Girraween High School

2025

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time: 10 minutes
- Working time: 2 Hours
- Write using a black pen
- NESA approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple-choice questions by completely colouring in the appropriate circle on your multiple-choice answer sheet.
- Answer questions in the writing booklet provided and show all relevant mathematical reasoning and/or calculations.

Total Marks: 70

Section 1 (Pages 1 – 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 6-11) **60 marks**

- Attempt Q11 - Q16
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10

Question 1

When the polynomial $P(x)=x^4+ax+2$ is divided by x^2+1, the remainder is $2x+3$. What is the value of a?

A) 1

B) 2

C) 0

D) 3

Question 2

A family of 6 adults and 4 children are randomly sitting around a circular table. What is the number of possible seating arrangements if none of the children sit together ?

A) 14 400

B) 21 600

C) 43 200

D) 86 200

Examination continues next page...

Question 3

What is the value of ${}^{4n}C_1 + {}^{4n}C_2 + \ldots + {}^{4n}C_{4n-1} + {}^{4n}C_{4n}$?

A) 2^{2n}

B) $2^{2n} - 1$

C) $16^n - 1$

D) 16^n

Question 4

The vector $\underset{\sim}{u}$ has a magnitude of 6 units and makes an angle of $135°$ with the horizontal. Which of the following represents the vector $\underset{\sim}{u}$?

A) $3\sqrt{2}\underset{\sim}{i} + 3\sqrt{2}\underset{\sim}{j}$

B) $-3\sqrt{2}\underset{\sim}{i} + 3\sqrt{2}\underset{\sim}{j}$

C) $2\sqrt{3}\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j}$

D) $-2\sqrt{3}\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j}$

Examination continues next page...

Question 5

A curve in the Cartesian plane is defined parametrically by the equations $x = t^3 + t$ and $y = t^4 + 2t^2$.

The gradient of the tangent to the curve at $t = 1$ is

A) 2

B) $\frac{1}{2}$

C) 3

D) $\frac{1}{3}$

Question 6

What is the value of $\lim_{x \to 0} \frac{\sin\left(\frac{1}{2}x\right)}{2x}$?

A) 0

B) $\frac{1}{4}$

C) 1

D) 4

Question 7

Which of the following is an expression for $\int \frac{3}{\sqrt{1-16x^2}}\,dx$?

A) $3\sin^{-1}(4x) + C$

B) $3\cos^{-1}(4x) + C$

C) $\frac{3}{4}\sin^{-1}(4x) + C$

D) $\frac{3}{4}\cos^{-1}(4x) + C$

Examination continues next page…

Question 8

If $y = \sin(\ln x)$, which is equal to $\dfrac{d^2y}{dx^2}$?

A) $\dfrac{\sin(\ln x) + \cos(\ln x)}{x^2}$

B) $\dfrac{\sin(\ln x) - \cos(\ln x)}{x^2}$

C) $\dfrac{\cos(\ln x) - \sin(\ln x)}{x^2}$

D) $\dfrac{-\sin(\ln x) - \cos(\ln x)}{x^2}$

Question 9

The function $f(x) = \log_e x^3$ has an inverse function $g(x)$. What is the equation of the normal to $y = g(x)$ at the point (0, 1)?

A) $y = \dfrac{1}{3}x - 1$

B) $y = \dfrac{1}{3}x + 1$

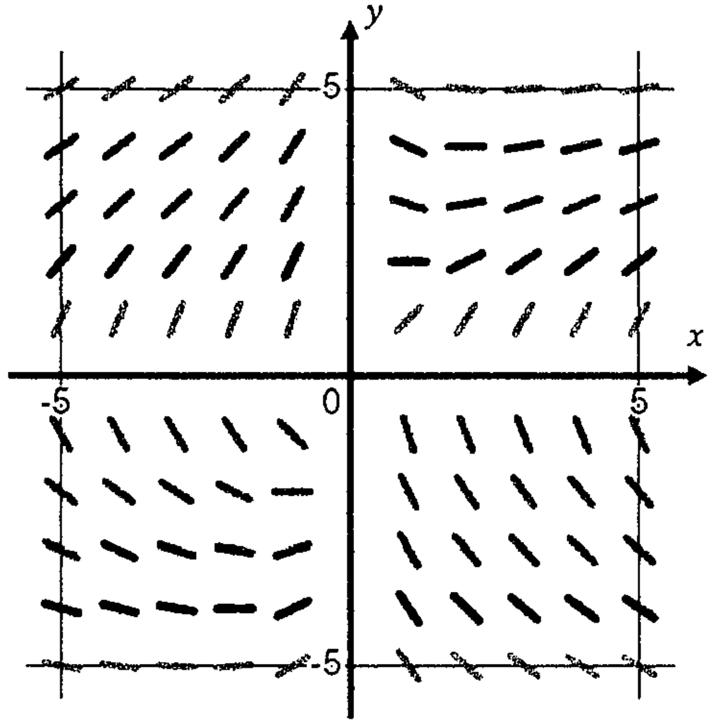
C) $y = -3x - 1$

D) $y = -3x + 1$

Examination continues next page…

Question 10

The differential equation which best represents the direction field given below is



A) $\dfrac{dy}{dx} = \dfrac{2x + y}{xy}$

B) $\dfrac{dy}{dx} = \dfrac{2x - y}{xy}$

C) $\dfrac{dy}{dx} = \dfrac{2x - y}{x + y}$

D) $\dfrac{dy}{dx} = \dfrac{2x - y}{x - y}$

Examination continues next page…

Section II

60 marks
Attempt Questions 11 - 16
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (10 marks) Start a new writing page

a) Solve $\dfrac{4+x}{2x} \geq 1$ 2

b) Find the coefficient of x^8 in the expansion of $\left(2x^3 - \dfrac{1}{x^2}\right)^{11}$ 2

c) Consider the polynomial $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$. It is known that $P(x)$ has a triple root at $x = \alpha$.

i) By using differentiation, find the value of α. 2

ii) Hence express $P(x)$ as the product of linear factors. 1

d) Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$, using the substitution $x = u^2 - 1$ 3

Examination continues next page…

Question 12 (10 marks) Start a new writing page

a) **i)** Express $\sqrt{12}\sin x + 2\cos x$ in the form $R\sin(x-\alpha)$ where $R > 0$ and $0 \le \alpha < 2\pi$ 3

ii) Hence, or otherwise, solve $\sqrt{12}\sin x + 2\cos x = 2$ for $0 \le x \le 2\pi$ 2

b) A freshly baked cake is cooling in a room of constant temperature of $20°C$. At time t minutes, its temperature T decreases according to the equation $\dfrac{dT}{dt} = -k(T-20)$ where k is a positive constant.

The initial temperature of the cake is $150°C$ and it cools to $100°C$ after 15 minutes.

i) Show that $T = 20 + Ae^{-kt}$ where A is a constant is a solution to this equation. 1

ii) Find the values A and k, giving k correct to 3 decimal places. 2

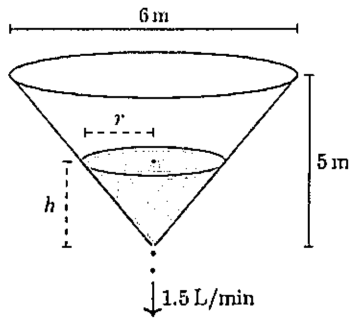
iii) How long will it take for the cake to cool to $25°C$? 2

(Use the value of k obtained in the previous part)

Examination continues next page…

Question 13 (10 marks) Start a new writing page

a) A tank in the shape of a cone is filled with water, which is leaking out from the bottom at a rate of 1.5 litres per minute. 3

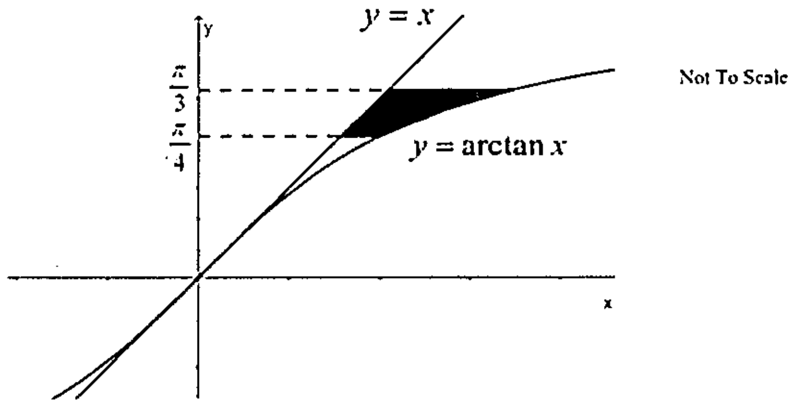


The volume of water in the tank is given by $V = \frac{1}{3}\pi r^2 h$, where h is the height in metres and r is the radius in metres.

Find the rate at which the water level is falling when the height of the water is 1 metre.

$\left[\text{Note: } 1\ m^3 = 1000L\right]$

b) Find the volume formed when the region between the curve $y = \tan^{-1} x$ and the lines $y = x$, $y = \frac{\pi}{3}$ and $y = \frac{\pi}{4}$, is rotated around the y – axis, correct to 2 decimal places. 3



c) i) Use mathematical induction to prove 3

$$4\left(1^3 + 2^3 + 3^3 + \ldots + n^3\right) = n^2\left(n+1\right)^2 \text{ for all positive integers } n.$$

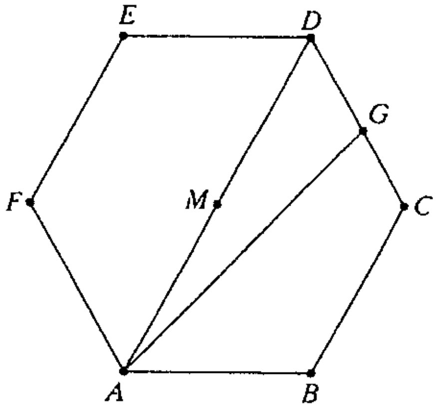
ii) Hence, find the value of $\lim\limits_{n \to \infty} \left(\frac{1^3 + 2^3 + 3^3 + \ldots + n^3}{n^4}\right)$. 1

Examination continues next page...

Question 14 (10 marks) Start a new writing page

a) Consider the vectors $\underset{\sim}{p} = \begin{pmatrix} t-8 \\ 6 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 3 \\ 2t \end{pmatrix}$. What are the possible values of t so that $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are parallel? 3

b) $ABCDEF$ is a regular hexagon. Let $\overrightarrow{AB} = \underset{\sim}{u}$, $\overrightarrow{BC} = \underset{\sim}{v}$ and $\overrightarrow{CD} = \underset{\sim}{w}$. G is the midpoint of CD. M is the midpoint of AD.

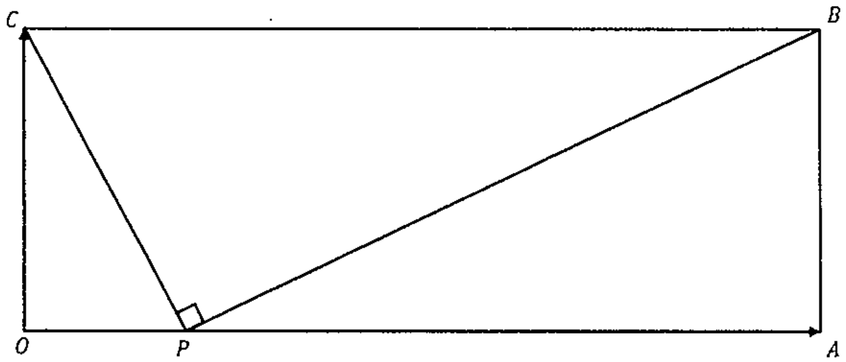


Find:

i) $\overrightarrow{AG}$ in terms of $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$. 1

ii) Hence, find $\overrightarrow{MG}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. 3

c) In the diagram, $OABC$ is a rectangle in which $\overrightarrow{OA} = \frac{5}{2}\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{j}$. P is a point on OA such that $\overrightarrow{OP} = \lambda \underset{\sim}{i}$ for some scalar parameter λ. Use vector methods, and the fact that $\angle CPB = 90°$, to show that $2\lambda^2 - 5\lambda + 2 = 0$, and hence find any values of λ. 3



Examination continues next page...

Question 15 (10 marks) Start a new writing page

a) Given that $y = e^{-7x}$ satisfies the second order differential equation $y + ky'' = 0$, find the value of k. 2

b) Consider the differential equation $\dfrac{dy}{dx} = x \sec y$. Find the solution that has a y – intercept of $\dfrac{\pi}{6}$, expressing your answer in the form $y = f(x)$. 3

c) A rabbit population grows according to the logistic equation $\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{20\,000}\right)$, where P is the number of rabbits after t months. The initial population is 1000.

i) Show that $\dfrac{20\,000}{P(20\,000 - P)} = \dfrac{1}{P} + \dfrac{1}{20\,000 - P}$ 1

ii) Solve $\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{20\,000}\right)$ and hence show that $P = \dfrac{20000e^{0.1t}}{19 + e^{0.1t}}$ 3

iii) Find the population of rabbits after seven months. 1

Examination continues next page…

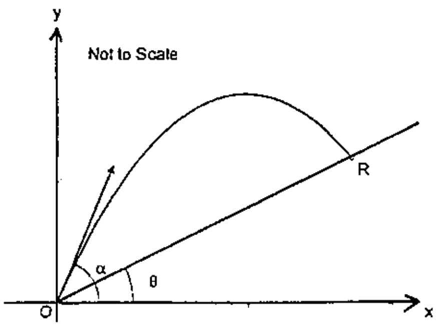
Question 16 (10 marks) Start a new writing page

a) Consider the function $f(x) = 2\sin^{-1}\left(\frac{x}{3}\right) + \frac{\pi}{2}$. Find the domain and range of $f(x)$. 2

b) **i)** Show that $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$ 2

ii) Hence, solve $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = 1$ where $0 \le \theta \le 2\pi$ 1

c) A stone is projected from O with velocity V at an angle α above the horizontal.
A straight road goes through O at an angle θ above the horizontal, where $\theta < \alpha$.
The stone strikes the road at R. Air resistance is to be ignored and the acceleration due to gravity is $g\ m/s^2$.



i) Given that the equations of motion of the stone are $x = vt\cos\alpha, y = vt\sin\alpha - \frac{gt^2}{2}$, 1

show that the cartesian equation for the motion is $y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{2v^2}$.

ii) If R is the point (x, y) express x and y in terms of RO and θ. 2

Hence show that the range RO of the stone up the road is given by

$$RO = \frac{2v^2\cos\alpha\sin(\alpha - \theta)}{g\cos^2\theta}$$

iii) Find an expression for α when RO is a maximum. 2

End of Examination.

12 Extension 1 Trial HSC 2025 – Solutions

1.

$$
\begin{array}{r}
x^2 - 1 \\
x^2+1 \overline{\big) \, x^4 + ax + 2} \\
\underline{x^4 + x^2 } \\
-x^2 + ax + 2 \\
\underline{-x^2 - 1} \\
ax + 3
\end{array}
$$

B

$a = 2$

2. Arrange 6 adults in a circle. This can be done in $5!$ ways. There are six places for the children. They can be arranged in $6 \times 5 \times 4 \times 3$ ways.

Total number of arrangements

$= 5! \times 6 \times 5 \times 4 \times 3$

$= 43200$

C

3. Sum of the coefficients in the n^{th} row of Pascal's triangle is

$${}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n$$

$${}^{4n}C_1 + {}^{4n}C_2 + \dots + {}^{4n}C_{4n}$$

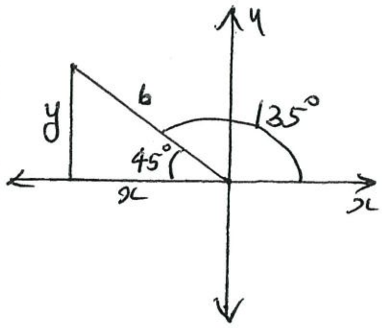
$= 2^{4n} - 1$

$= (2^4)^n - 1$

C

$= 16^n - 1$

4



$\sin 45 = \frac{y}{6}$

$y = 6\sin 45 = \frac{6}{\sqrt{2}}$

$= \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

$\cos 45 = \frac{x}{6}$

$x = 6\cos 45$

$= \frac{6}{\sqrt{2}} = 3\sqrt{2}$

B

$x + iy = -3\sqrt{2}\,\underset{\sim}{i} + 3\sqrt{2}\,\underset{\sim}{j}$

5\. $x = t^3 + t$ $\quad\quad y = t^4 + 2t^2$

$\frac{dx}{dt} = 3t^2 + 1$ $\quad\quad \frac{dy}{dt} = 4t^3 + 4t$

$\frac{dy}{dx} = \frac{4t^3 + 4t}{3t^2 + 1}$

when $t = 1$, $\frac{dy}{dx} = \frac{4+4}{3+1} = 2$

A

6 $\lim_{x \to 0} \dfrac{\sin \frac{1}{2}x}{2x}$

$= \lim_{x \to 0} \dfrac{\sin \frac{1}{2}x}{\frac{1}{2}x} \times \dfrac{\frac{1}{2}x}{2x}$

$= \dfrac{1}{4} \lim_{x \to 0} \dfrac{\sin \frac{1}{2}x}{\frac{1}{2}x}$

B

$= \dfrac{1}{4} \times 1 = \dfrac{1}{4}$

7. $\int \dfrac{3}{\sqrt{1-16x^2}} = 3 \int \dfrac{dx}{16(\frac{1}{16} - x^2)}$

C

$= \dfrac{3}{4} \sin^{-1}(4x) + C$

8 $y = \sin(\ln x)$

$y' = \cos(\ln x) \times \frac{1}{x} = \frac{\cos(\ln x)}{x}$

$y'' = \frac{x \times -\sin(\ln x) \times \frac{1}{x} - \cos(\ln x) \times 1}{x^2}$

$= \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}$ (D)

9 $f(x) = \log_e x^3$

$y = \log_e x^3$

$x = \log_e y^3$

$e^x = y^3$

$y = (e^x)^{\frac{1}{3}}$

$y' = e^{\frac{x}{3}} \times \frac{1}{3}$

$= \frac{1}{3} e^{\frac{x}{3}}$

when $x = 0$

$y' = \frac{1}{3}$

Gradient of the normal $= -3$

Equation of the normal

$y - 1 = -3(x - 0)$

$y = -3x + 1$ (D)

10 $\frac{dy}{dx} = 0$ at $(1,2)$

Test $(1,2)$

A) $\frac{2 \times 1 + 2}{1 \times 2} = \frac{4}{2} = 2$

B, C, D $= 0$

A ✗

$\frac{dy}{dx} = 1$ at $(1,1)$

Test $(1,1)$

B $\frac{2-1}{1} =$

✗ C $\frac{2-1}{2} = \frac{1}{2}$

✗ D $\frac{2-1}{0}$

(B)

Question 11

(a) $\dfrac{4+x}{2x} \geq 1$

$\dfrac{x^2(4+x)}{2x} \geq x^2$

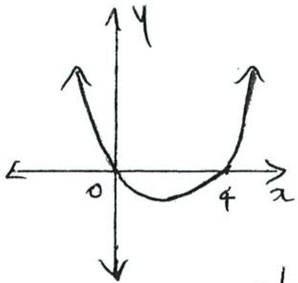
$x(4+x) \geq 2x^2$

$0 \geq 2x^2 - x(4+x)$

$0 \geq 2x^2 - 4x - x^2$

$0 \geq x^2 - 4x$

$0 \geq x(x-4)$ 1



From the graph

$x(x-4) \leq 0$

when $0 \leq x \leq 4$

But $x \neq 0$

$0 < x \leq 4$ 1

b) $\left(2x^3 - \dfrac{1}{x^2}\right)^{11}$

$T_{r+1} = (-1)^r \, {}^{11}C_r \, (2x^3)^{11-r} \left(\dfrac{1}{x^2}\right)^r$

$= (-1)^r \, {}^{11}C_r \, 2^{11-r} \, x^{33-5r}$

$33 - 5r = 8$

$25 = 5r$

$r = 5$ 1

Coefficient of $x^8 = (-1)^5 \, {}^{11}C_5 \, 2^{11-5}$

$= -\,{}^{11}C_5 \, 2^6$

$= -29568$ 1

c) (i) $p(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$

$p'(x) = 4x^3 - 9x^2 - 12x + 28$

$p''(x) = 12x^2 - 18x - 12$

For a triple root $x = \alpha$

$p(\alpha) = p'(\alpha) = p''(\alpha) = 0$

$p''(\alpha) = 0$

$12\alpha^2 - 18\alpha - 12 = 0$

$6(2\alpha^2 - 3\alpha - 2) = 0$

$2\alpha^2 - 3\alpha - 2 = 0$

$(\alpha - 2)(2\alpha + 1) = 0$

$\alpha = 2$ or $\frac{-1}{2}$ 1

$p(2) = 2^4 - 3\times 2^3 - 6\times 2^2 + 28\times 2 - 24$

$= 16 - 24 - 24 + 56 - 24 = 0$

$\therefore$ $\alpha = 2$ is the triple root. 1

(ii) Let 2, 2, 2 and β be the roots of $p(x)$

$6 + \beta = 3$ (sum of roots)

$\beta = -3$

$p(x) = (x-2)^3 (x+3)$ 1

d) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$

$x = u^2 - 1$

$\dfrac{dx}{du} = 2u$

$dx = 2u\,du$

$x+1 = u^2-1+1 = u^2$

when $x = 0$

$0 = u^2 - 1$

$u^2 = 1$

$u = 1$

when $x = 3$

$3 = u^2 - 1$

$u^2 = 4$

$u = 2$

$I = \displaystyle\int_1^2 \frac{(u^2-1)\,2u\,du}{\sqrt{u^2}}$ 1

$= \displaystyle\int_1^2 2(u^2-1)\,du$

$= 2\left[\dfrac{u^3}{3} - u\right]_1^2$ 1

$= 2\left\{\left(\dfrac{8}{3} - 2\right) - \left(\dfrac{1}{3} - 1\right)\right\}$

$= 2\left(\dfrac{8}{3} - 2 - \dfrac{1}{3} + 1\right)$

$= 2\left(\dfrac{8}{3} - \dfrac{1}{3} - 1\right)$

$= 2 \times \dfrac{4}{3} = \underline{\underline{\dfrac{8}{3}}}$ 1

Question 12

a) (i) $\sqrt{12}\sin x + 2\cos x = R\sin(x-\alpha)$

$= R[\sin x\cos\alpha - \cos x\sin\alpha]$

$= R\sin x\cos\alpha - R\cos x\sin\alpha$

Equating coefficients of $\sin x$ and $\cos x$

$R\cos\alpha = \sqrt{12}$, $-R\sin\alpha = 2$

$R\cos\alpha = \sqrt{12}$ —① 1

$R\sin\alpha = -2$ —②

squaring and adding ① and ②

$R^2\cos^2\alpha = 12$ $R^2\sin^2\alpha = 4$

$R^2(\sin^2\alpha + \cos^2\alpha) = 16$

$R^2 = 16$

$R = 4$ $(\because R > 0)$ 1

$\sin\alpha = \frac{-2}{4} = \frac{-1}{2}$, $\cos\alpha = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$

Since $\sin\alpha$ is negative and $\cos\alpha$ positive α is in the 4th quadrant.

acute angle $= \frac{\pi}{6}$ 1

IVth quadrant $\angle = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$\therefore \sqrt{12}\sin x + 2\cos x = 4\sin\left(x - \frac{11\pi}{6}\right)$

ii) $4\sin\left(x - \frac{11\pi}{6}\right) = 2$

$\sin\left(x - \frac{11\pi}{6}\right) = \frac{1}{2}$ $\quad 0 \le x \le 2\pi$

Let $u = x - \frac{11\pi}{6}$

$\sin u = \frac{1}{2}$ $\quad -\frac{11\pi}{6} \le u \le \frac{\pi}{6}$

$u = \frac{\pi}{6}, \overset{\times}{\frac{5\pi}{6}}, \frac{\pi}{6} - 2\pi, \frac{5\pi}{6} - 2\pi$

$= \frac{\pi}{6}, -\frac{11\pi}{6}, -\frac{7\pi}{6}$

$x - \frac{11\pi}{6} = \frac{\pi}{6}$ $\quad x - \frac{11\pi}{6} = -\frac{11\pi}{6}$

$x = 2\pi$ $\quad x = 0$

$x - \frac{11\pi}{6} = -\frac{7\pi}{6}$

$x = \frac{2\pi}{3}$

$x = 0, \frac{2\pi}{3}, 2\pi$

3 correct - 2 mark

2 correct - 1 mark

1 correct - 0 mark.

b) i) $LHS = \frac{dT}{dt} = Ae^{-kt} \times -k$

$= -kAe^{-kt}$

$RHS = -k(T-20)$

$= -k \times Ae^{-kt}$

$= -kAe^{-kt}$

$LHS = RHS$

$\therefore T = 20 + Ae^{-kt}$ is a solution.

ii) $T = 20 + Ae^{-kt}$

when $t=0$, $T=150$

$150 = 20 + Ae^{0}$

$A = 130$

$\therefore T = 20 + 130e^{-kt}$

when $t=15$, $T=100$

$100 = 20 + 130e^{-15k}$

$e^{-15k} = \frac{80}{130} = \frac{8}{13}$

$-15k = \log_e\left(\frac{8}{13}\right)$

$k = \frac{-1}{5}\log_e\left(\frac{8}{13}\right) = \underline{\underline{0.032}}$

(iii) $25 = 20 + 130e^{-0.032t}$ 1

$$\frac{5}{130} = e^{-0.032t}$$

$$-0.032t = \log_e\left(\frac{5}{130}\right)$$

$$t = \frac{-1}{0.032}\log_e\left(\frac{5}{130}\right)$$

$$= \underline{\underline{101.82 \text{ minutes}}}$$ 1

Question 13

$$\frac{dv}{dt} = -1.5 \text{ litres/min}$$

$$= -0.0015\ m^3/\text{minute}$$

$$V = \frac{1}{3}\pi r^2 h$$

By similarity $\frac{r}{h} = \frac{3}{5}$

$$r = \frac{3h}{5}$$

$$V = \frac{1}{3}\pi \times \frac{9h^2}{25} \times h$$

$$= \frac{3\pi h^3}{25}$$

$$\frac{dV}{dh} = \frac{9\pi h^2}{25} \qquad 1$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$-0.0015 = \frac{9\pi h^2}{25} \times \frac{dh}{dt} \qquad 1$$

$$\frac{dh}{dt} = -\frac{0.0015 \times 25}{9\pi h^2}$$

when $h = 1$ metre

$$\frac{dh}{dt} = -\frac{0.0015 \times 25}{9\pi} \qquad 1$$

$$= -0.0013 \text{ m/minute}$$

b) $$V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\tan^2 y - y^2)\, dy \qquad 1$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec^2 y - 1 - y^2)\, dy$$

$$= \pi \left[\tan y - y - \frac{y^3}{3} \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \qquad 1$$

$= \pi \left\{ \left(\tan\frac{\pi}{3} - \frac{\pi}{3} - \frac{\pi^3}{27} \times \frac{1}{3} \right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi^3}{64} \times \frac{1}{3} \right) \right\}$

$= \pi \left(\sqrt{3} - \frac{\pi}{3} - \frac{\pi^3}{81} - 1 + \frac{\pi}{4} + \frac{\pi^3}{192} \right)$

$= \underline{\underline{0 \cdot 78 \text{ cubic units.}}}$ 1

c) (i) $\underline{n = 1}$

LHS $= 4 \times 1^3 = 4$

RHS $= 1^2(1+1)^2 = 4$

LHS = RHS $\therefore$ true for $n = 1$

Assume true for $n = k$

$4(1^3 + 2^3 + 3^3 + \dots + k^3) = k^2(k+1)^2$ — 1

To prove true for $n = k+1$

$4(1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3) = (k+1)^2(k+2)^2$ 2

1

LHS of 2 $= 4(1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3)$

$= 4(1^3 + 2^3 + 3^3 + \dots + k^3) + 4(k+1)^3$ 1

$= k^2(k+1)^2 + 4(k+1)^3$ by assumption 1

$= (k+1)^2 \left(k^2 + 4(k+1)\right)$

$= (k+1)^2 \left(k^2 + 4k + 4\right)$

$= (k+1)^2 (k+2)^2$

$=$ RHS of 2

If the result is true for $n=k$ then it is true for $n=k+1$.

Hence by the principle of mathematical induction, the result is true for all positive integers.

ii) From i)

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2 (n+1)^2}{4}$$

$$\frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \frac{n^2 (n+1)^2}{4n^4}$$

$$= \frac{(n+1)^2}{4n^2} = \frac{n^2 + 2n + 1}{4n^2}$$

$$\lim_{n \to \infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \lim_{n \to \infty} \frac{n^2 + 2n + 1}{4n^2}$$

$$= \lim_{n \to \infty} \frac{1 + \frac{2n}{n^2} + \frac{1}{n^2}}{4}$$

$$= \lim_{n \to \infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4}$$

$$= \frac{1}{4} \qquad 1$$

Question 14

a) For $\underset{\sim}{p}$ and $\underset{\sim}{q}$ to be parallel

$\underset{\sim}{p} = \lambda \underset{\sim}{q}$ where λ is a constant

$$\begin{pmatrix} t-8 \\ 6 \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 2t \end{pmatrix}$$

$t - 8 = 3\lambda$ — ①

$6 = 2\lambda t$ — ② $\qquad 1$

From ② $\lambda = \frac{6}{2t} = \frac{3}{t}$ — ③

substitute ③ into ①

$t - 8 = 3 \times \frac{3}{t}$

$t - 8 = \frac{9}{t}$

$t^2 - 8t = 9$

$t^2 - 8t - 9 = 0$

$(t-9)(t+1) = 0$

$t = -1, 9 \qquad 2$

b) i) $\overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CD}$

$= \underset{\sim}{u} + \underset{\sim}{v} + \frac{1}{2}\underset{\sim}{w}$

ii) $\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{AG}$

$\overrightarrow{MA} = \frac{1}{2}\overrightarrow{DA}$

$= \frac{1}{2}\left(\overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BA}\right)$

$= \frac{1}{2}\left(-\underset{\sim}{w} - \underset{\sim}{v} - \underset{\sim}{u}\right)$

$= -\frac{1}{2}\underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{w}$

$\overrightarrow{MG} = \left(-\frac{1}{2}\underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{w}\right) + \left(\underset{\sim}{u} + \underset{\sim}{v} + \frac{1}{2}\underset{\sim}{w}\right)$

$= -\frac{1}{2}\underset{\sim}{u} + \underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v} + \underset{\sim}{v} - \frac{1}{2}\underset{\sim}{w} + \frac{1}{2}\underset{\sim}{w}$

$= \frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v}$

$= \frac{1}{2}\left(\underset{\sim}{u} + \underset{\sim}{v}\right)$

c) Since $\overrightarrow{PB} \perp \overrightarrow{PC}$, $\overrightarrow{PC} \cdot \overrightarrow{PB} = 0$

$$\overrightarrow{PC} = \overrightarrow{OC} - \overrightarrow{OP}$$
$$= \underset{\sim}{j} - \lambda \underset{\sim}{i} = -\lambda \underset{\sim}{i} + \underset{\sim}{j}$$

$$\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP}$$
$$= \frac{5}{2} \underset{\sim}{i} + \underset{\sim}{j} - \lambda \underset{\sim}{i}$$
$$= \left(\frac{5}{2} - \lambda\right) \underset{\sim}{i} + \underset{\sim}{j}$$

1

$$\overrightarrow{PC} \cdot \overrightarrow{PB} = \left(-\lambda \underset{\sim}{i} + \underset{\sim}{j}\right) \cdot \left(\frac{5}{2} - \lambda\right) \underset{\sim}{i} + \underset{\sim}{j}$$

$$= -\lambda\left(\frac{5}{2} - \lambda\right) + 1$$ 1

$$-\lambda\left(\frac{5}{2} - \lambda\right) + 1 = 0$$
$$-\lambda\left(\frac{5 - 2\lambda}{2}\right) + 1 = 0$$
$$-\lambda(5 - 2\lambda) + 2 = 0$$
$$2\lambda^2 - 5\lambda + 2 = 0$$
$$2\lambda^2 - 4\lambda - \lambda + 2 = 0$$
$$2\lambda(\lambda - 2) - (\lambda - 2) = 0$$
$$(\lambda - 2)(2\lambda - 1) = 0$$ 1
$$\lambda = 2 \text{ or } \lambda = \frac{1}{2}$$

Question 15

a)

$$y = e^{-7x}$$

$$y' = -7e^{-7x}$$

$$y'' = -7 \times -7e^{-7x} = 49e^{-7x}$$ 1

$$y + ky'' = 0$$

$$e^{-7x} + k \times 49e^{-7x} = 0$$

$$e^{-7x}(1 + 49k) = 0$$

$$1 + 49k = 0 \quad (\because e^{-7x} \neq 0)$$

$$k = \frac{-1}{49}$$ 1

b)

$$\frac{dy}{dx} = x \sec y$$

$$\frac{dy}{\sec y} = x\,dx$$

$$\int \frac{dy}{\sec y} = \int x\,dx$$

$$\int \cos y\,dy = \int x\,dx$$

$$\sin y = \frac{x^2}{2} + C$$ 1

when $x = 0$, $y = \frac{\pi}{6}$

$$\sin\frac{\pi}{6} = 0 + C$$

$$C = \frac{1}{2}$$ 1

$$\sin y = \frac{x^2}{2} + \frac{1}{2}$$

$$= \frac{x^2+1}{2}$$

$$y = \sin^{-1}\left(\frac{x^2+1}{2}\right)$$ 1

c) i) $RHS = \frac{1}{p} + \frac{1}{20000-p}$

$= \frac{20000-p+p}{p(20000-p)}$ 1

$= \frac{20000}{p(20000-p)} = LHS$

ii) $\frac{dp}{dt} = 0 \cdot 1 p \left(1 - \frac{p}{20000}\right)$

$= 0 \cdot 1 p \left(\frac{20000-p}{20000}\right)$

$\frac{dp}{dt} \times \frac{20000}{20000-p} = 0 \cdot 1 p$

$\frac{20000}{p(20000-p)} dp = 0 \cdot 1 \, dt$

$\int 0 \cdot 1 \, dt = \int \frac{20000}{p(20000-p)} dp$

$0 \cdot 1 t + C = \int \left(\frac{1}{p} + \frac{1}{20000-p}\right) dp$

$= \ln(p) - \ln(20000-p)$ 1

$$= \ln \frac{P}{20000-P}$$

when $t=0$, $P=1000$

$$0 + C = \log_e \frac{1000}{20000-1000}$$

$$= \log_e \frac{1000}{19000} = \log \frac{1}{19}$$

$$C = \log_e \frac{1}{19}$$

$$0.1t + \log_e \frac{1}{19} = \log_e \left(\frac{P}{20000-P} \right)$$

$$\frac{P}{20000-P} = e^{0.1t + \ln \frac{1}{19}}$$

$$= e^{0.1t} \times e^{\ln \frac{1}{19}}$$

$$\frac{P}{20000-P} = e^{0.1t} \times \frac{1}{19} = \frac{e^{0.1t}}{19}$$

$$19P = e^{0.1t}(20000-P)$$

$$19P = 20000e^{0.1t} - Pe^{0.1t}$$

$$19P + Pe^{0.1t} = 20000e^{0.1t}$$

$$P(19 + e^{0.1t}) = 20000e^{0.1t}$$

$$P = \frac{20000e^{0.1t}}{19 + e^{0.1t}}$$ 1

(iii) when $t = 7$

$$P = \frac{20000e^{0.1\times 7}}{19 + e^{0.1\times 7}}$$

$$= 1916.605$$ 1

The population of rabbits after seven months is 1916

Question 16

a) $f(x) = 2\sin^{-1}\left(\frac{x}{3}\right) + \frac{\pi}{2}$

$-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$

$\underline{\underline{D: \ -3 \le x \le 3}}$ 1

$2 \times \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} \le 2\sin^{-1}\left(\frac{x}{3}\right) + \frac{\pi}{2}$

$\le 2 \times \frac{\pi}{2} + \frac{\pi}{2}$

$\frac{-\pi}{2} \le f(x) \le \frac{3\pi}{2}$

$\underline{\underline{R: \ -\frac{\pi}{2} \le f(x) \le \frac{3\pi}{2}}}$ 1

b i) $LHS = \frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta}$

$= \frac{(\sin\theta + \sin 5\theta) + \sin 3\theta}{(\cos\theta + \cos 5\theta) + \cos 3\theta}$ 1

$= \frac{2\sin\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) + \sin 3\theta}{2\cos\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) + \cos 3\theta}$

$$= \frac{2\sin 3\theta \cos 2\theta + \sin 3\theta}{2\cos 3\theta \cos 2\theta + \cos 3\theta}$$

1

$$= \frac{\sin 3\theta \,(2\cos 2\theta + 1)}{\cos 3\theta \,(2\cos 2\theta + 1)}$$

$= \tan 3\theta = RHS$

ii) $\tan 3\theta = 1, \quad 0 \le \theta \le 2\pi$

$u = 3\theta$

$\tan u = 1, \quad 0 \le 3\theta \le 6\pi$

$u = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$ 1

$\theta = \frac{u}{3} = \underline{\underline{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}}}$

c) i) $x = Vt\cos\alpha$ —— (1)

$y = Vt\sin\alpha - \frac{gt^2}{2}$ —— (2)

From (1) $t = \frac{x}{V\cos\alpha}$

substitute in (2)

$$y = \frac{Vx \sin\alpha}{V\cos\alpha} - \frac{g}{2} \times \frac{x^2}{V^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$$ 1

ii) $\sin\theta = \frac{y}{R\theta}$ $\cos\theta = \frac{x}{R\theta}$

$x = R\theta\cos\theta$

$y = R\theta\sin\theta$

substitute $x = R\theta\cos\theta$ and

$y = R\theta\sin\theta$ in

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$$

$$R\theta\sin\theta = R\theta\cos\theta\tan\alpha - \frac{g}{2V^2} \times R\theta^2\cos^2\theta\sec^2\alpha$$

$$= R\theta\cos\theta\frac{\sin\alpha}{\cos\alpha} - \frac{g}{2V^2} \times R\theta^2\cos^2\theta \times \frac{1}{\cos^2\alpha}$$ 1

Multiply by $\cos^2\alpha$

$$R\theta\sin\theta\cos^2\alpha = R\theta\cos\theta\sin\alpha\cos\alpha - \frac{g}{2V^2} \times R\theta^2\cos^2\theta$$

divide by $R\theta$

$$\sin\theta \cos^2\alpha = \cos\theta \sin\alpha \cos\alpha - \frac{g}{2V^2} \times RO \cos^2\theta$$

$$\frac{g}{2V^2} RO \cos^2\theta = \cos\theta \sin\alpha \cos\alpha - \sin\theta \cos^2\alpha$$

$$= \cos\alpha \left(\cos\theta \sin\alpha - \sin\theta \cos\alpha\right)$$

$$= \cos\alpha \sin(\alpha - \theta)$$

$$g \times RO \times \cos^2\theta = 2V^2 \cos\alpha \sin(\alpha - \theta)$$

$$RO = \frac{2V^2 \cos\alpha \sin(\alpha - \theta)}{g\cos^2\theta}$$

iii $$\frac{dRO}{d\alpha} = \frac{2V^2}{g\cos^2\theta}\left[\cos\alpha \cos(\alpha-\theta) + \sin(\alpha-\theta) \times -\sin\alpha\right]$$

$$\frac{dRO}{d\alpha} = 0 \Rightarrow \cos\alpha \cos(\alpha-\theta) - \sin\alpha \sin(\alpha-\theta) = 0$$

ie $\cos(\alpha + \alpha - \theta) = 0$

$$\cos(2\alpha - \theta) = 0$$

$$2\alpha - \theta = \frac{\pi}{2}$$

$$2\alpha = \theta + \frac{\pi}{2}$$

$$\alpha = \frac{\theta}{2} + \frac{\pi}{4}$$

$$\frac{d^2 RO}{d\alpha^2} = \frac{2V^2}{g\cos^2\theta} \times -\sin(2\alpha-\theta)\times 2$$

$$= \frac{-4V^2}{g\cos^2\theta}\sin(2\alpha-\theta)$$

when $\alpha = \frac{\theta}{2} + \frac{\pi}{4}$

$$\frac{d^2 RO}{d\alpha^2} = \frac{-4V^2}{g\cos^2\theta}\sin\left[\theta + \frac{\pi}{2} - \theta\right]$$

$$= \frac{-4V^2}{g\cos^2\theta}\sin\frac{\pi}{2}$$

$$= \frac{-4V^2}{g\cos^2\theta} < 0$$

$\therefore$ RO is maximum when

$\alpha = \frac{\theta}{2} + \frac{\pi}{4}$