

iii) $x > 1$

d) $\frac{dy}{dx} = -\frac{3}{\sqrt{16 - 9x^2}}$

$y = \int \frac{-3}{\sqrt{9\left(\frac{16}{9} - x^2\right)}}\, dx$

(*) $y = \int \frac{-\cancel{3}}{\cancel{3}\sqrt{\left(\frac{4}{3}\right)^2 - x^2}}\, dx$

$y = \left[\cos^{-1}\frac{x}{4/3}\right] + c$

$y = \left[\cos^{-1}\frac{3x}{4}\right] + c$

when $x = 0, \; y = -\frac{\pi}{2}$

$-\frac{\pi}{2} = \underbrace{\cos^{-1} 0}_{\frac{\pi}{2}} + c$

$c = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi$

So $\underline{y = \cos^{-1}\left(\frac{3x}{4}\right) - \pi}$

alternatively : from (*)

$y = -\int \frac{1}{\ldots}\, dx \qquad \therefore c = -\frac{\pi}{2}$

alternatively: from ⊛

$$y = -\int \frac{1}{\sqrt{\left(\frac{4}{3}\right)^2 - x^2}}\,dx$$

$$y = -\sin^{-1}\left[\frac{x}{4/3}\right] + c$$

when $x = 0$, $y = -\frac{\pi}{2}$

$$-\frac{\pi}{2} = -\sin^{-1} 0 + c$$

$\therefore c = -\frac{\pi}{2}$

$\therefore y = -\sin^{-1}\left(\frac{3x}{4}\right) - \frac{\pi}{2}$

another way to set this out...

$$-\int \frac{3\,dx}{\sqrt{4^2 - (3x)^2}} \qquad f(x) = 3x,\ a = 4$$

$$y = -\sin^{-1}\left(\frac{3x}{4}\right) + c$$

when $x = 0$, $y = -\frac{\pi}{2}$

$$-\frac{\pi}{2} = -\sin^{-1} 0 + c$$

$\therefore c = -\frac{\pi}{2}$

So $f(x) = -\sin^{-1}\left(\frac{3x}{4}\right) - \frac{\pi}{2}$

e) If $n^3 + 2n$ is divisible by 3.

Show true for $n = 1$: $\quad 1^3 + 2 \times 1 = 3$ which is divisible by 3.

Assume true for $n = k$: $\quad k^3 + 2k$ is divisible by 3

ie $k^3 + 2k = 3M$ [where M is an integer]

Show true for $n = k+1$: $\quad (k+1)^3 + 2(k+1)$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= \underbrace{k^3 + 2k} + (3k^2 + 3k + 3)$$

$$= 3M + 3(k^2 + k + 1)$$

$$= 3[M + k^2 + k + 1] = 3N$$

as k is integer, M is integer, then

$N = M + k^2 + k + 1$ is integer

$\therefore (k+1)^3 + (k+1)$ is divisible by 3.

$\therefore$ Since true for $n = 1$, and if true for $n = k$ it is true for $n = k+1$.

QUESTION 12

a) $y = Ae^{3x} + Be^{-2x}$

$y' = 3Ae^{3x} - 2Be^{-2x}$

$y'' = 9Ae^{3x} + 4Be^{-2x}$

$y'' - y' - 6y$

$$= 9Ae^{3x} + 4Be^{-2x} - \left(3Ae^{3x} - 2Be^{-2x}\right) - 6\left(Ae^{3x} + Be^{-2x}\right)$$

$$= \cancel{9Ae^{3x}} + \cancel{4Be^{-2x}} - \cancel{3Ae^{3x}} + \cancel{2Be^{-2x}} - \cancel{6Ae^{3x}} - \cancel{6Be^{-2x}}$$

$= 0$ as required.

b) $$\lim_{x \to 0} \frac{\sin^2 4x}{4x^2}$$

b) $\lim_{x\to 0} \frac{\sin^2 4x}{4x^2}$

$= \lim_{x\to 0} \frac{\sin 4x}{4x} \times \frac{\sin 4x}{4x} \times 4$

$= 4$

c) $\int \cos x \cdot \sin^2 x \, dx$

$= \frac{\sin^3 x}{3} + c$

$\int f'(x) \cdot [f(x)]^n \, dx = \frac{f(x)^{n+1}}{n+1} + c.$

d) $\frac{dy}{dx} = e^x \cdot \cos^2 y.$

$\frac{dy}{\cos^2 y} = e^x \, dx$

$\int \sec^2 y \, dy = \int e^x \, dx$

$\tan y = e^x + c$

$y = \tan^{-1}(e^x + c)$

but $(0,0)$ satisfies

$0 = \tan^{-1}(1 + c)$

$\therefore 1 + c = 0$

$c = -1$

$\therefore y = \tan^{-1}(e^x - 1)$

e) $\overrightarrow{AB} = -\underset{\sim}{a} + \underset{\sim}{b}$

$= \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{AP} = \frac{m}{m+n}(\overrightarrow{AB})$

$= \frac{m(\underset{\sim}{b} - \underset{\sim}{a})}{m+n}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \underset{\sim}{a} + \frac{m(\underset{\sim}{b} - \underset{\sim}{a})}{m+n}$

$= \frac{\underset{\sim}{a}(m+n) + m\underset{\sim}{b} - m\underset{\sim}{a}}{m+n}$

$= \frac{\cancel{\underset{\sim}{a}m} + \underset{\sim}{a}n + \underset{\sim}{b}m - \cancel{m\underset{\sim}{a}}}{m+n}$

$= \frac{n\underset{\sim}{a} + m\underset{\sim}{b}}{m+n}$ as required.

f) $\frac{dT}{dt} = k(200 - T)$

$\frac{dt}{dT} = \frac{1}{k(200 - T)}$

$t = -\int \frac{-1}{k(200-T)} \, dT$

$t = \frac{-1}{k}\left[\ln|200 - T|\right] + c$

$t - c = -\frac{1}{k} \ln|200 - T|$

$-k(t - c) = \ln|200 - T|$

$e^{-kt + kc} = |200 - T|$

ii) $T = 200 + Ae^{-kt}$

when $t = 0$, $T = 0$

$0 = 200 + Ae^0$

$\therefore A = -200$

$T = 200 - 200e^{-kt}$

when $t = 10$, $T = 20$

$-k(t-c) = \ln|200-T|$

$e^{-kt+kc} = |200-T|$

$e^{-kt} \cdot e^{kc} = |200-T|$

$\therefore 200-T = \pm e^{kc} \cdot e^{-kt}$

$T = 200 \mp e^{kc} \cdot e^{-kt}$

Let $A = \pm e^{kc}$

$\therefore T = 200 + Ae^{-kt}$

when $t=10$, $T=20$

$20 = 200 - 200e^{-10k}$

$-180 = -200e^{-10k}$

$\frac{180}{200} = e^{-10k}$

$0.9 = e^{-10k}$

$-10k = \ln 0.9$

$k = \frac{\ln 0.9}{-10}$

$= -\frac{1}{10}\ln\left(\frac{9}{10}\right)$

or $\frac{1}{10}\ln\left(\frac{10}{9}\right)$

iii) find t when $T=60$

$60 = 200 - 200e^{-k \times t}$

$-140 = -200e^{-kt}$

$\frac{7}{10} = e^{-kt}$

$\ln\left(\frac{7}{10}\right) = -kt$

$t = -\frac{1}{k}\left[\ln\frac{7}{10}\right]$

$t = 33.85$ mins

time is

$33.85 - 10 = 23.85$

$\doteqdot 24$ minutes

after she first checked

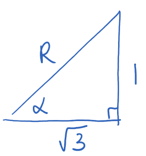
QUESTION 13

a) $\sqrt{3}\sin\theta - \cos\theta - 1 = 0 \qquad 0 \le \theta \le 2\pi$

Let $\sqrt{3}\sin\theta - \cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$R\cos\alpha = \sqrt{3} \qquad R\sin\alpha = 1$



$R = 2 \qquad \tan\alpha = \frac{1}{\sqrt{3}}$

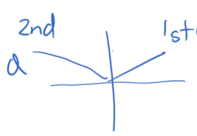
$\alpha = \frac{\pi}{6}$

$\therefore \sqrt{3}\sin\theta - \cos\theta = 1$

becomes $2\sin\left(\theta - \frac{\pi}{6}\right) = 1$

$\sin\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore \left(\theta - \frac{\pi}{6}\right) = \frac{\pi}{6}, \frac{5\pi}{6}$



$\therefore \theta = \frac{2\pi}{6}, \frac{6\pi}{6}$

$\theta = \frac{\pi}{3}, \pi$

Alternatively: use t subst.

$\sqrt{3}\cdot\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} - 1 = 0$

$\sqrt{3}\cdot 2t - 1 + \cancel{t^2} - 1 - \cancel{t^2} = 0$

$2\sqrt{3}\cdot t - 2 = 0$

$t = \frac{2}{2\sqrt{3}}$

$t = \frac{1}{\sqrt{3}}$

$\tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$ for $0 \le \theta \le 2\pi$, $0 \le \frac{\theta}{2} \le \pi$

$\frac{\theta}{2} = \frac{\pi}{6}$

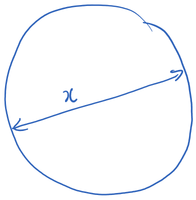
$\theta = \frac{2\pi}{6} = \frac{\pi}{3}$

also check $\theta = \pi$ is a solution when using t method

$\sqrt{3}\sin\pi - \cos\pi - 1 = 0$ ✓

$\therefore \theta = \frac{\pi}{3}$ or π

b) Let diameter be x: $\frac{dx}{dt} = 0.005$ m/s



$$A = \pi r^2$$
$$= \pi \times \left(\frac{x}{2}\right)^2$$
$$= \frac{1}{4}\pi x^2$$

$$\frac{dA}{dx} = \frac{1}{4}\pi \cdot 2x$$
$$= \frac{x\pi}{2}$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$
$$= \frac{x\pi}{2} \times 0.005$$

when $x = 6$

$$\frac{dA}{dt} = \frac{6}{2}\pi \times 0.005$$
$$= 0.04712\ldots$$

$\therefore \frac{dA}{dt} = 0.047$ m^2/s (2 sig figs).

Alternatively, work using the radius

$$\frac{dr}{dt} = 0.005 \div 2 = 0.0025$$

(you can only $\div 2$ as diameter/radius are linear measurements)

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$= 2\pi r \times 0.0025$$

when $r = 6 \div 2 = 3$

$\frac{dA}{dt} = 0.047$ m^2/s

c) $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ $\quad \underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$

$|\underset{\sim}{a}| = \sqrt{m^2+1}$

$|\underset{\sim}{b}| = \sqrt{1+m^2}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} m \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ m \end{bmatrix}$$
$$= m \cdot 1 + 1 \cdot m$$
$$= 2m.$$

$$\therefore 2m = (1+m^2)\cos 60°$$
$$2m = (1+m^2) \cdot \frac{1}{2}$$
$$4m = 1+m^2$$
$$m^2 - 4m + 1 = 0$$

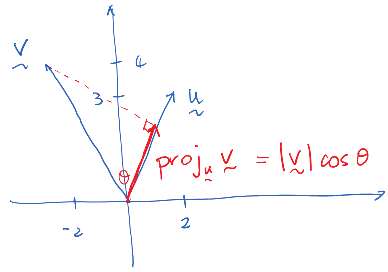
$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.1}}{2}$$
$$= \frac{4 \pm \sqrt{12}}{2}$$
$$= \frac{4 \pm 2\sqrt{3}}{2}$$
$$m = 2 \pm \sqrt{3}$$

$\therefore$ exact values are

$m = 2+\sqrt{3}$

$m = 2-\sqrt{3}$.

d) $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ $\quad \underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$



$$\text{proj}_{\underset{\sim}{u}}\underset{\sim}{v} = \text{scalar proj} \times \hat{u}$$

$$= \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}|} \cdot \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$$

$$= \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}|^2} \cdot \underset{\sim}{u}$$

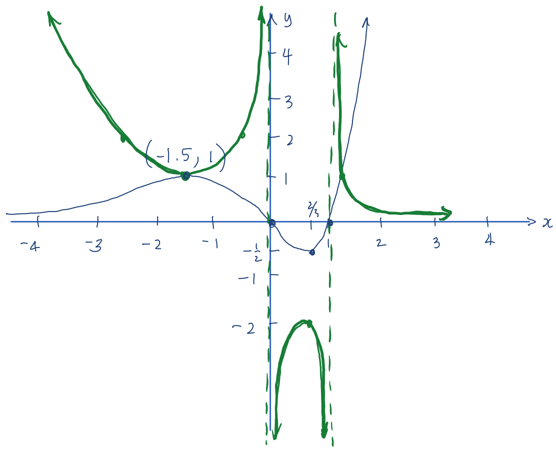
$$= \frac{2\times -2 + 3\times 4}{\left(\sqrt{2^2+3^2}\right)^2} \cdot \left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$

$$= \frac{8}{13}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$

since $\underset{\sim}{u}\cdot\underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$

$$|\underset{\sim}{v}|\cos\theta = \underbrace{\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}|}}_{\text{scalar projection.}}$$

e)



$$y = \frac{1}{f(x)}.$$

f) Arrangements have order ∴ permutations ....

Sarah $\quad {}^{2n+2}P_n = \frac{(2n+2)!}{(2n+2-n)!} = \frac{(2n+2)!}{(n+2)!}$

Nikki $\quad {}^{2n}P_n = \frac{(2n)!}{(2n-n)!} = \frac{(2n)!}{n!}$

$$\therefore \frac{\text{Sarah}}{\text{Nikki}} = \frac{18}{5}$$

$$\frac{(2n+2)!}{(n+2)!} \div \frac{(2n)!}{n!} = \frac{18}{5}$$

$$\frac{(2n+2)!}{(n+2)!} \times \frac{n!}{(2n)!} = \frac{18}{5}$$

$$\frac{(2n+2)(2n+1)}{(n+2)(n+1)} = \frac{18}{5} \xrightarrow{\text{alternatively take out c.f of 2}} \frac{2\cancel{(n+1)}(2n+1)}{(n+2)\cancel{(n+1)}} = \frac{18}{5}$$

$$5(2n+2)(2n+1) = 18(n+2)(n+1)$$

$$20n^2 + 30n + 10 = 18n^2 + 54n + 36$$

$$10(2n+1) = 18(n+2)$$

$$20n + 10 = 18n + 36$$

$$20n^2 + 30n + 10 = 18n^2 + 54n + 36$$
$$2n^2 - 24n - 26 = 0$$
$$n^2 - 12n - 13 = 0$$
$$(n-13)(n+1) = 0$$
$$\therefore n = 13 \text{ or } n = -1 \quad (\text{but } n > 0)$$
$$\therefore \underline{n = 13}$$

$$20n + 10 = 18n + 36$$
$$2n = 26$$
$$n = 13$$

QUESTION 14:

a) i) $$\frac{\sec^2 x}{\tan x} = \frac{1}{\cos^2 x} \div \frac{\sin x}{\cos x}$$
$$= \frac{1}{\cos^2 x} \times \frac{\cos x}{\sin x}$$
$$= \frac{1}{\sin x} \times \frac{1}{\cos x}$$
$$= \frac{\operatorname{cosec} x}{\cos x} \quad \text{as required.}$$

ii) $u = \tan x \quad \therefore du = \sec^2 x \, dx$

when $x = \frac{\pi}{6}$, $u = \tan\frac{\pi}{6}$

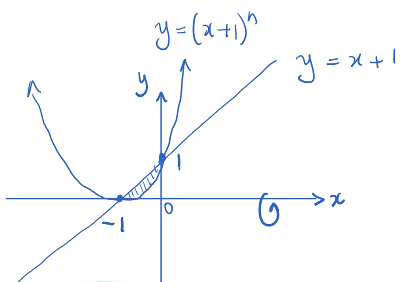
$u = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{3}$, $u = \tan\frac{\pi}{3}$

$u = \sqrt{3}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\operatorname{cosec} x}{\cos x}\, dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\, dx$$
$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{du}{u}$$
$$= \Big[\ln|u|\Big]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln\sqrt{3}^{-1}$$
$$= \ln\sqrt{3} + \ln\sqrt{3}$$
$$= 2\ln\sqrt{3} = \ln\sqrt{3}^{2} = \ln 3.$$

b) (i)



$$\boxed{V = \pi\int_a^b y^2\, dx}$$

NOTE: $y = (x+1)^n$ is below $y = (x+1)$ for $n \geq 2$

$$V = \pi\int_{-1}^{0} \left((x+1)^2 - (x+1)^{2n}\right) dx$$
$$= \pi\left[\frac{(x+1)^3}{3} - \frac{(x+1)^{2n+1}}{2n+1}\right]_{-1}^{0}$$
$$= \pi\left[\left(\frac{1}{3} - \frac{1^{2n+1}}{2n+1}\right) - (0-0)\right]$$
$$= \pi\left[\frac{1}{3} - \frac{1}{2n+1}\right] \quad \text{as required.}$$

$y=(x+1)$ is below

$y=(x+1)^n$ for $n \geq 2$

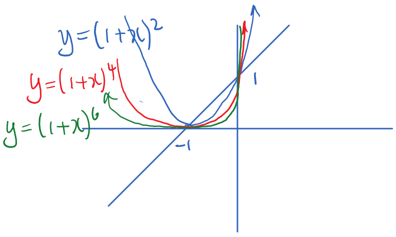
where $-1 \leq x \leq 0$

$= \pi\left[\frac{1}{3} - \frac{1}{2n+1}\right]$ as required.

(ii) $\lim_{n\to\infty} V_n = \lim_{n\to\infty} \pi\left[\frac{1}{3} - \frac{1}{2n+1}\right]$ $\to 0$

$= \frac{\pi}{3}$ units^3.

iii) As $n \to \infty$ the graph of $y=(1+x)^n$ sits closer to the axes (n even) $\therefore$ Area between $y=(1+x)^n$ & the axes approaches 0 as $n \to \infty$



$\therefore$ Volume generated is just the cone formed by rotating $y = x+1$ $(r=1, h=1)$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 1^2 \times 1 = \frac{1}{3}\pi$ units^3.

c)

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t=0$, $\dot{y}=0 \therefore c=0$

$\dot{y} = -10t$

$y = -5t^2 + c_1$

$t=0$, $y=h$ $\therefore c_1 = h$

$y = -5t^2 + h$

$\ddot{x} = 0$

$\dot{x} = k = V$

$x = Vt + k_1$

$t=0, x=0 \therefore k_1 = 0$

$x = Vt$

i) $\therefore \tilde{S} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$

$= Vt\,\underset{\sim}{i} + (-5t^2+h)\,\underset{\sim}{j}$

ii) $V = 252$ $h = 150$

$\begin{cases} x = 252t \\ y = -5t^2 + 150 \end{cases}$

hits water when $y=0$

$-5t^2 + 150 = 0$

$t^2 = 30$

$t = \sqrt{30}$ seconds

iii) 252 km/h

$= \frac{252\,000 \text{ m}}{3600 \text{ sec}}$

when $t = \sqrt{30}$ $x = 70t$

$= 70\sqrt{30}$ m

$= \dfrac{252\,000 \text{ m}}{3600 \text{ sec}}$

$= \underline{70 \text{ m/s}}$

When $t = \sqrt{30}$ $x = 70t$

$= 70\sqrt{30}$ m

$\doteqdot \underline{383.4 \text{ m}}$

In that time it drifts $0.5 \times \sqrt{30}$

$= \underline{2.7 \text{ m}}$

$\therefore$ $383.4 - 40 < (S + 2.7) < 383.4 + 40$

$340.7 < S < 420.7$

So plane must drop package between 340.7 m & 420.7 m

12.28