

GOSFORD HIGH SCHOOL

2020

Trial HSC Examination Mathematics Extension 1

General Instructions

Total Marks – 70

All questions may be attempted

Section I (10 Marks)

Answer questions 1-10 on the Multiple Choice answer sheet provided.

Questions 1-10 are of equal values

Section II (60 Marks)

For Questions 11-14, start a new answer booklet for each question.

Questions 11-14 are of equal values

- Reading time -10 minutes
- Writing time - 2 Hours
- Writing using a black pen
- NESA approved calculators may be used
- Leave your answers in the simplest exact form, unless otherwise stated
- Marks may be deducted for careless or badly arranged work
- All necessary working should be shown
- A Reference Sheet is provided

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.

Candidate Number

Section I – Multiple Choice (10 marks)

Attempt Questions 1 – 10

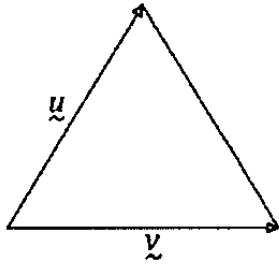
Read each question and choose an answer A, B, C or D.

Allow about 15 minutes for this section.

Use multiple-choice answer sheet for Question 1- 10

1. What is the value of $\sin 2x$ given that $\sin x = \frac{2\sqrt{3}}{4}$ and x is obtuse?
(A) $-\frac{\sqrt{3}}{4}$
(B) $-\frac{\sqrt{3}}{2}$
(C) $\frac{\sqrt{3}}{4}$
(D) $\frac{\sqrt{3}}{2}$
2. A ball is thrown from the origin O with a velocity V and an angle of elevation of θ , where $\theta \neq \frac{\pi}{2}$. What is the Cartesian equation of the flight path? Take $g = 10 \text{ ms}^{-1}$.
(A) $y = x \tan \theta - \frac{5x^2}{V^2} (1 + \tan^2 \theta)$
(B) $y = x \tan \theta - \frac{10x^2}{V^2} (1 + \tan^2 \theta)$
(C) $x = V \cos \theta t$ and $y = -5t^2 + V \sin \theta t$
(D) $x = V \cos \theta t$ and $y = -10t^2 + V \sin \theta t$
3. An examination consists of 30 multiple-choice questions, each question having five possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is $E(X)$?
(A) 5
(B) 6
(C) 9
(D) 15

4. An equilateral triangle of side 3 units is shown below.
 Vectors $\underline{u}$ and $\underline{v}$ are represented in the diagram.



What is the value of $\underline{u} \cdot \underline{v}$?

- (A) 0
 (B) $\frac{9}{\sqrt{2}}$
 (C) $\frac{9}{2}$
 (D) 9
5. If $y = \sin^{-1} \frac{a}{x}$ then $\frac{dy}{dx}$ equals:

- (A) $\frac{-a}{x^2\sqrt{x^2-a^2}}$
 (B) $\frac{x}{\sqrt{x^2-a^2}}$
 (C) $\frac{-x}{\sqrt{x^2-a^2}}$
 (D) $\frac{-a}{x\sqrt{x^2-a^2}}$

6. The equation $y = e^{ax}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

What are the possible values of a ?

- (A) $a = -2$ or $a = 3$
 (B) $a = -1$ or $a = 6$
 (C) $a = 2$ or $a = -3$
 (D) $a = 1$ or $a = -6$
7. Which of the following is an expression for $\int \frac{x}{\sqrt{9-x^2}} dx$? Use the substitution $u = 9 - x^2$.
- (A) $-\sqrt{9-x^2} + C$
 (B) $-2\sqrt{9-x^2} + C$
 (C) $\sqrt{9-x^2} + C$
 (D) $2\sqrt{9-x^2} + C$

8. Let $\underline{u} = \underline{i} + \underline{j}$ and $\underline{v} = \underline{i} - \underline{j}$. What is the angle between the two vectors.
- (A) $\frac{\pi}{2}$
 (B) $\frac{\pi}{4}$
 (C) π
 (D) 2π
9. Which of the following expressions represents the area of the region bounded by the curve $y = \sin^3 x$ and the x -axis from $x = -\pi$ to $x = 2\pi$? Use the substitution $u = \cos x$.
- (A) $-\int_{-\pi}^{2\pi} (1 - u^2) du$
 (B) $-3 \int_0^{\pi} (1 - u^2) du$
 (C) $-\int_{-1}^1 (1 - u^2) du$
 (D) $3 \int_{-1}^1 (1 - u^2) du$
10. Emma made an error proving that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. The proof is shown below.
- Step 1: To prove $2^n + (-1)^{n+1}$ is divisible by 3 (n is an integer)
 To prove true for $n = 1$
 $2^1 + (-1)^{1+1} = 2 + 1$
 $= 3 \times 1$ Line 1
 Result is true for $n = 1$
- Step 2: Assume true for $n = k$
 $2^k + (-1)^{k+1} = 3m$ (m is an integer) Line 2
- Step 3: To prove true for $n = k + 1$
 $2^{k+1} + (-1)^{k+1+1} = 2(2^k) + (-1)^{k+2}$ Line 3
 $= 2[3m + (-1)^{k+1}] + (-1)^{k+2}$ Line 4
 $= 2 \times 3m + 2 \times (-1)^{k+1} + (-1)^{k+2}$
 $= 3[2m + (-1)^{k+2}]$
- Which is a multiple of 3 since m and k are integers.
 Step 4: True by induction

In which line did Emma make an error?

- (A) Line 1
 (B) Line 2
 (C) Line 3
 (D) Line 4

----- End of Section I -----

Section II (60 marks)

Attempt Questions 11 – 14. Allow about 1 hour and 45 minutes for this section

Questions 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u = \cos x + 1$, find the exact value of the following integral:

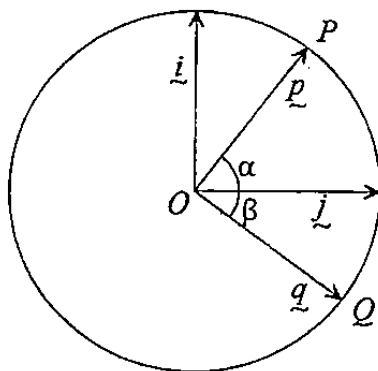
$$\int_0^{\frac{\pi}{2}} e^{\cos x + 1} \sin x \, dx \quad 2$$

(b) (i) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$, and $0 \leq \alpha < 2\pi$.

Give α correct to 1 decimal place. 2

(ii) Hence calculate the maximum value of $\frac{4}{12 + 6\cos\theta + 8\sin\theta}$ 2

(c) Use the unit circle and the vectors in the diagram to derive the expansion of $\cos(\alpha + \beta)$



3

A man throws a ball from a point O on the horizontal ground so that it lands on the ground at a point P distant 80 m from him. The ball reaches the highest point 20 m above the ground. Neglect the sizes of the man and the ball. Take $g = 10 \text{ m/s}^2$

(d) (i) Find the vertical and horizontal components of velocity of projection of the ball. 2

(ii) Find the velocity of projection of the ball. 2

(iii) Find the further distance from the other boy B who starts running at the speed of 5 m/s

to catch the ball at the point P simultaneously. 2

Questions 12 (15 marks) Use a SEPARATE writing booklet

(a) The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β, γ

(i) Show that $\beta + \gamma = 1 - \alpha$ 1

(ii) Using part (i), and similar results, evaluate $\frac{\beta + \gamma}{\alpha} + \frac{\gamma + \alpha}{\beta} + \frac{\alpha + \beta}{\gamma}$ 2

(b) Use mathematical induction to prove that $15^n + 2^{3n} - 2$ is a multiple of 7 for $n \geq 1$ 3

(c) Points A, B have position vectors $\overrightarrow{OA} = 3i - 2j, \overrightarrow{OB} = -i + j$

(i) Find the unit vector along $\overrightarrow{AB}$ 1

(ii) Suppose P is a point on AB such that $\overrightarrow{OP} \perp \overrightarrow{AB}$, Find $\overrightarrow{OP}$ 2

(d) A particle is projected across horizontal ground from the origin O . Its initial velocity vector is $12i + 5j$ and its acceleration vector is $0i - 10j$.

Find:

(i) the initial speed of the particle 1

(ii) the angle of projection, correct to the nearest minute 1

(iii) Beginning with its acceleration vector, derive the velocity at any time t in the form

$$\underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j} \quad \text{2}$$

(iv) Show that its displacement at time t is $\underline{r} = (12t)\underline{i} + (5t - 5t^2)\underline{j}$ 1

(v) Find the maximum height of the particle. 1

Questions 13 (15 marks) Use a SEPARATE writing booklet

(a) $f(x) = \sqrt{4 - \sqrt{x}}$

(i) Explain why the domain of $f(x)$ is $0 \leq x \leq 16$ 1

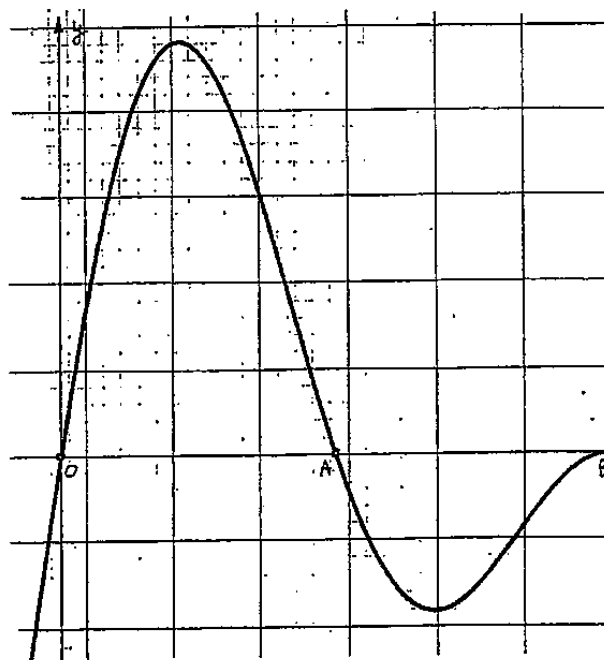
(ii) Prove $f(x)$ is a decreasing function and find its range. 2

(iii) Since $f(x)$ is monotonic, an inverse function exists. Find the domain and range of $f^{-1}(x)$. Hence find $f^{-1}(x)$. 2

(b) (i) Given that $\frac{d}{dx}(\sin 2x - 2x \cos 2x) = 4x \sin 2x$. The curve shown below is part of the

function $y = x \sin 2x$. Write down the coordinates of the points A and B . 1

(ii) If the area bounded by the curve and the line AB is k times that of the area of the region bounded by the curve and the line OA . Determine the value of k . 2



(iii) Show that $\int_0^{\frac{\pi}{8}} x \sin 2x \, dx = \frac{\sqrt{2}}{32}(4 - \pi)$ 2

Questions 13 (Continued)

- (c) On a roulette wheel, there are 18 red numbers, 18 black numbers, and 1 green number.

A ball is dropped onto the spinning wheel and lands on one of the numbers randomly.

Each result is independent. A gambler bets that the ball will land on any of the black numbers.

- (i) Define the gambler's bet as a Bernoulli random variable X and give its mean and variance. 1
- (ii) If the gambler makes the same bet five times, let the random variable Y be the number of times the gambler wins. Describe the distribution of Y , and give its mean and variance. 2
- (iii) If the gambler makes the same bet five times, what is the probability he will win more times than he loses? Give your answer correct to three decimal places. 2

Questions 14 (15 marks) Use a SEPARATE writing booklet

- (a) Given that $t = \tan \frac{\theta}{2}$, then $\sin \theta = \frac{2t}{1+t^2}$ and $\cos \theta = \frac{1-t^2}{1+t^2}$ for any angle θ .

(DO NOT PROVE THIS)

- (i) Show that the equation $7 \cos \theta + 4 \sin \theta + 5 = 0$ can be written as $t^2 - 4t - 6 = 0$ 2

- (ii) θ_1, θ_2 are the solutions of the equation $7 \cos \theta + 4 \sin \theta + 5 = 0$ for $(\theta_1 > 0, 0 < \theta_2 < 2\pi)$

Then without solving for θ , show that $\cot \frac{\theta_1}{2} + \cot \frac{\theta_2}{2} = -\frac{2}{3}$ 2

- (b) Show that, among any four integers, there are two integers whose difference is divisible by 3. 2

- (c) A tank contains 2500 litres of water and 25 Kg of dissolved salt. Fresh water enters the tank at a rate of 20 litres per minute. The solution is thoroughly mixed at all times and is drained from the tank at a rate of 15 litres per minute.

- (i) Using y for the amount of salt in the tank in kilograms (as a function of time), and t for time in minutes, show that the concentrations of salt in the tank at time t can be

given by $C = \frac{y}{2500 + 5t}$ 1

- (ii) Explain why the rate of change of salt in the tank can be given by $y' = -15C$ 1

- (iii) Find y , the amount of salt in the tank as a function of t . 4

- (d) Find the six solutions of the equation:

$$\sin\left(2 \cos^{-1}\left(\cot\left(2 \tan^{-1} x\right)\right)\right) = 0$$
 3

Give your answers as simplified surds

Ext 1 Trial 2020 (CHS)

1. B 2. A 3. B 4. C 5. D 6. C 7. A 8. A 9. D 10. D

Q11

a) $\int_0^{\pi/2} e^{\cos x + 1} \sin x \, dx$

$$u = \cos x + 1$$

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{du}{-\sin x}$$

$$x = \frac{\pi}{2} \quad u = 1$$

$$x = 0 \quad u = 2$$

$$I = \int_1^2 e^u \cdot -du$$

$$= - \int_1^2 e^u \, du$$

$$= - [e^u]_1^2$$

$$= -e^2 + e$$

(2)

b) i) $6 \cos \theta + 8 \sin \theta$

$$= R \cos(\theta - \alpha)$$

$$\therefore R \cos \alpha = 6$$

$$R \sin \alpha = 8$$

$$\therefore \tan \alpha = \frac{4}{3} \approx 0.9$$

$$R^2 (\sin^2 \alpha + \cos^2 \alpha) = 36 + 64$$

$$R = 10$$

$$\therefore 10 \cos(\theta - 0.9)$$

(2)

ii) $\frac{4}{12 + 6 \cos \theta + 8 \sin \theta}$

$$= \frac{4}{12 + 10 \cos(\theta - 0.9)}$$

$$\text{max at } 10 \cos(\theta - 0.9) = -1$$

$$\therefore \text{max} = \frac{4}{12 - 10} = \frac{4}{2} = 2$$

(2)

c) $\cos(\alpha + \beta) = \frac{\underline{p} \cdot \underline{q}}{|\underline{p}| |\underline{q}|}$

$$\text{and } |\underline{p}| = |\underline{q}| = 1$$

$$\therefore \cos(\alpha + \beta) = \underline{p} \cdot \underline{q}$$

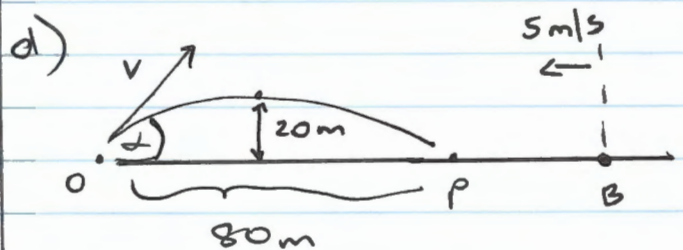
$$\underline{p} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$$

$$\underline{q} = \cos \beta \underline{i} - \sin \beta \underline{j}$$

$$\underline{p} \cdot \underline{q} = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

(3)



di)
horiz

$$\ddot{x} = 0$$

$$\dot{x} = 0t + c$$

$$\text{at } t=0 \dot{x} = v \cos \alpha = c$$

$$\dot{x} = v \cos \alpha$$

$$x = vt \cos \alpha + c_1$$

$$\text{at } t=0 \quad x=0$$

$$\therefore x = vt \cos \alpha$$

vert

$$\ddot{y} = -g$$

$$\dot{y} = -gt + c_2$$

$$\text{at } t=0 \quad \dot{y} = v \sin \alpha = c_2$$

$$\therefore \dot{y} = -10t + v \sin \alpha$$

$$y = -5t^2 + vt \sin \alpha + c_3$$

$$\text{at } t=0 \quad y=0 = c_3$$

$$\therefore y = -5t^2 + vt \sin \alpha$$

$$\text{max height at } \frac{dy}{dt} = 0$$

$$\therefore -10t + v \sin \alpha = 0$$

$$\therefore t = \frac{v \sin \alpha}{10}$$

$$y = -5 \left(\frac{v^2 \sin^2 \alpha}{100} \right) + v \sin \alpha \left(\frac{v \sin \alpha}{10} \right)$$

$$20 = -\frac{v^2 \sin^2 \alpha}{20} + \frac{v^2 \sin^2 \alpha}{10}$$

$$\therefore v^2 \sin^2 \alpha = 400 \quad \text{as } v \sin \alpha = 20$$

$$v \sin \alpha = 20$$

* at $y=0$

$$t(-5t + v \sin \alpha) = 0 \quad t \neq 0$$

$$\therefore t = \frac{v \sin \alpha}{5}$$

$$\therefore x = v \left(\frac{v \sin \alpha}{5} \right) \cos \alpha$$

$$\therefore 80 = \frac{v^2 \sin \alpha \cos \alpha}{5}$$

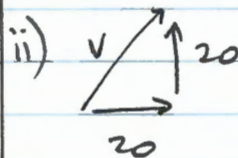
$$* v \sin \alpha = 20$$

$$\therefore 80 = \frac{v \cos \alpha \cdot 20}{5}$$

$$\therefore v \cos \alpha = 20$$

(2)

vert & horiz. components
of velocity = 20 m/s



$$v^2 = 20^2 + 20^2$$
$$v = 20\sqrt{2} \text{ m/s}$$

(2)

iii) Using $x = vt \cos \alpha$

$$v \cos \alpha = 20, \quad x = 80$$

$$\therefore \text{time of flight} = \frac{80}{20}$$

$$t = 4 \text{ s}$$



$$\therefore \text{Dist} = 5 \times 4$$

$$= 20 \text{ m}$$

(2)

or 100 m from nan

Q12

a) $x^3 - x^2 - 5x + 2 = 0$

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$= \frac{-(-1)}{1}$$

$$= 1$$

$\therefore \beta + \gamma = 1 - \alpha$

①

b) $\frac{\beta + \gamma}{\alpha} + \frac{\gamma + \alpha}{\beta} + \frac{\alpha + \beta}{\gamma}$

from ①

$$\beta + \gamma = 1 - \alpha$$

$$\alpha + \beta = 1 - \gamma$$

$$\alpha + \gamma = 1 - \beta$$

$$\therefore \frac{1 - \alpha}{\alpha} + \frac{1 - \beta}{\beta} + \frac{1 - \gamma}{\gamma}$$

$$= \frac{1}{\alpha} - 1 + \frac{1}{\beta} - 1 + \frac{1}{\gamma} - 1$$

$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) - 3$$

$$= \frac{\alpha\beta\gamma}{\alpha\beta\gamma} - 3$$

$$= \frac{-5}{-2} - 3$$

$= -\frac{1}{2}$

②

b) RTP $15^n + 2^{3n} - 2 = 7p$

p is true int
 $n \geq 1$

① Prove $n=1$

$$\text{LHS} = 15^1 + 2^{3(1)} - 2$$

$$= 15 + 8 - 2$$

$$= 21$$

$\therefore$ divisible by 7

$\therefore$ true $n=1$

② Assume true $n=k, k \geq 1$

ie

$$15^k + 2^{3k} - 2 = 7q \quad q \text{ true int}$$

$$15^k = 7q - 2^{3k} + 2$$

③ RTP $n=k+1$

ie $15^{k+1} + 2^{3(k+1)} - 2 = 7m$

LHS

m is true int

$$= 15 \cdot 15^k + 2^{3k} \cdot 2^3 - 2$$

$$= 15(7q - 2^{3k} + 2) + 8 \cdot 2^{3k} - 2$$

$$= 7 \cdot 15q - 15 \cdot 2^{3k} + 30 + 8 \cdot 2^{3k} - 2$$

$$= 7 \cdot 15q - 7 \cdot 2^{3k} + 28$$

$$= 7(15q - 2^{3k} + 4)$$

$= 7m$

③

$$c) \underline{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$i) \underline{c} = \underline{AB} = -\underline{a} + \underline{b}$$

$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

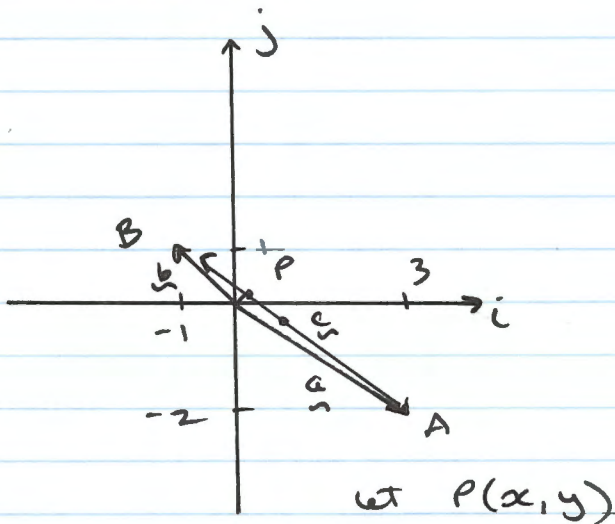
$$= \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\hat{\underline{c}} = \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad |\underline{c}| = \sqrt{16+9} = 5$$

$$= \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$$

(1)

ii)



$$\vec{PO} \cdot \vec{AB} = 0 \quad \vec{OP} \perp \vec{AB}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix} = 0$$

$$\therefore -4x + 3y = 0 \quad (1)$$

$$\star \vec{PB} = \lambda \vec{AB}$$

$$ii) \text{ slope } \vec{PB} = \text{ slope } \vec{AB}$$

$$\frac{1-y}{-1-x} = \frac{-2-1}{3+1} = \frac{-3}{4}$$

$$4-4y = 3+3x$$

$$3x + 4y - 1 = 0 \quad (2)$$

$$-12x + 9y = 0 \quad (1) \times 3$$

$$+12x + 16y - 4 = 0 \quad (2) \times 4$$

$$25y - 4 = 0$$

$$y = \frac{4}{25} \quad x = \frac{13}{25}$$

$$P\left(\frac{13}{25}, \frac{4}{25}\right) \quad (2)$$

$$d) \underline{v} = 12\underline{i} + 5\underline{j}$$

$$\underline{a} = 0\underline{i} - 10\underline{j}$$

at $t=0$

$$\underline{v} = 12\underline{i} + 5\underline{j} \quad \underline{a} = 0\underline{i} - 10\underline{j}$$

$$i) \text{ at } t=0$$

$$|\underline{v}| = \sqrt{12^2 + 5^2}$$

$$= 13 \text{ m/s} \quad (1)$$

$$ii) \tan \theta = \frac{5}{12}$$

$$\therefore \theta = 22^\circ 37' \quad (1)$$

$$iii) \underline{a} = 0\underline{i} - 10\underline{j}$$

$$\ddot{x} = 0$$

$$\ddot{y} = -10$$

$$\dot{x} = 0t + c$$

$$\dot{y} = -10t + c_1$$

$$\text{at } t=0 \quad \dot{x} = 12, \dot{y} = 5$$

$$\therefore c = 12, \quad c_1 = 5$$

$$\dot{x} = 12$$

$$\dot{y} = -10t + 5$$

$$\therefore \underline{v} = 12\underline{i} + (-10t + 5)\underline{j} \quad (2)$$

iv)

$$x = 12t + c_2 \quad y = -\frac{10t^2}{2} + 5t + c_3$$

$$\text{at } t = 0 \quad x = 0 \quad y = 0$$

$$\therefore c_2 = 0 = c_3$$

$$x = 12t$$

$$y = -5t^2 + 5t$$

$$\therefore \underline{r = 12t \hat{i} + (5t - 5t^2) \hat{j}} \quad (1)$$

v) max height at $\dot{y} = 0$

$$\therefore -10t + 5 = 0$$

$$10t = 5$$

$$t = 0.5$$

$$\therefore y = -5(0.5)^2 + 5(0.5)$$

$$= 1.25 \text{ m}$$

(1)

Q 13.

a) (i) $f(x) = \sqrt{4 - \sqrt{x}}$

For $\sqrt{x}$: $x \geq 0$

For $\sqrt{4 - \sqrt{x}}$: $4 - \sqrt{x} \geq 0$

$$4 \geq \sqrt{x}$$

$$16 \geq x \quad (x \geq 0)$$

$$\therefore 0 \leq x \leq 16$$

(1 mark)

(ii) $f(x) = (4 - x^{\frac{1}{2}})^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2} (4 - x^{\frac{1}{2}})^{-\frac{1}{2}} \cdot -\frac{1}{2} x^{-\frac{1}{2}}$$

$$= -\frac{1}{4\sqrt{x}\sqrt{4-\sqrt{x}}}$$

$$\sqrt{x} \geq 0 \quad \forall x$$

$$\sqrt{4 - \sqrt{x}} \geq 0 \quad \forall x$$

$$\therefore f'(x) = \frac{-1}{4 (+)(+)}$$

$$< 0$$

$\therefore f(x)$ is a decreasing function

$\Rightarrow$ Range can be found by evaluating $f(x)$ at endpoints

$$f(0) = 2$$

$$f(16) = 0$$

$$\therefore \text{Range } y \in [0, 2]$$

(2 marks)

$$(iii) f(x) \quad D: x \in [0, 16]$$

$$R: y \in [0, 2]$$

$$f^{-1}(x) \quad D: x \in [0, 2]$$

$$R: y \in [0, 16]$$

$$f: y = \sqrt{4 - \sqrt{x}}$$

$$f^{-1}: x = \sqrt{4 - \sqrt{y}}$$

$$x^2 = 4 - \sqrt{y}$$

$$x^2 - 4 = -\sqrt{y}$$

$$\sqrt{y} = 4 - x^2$$

$$y = (4 - x^2)^2 \quad \text{for } (0 \leq x \leq 2)$$

(2 marks)

$$b) (i) \quad x \sin 2x = 0$$

$$x = 0 \quad \text{or} \quad \sin 2x = 0$$

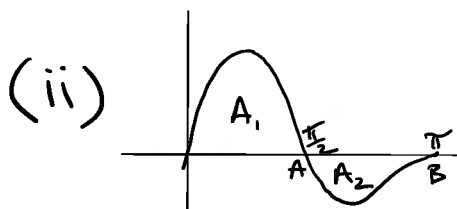
$$2x = 0, \pi, 2\pi, \dots$$

$$x = 0, \frac{\pi}{2}, \pi, \dots$$

$$\therefore A \left(\frac{\pi}{2}, 0 \right)$$

$$B (\pi, 0)$$

(1 mark)



$$A_2 = k A_1$$

$$-\int_{\frac{\pi}{2}}^{\pi} x \sin 2x \, dx = k \int_0^{\frac{\pi}{2}} x \sin 2x \, dx$$

$$-\frac{1}{4} \left[\sin 2x - 2x \cos x \right]_{\frac{\pi}{2}}^{\pi} = \frac{k}{4} \left[\sin 2x - 2x \cos x \right]_0^{\frac{\pi}{2}}$$

$$\vdots$$

$$\frac{3\pi}{4} = \frac{k\pi}{4}$$

$$\therefore k = 3$$

(2 marks)

$$\begin{aligned}
 \text{(iii)} \quad & \int_0^{\pi/8} x \sin 2x \, dx \\
 &= \frac{1}{4} [\sin 2x - 2x \cos 2x]_0^{\pi/8} \\
 &= \frac{1}{4} \left[\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4} \times \frac{1}{\sqrt{2}} \right) \right] \\
 &= \frac{1}{4\sqrt{2}} \left(1 - \frac{\pi}{4} \right) \\
 &= \frac{\sqrt{2}}{8} \left(\frac{4-\pi}{4} \right) \\
 &= \frac{\sqrt{2}}{32} (4-\pi)
 \end{aligned}$$

(2 marks)

(c) (i) This is a Bernoulli random variable with $p = 18/37$. This may be written

$$X \sim \text{Ber}\left(\frac{18}{37}\right) \text{ or } X \sim \text{Bin}\left(\frac{18}{37}, 1\right).$$

$$\mu = \frac{18}{37}, \quad \sigma^2 = \frac{18}{37} \times \frac{19}{37} = \frac{342}{1369}$$

(1 mark)

(ii) This is a binomial random variable with $p = 18/37$ and $n = 5$. This may be written

$$Y \sim \text{Bin}\left(\frac{18}{37}, 5\right)$$

$$\mu = 5 \times \frac{18}{37} = \frac{90}{37} \approx 2.43$$

$$\sigma^2 = 5 \times \frac{18}{37} \times \frac{19}{37} = \frac{1710}{1369} \approx 1.25$$

(2 marks)

(iii) This is a binomial probability, and we are looking for $P(W \geq 3)$, where W is the number of wins in five bets

$$P(W \geq 3) = P(W=3) + P(W=4) + P(W=5)$$

$$= \binom{5}{3} \left(\frac{18}{37}\right)^3 \left(\frac{19}{37}\right)^2 + \binom{5}{4} \left(\frac{18}{37}\right)^4 \left(\frac{19}{37}\right)^1$$

$$+ \binom{5}{5} \left(\frac{18}{37}\right)^5 \left(\frac{19}{37}\right)^0$$

$$= 0.475$$

(2 marks)

2020 Trial Extension 1 Q14 Solutions

a) i) $7 \times \left(\frac{1-t^2}{1+t^2} \right) + 4 \times \left(\frac{2t}{1+t^2} \right) + 5 = 0$

$$7(1-t^2) + 8t + 5(1+t^2) = 0$$

$$7 - 7t^2 + 8t + 5 + 5t^2 = 0$$

$$12 + 8t - 2t^2 = 0$$

$$-2(t^2 - 4t - 6) = 0$$

$$\therefore t^2 - 4t - 6 = 0 \text{ as Req.}$$

ii) Solution 1

$$t = \frac{4 \pm \sqrt{16 - 4 \times 1 \times (-6)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{40}}{2}$$

$$= 2 \pm \sqrt{10}$$

Since $t = \tan \frac{\theta}{2}$, then $\frac{1}{t} = \cot \frac{\theta}{2}$

So $\cot \frac{\theta_1}{2} + \cot \frac{\theta_2}{2} = \frac{1}{2+\sqrt{10}} + \frac{1}{2-\sqrt{10}}$

$$= \frac{2-\sqrt{10} + 2+\sqrt{10}}{4-10}$$

$$= \frac{4}{-6}$$

$$= -\frac{2}{3} \text{ as Req.}$$

Solution 2

Since roots are θ_1 and θ_2 , let $t_1 = \tan \left(\frac{\theta_1}{2} \right)$ and $t_2 = \tan \left(\frac{\theta_2}{2} \right)$

hence $\cot \left(\frac{\theta_1}{2} \right) = \frac{1}{t_1}$ and $\cot \left(\frac{\theta_2}{2} \right) = \frac{1}{t_2}$

If t_1 and t_2 are the roots of $t^2 - 4t - 6 = 0$,

then $t_1 + t_2 = -(-4) = 4$

and $t_1 \times t_2 = -6$

So $\cot \left(\frac{\theta_1}{2} \right) + \cot \left(\frac{\theta_2}{2} \right) = \frac{1}{t_1} + \frac{1}{t_2}$

$$= \frac{t_1 + t_2}{t_1 \times t_2}$$

$$= \frac{4}{-6} = -\frac{2}{3} \text{ as Req.}$$

b) Integers can go into 3 pigeon holes. When the integer is divided by 3

Box 0

Box 1

Box 2

Remainder = 0

Remainder = 1

Remainder = 2

By Pigeonhole Principle at least 1 box contains 2 integers.

If the 2 integers are in Box (x) where $x = 0, 1, 2$ then the integers can be written as $3M+x$ and $3N+x$, where M and N are integers.

$\therefore$ The difference between the integers is $(3M+x) - (3N+x) = 3(N-M)$

Hence the difference is divisible by 3.

c) i) Amount of liquid in the tank, at t minutes is

$$A = 2500 + 20t - 15t \\ = 2500 + 5t$$

$$\text{Concentration of Salt} = \frac{\text{Amount of Salt}}{\text{Amount of Water}}$$

$$\therefore C = \frac{y}{2500 + 5t}$$

ii) Rate of Change of Salt = Concentration of Salt per litre $\times$ Amount of liquid leaving the tank.
 $\therefore y' = C \times (-15)$
 $= -15C$

$$\text{iii)} \quad \frac{dy}{dt} = -15C = \frac{-15y}{2500 + 5t}$$

$$\therefore dy \times \frac{1}{y} = \frac{-15}{2500 + 5t} \times dt$$

$$\text{Hence } \int \frac{1}{y} \cdot dy = -3 \int \frac{5}{2500 + 5t} \cdot dt$$

$$\text{So } \ln|y| = -3 \ln|2500 + 5t| + k$$

$$\ln|y| + 3 \ln|2500 + 5t| = k$$

$$\ln|y \times (2500 + 5t)^3| = k$$

$$\therefore y \times (2500 + 5t)^3 = e^k$$

$$y = \frac{e^k \times 1}{(2500 + 5t)^3}, \text{ let } A = e^k$$

$$\therefore y = \frac{A}{(2500 + 5t)^3}$$

$$\text{When } t=0, y=25$$

$$\text{So } 25 = \frac{A}{(2500 + 5 \times 0)^3}$$

$$\therefore A = 25 \times 2500^3$$

$$\text{Hence } y = \frac{25 \times 2500^3}{(2500 + 5t)^3}$$

d) Let $\alpha = \tan^{-1} x$ then $x = \tan \alpha$

Now $\cot(2 \tan^{-1} x) = \frac{1}{\tan(2 \tan^{-1} x)} = \frac{1}{\tan 2\alpha}$

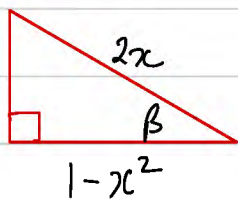
$$\cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{1 - \tan^2 \alpha}{2 \tan \alpha}$$

$$\therefore \cot 2\alpha = \frac{1 - x^2}{2x}$$

Hence $\sin(2 \cos^{-1}(\frac{1-x^2}{2x})) = 0$

Let $\beta = \cos^{-1}(\frac{1-x^2}{2x})$ then $\cos \beta = \frac{1-x^2}{2x}$ and $\sin 2\beta = 0$

Now $\sin 2\beta = 2 \sin \beta \cos \beta$



$$\begin{aligned} \therefore \text{3rd Side} &= \sqrt{(2x)^2 - (1-x^2)^2} \\ &= \sqrt{4x^2 - (1 - 2x^2 + x^4)} \\ &= \sqrt{-x^4 + 6x^2 - 1} \end{aligned}$$

$$\text{Hence } \sin \beta = \frac{\sqrt{-x^4 + 6x^2 - 1}}{2x}$$

So $\sin 2\beta = 2 \sin \beta \cos \beta = 2x \frac{\sqrt{-x^4 + 6x^2 - 1}}{2x} \times \frac{1-x^2}{2x} = 0$

Which makes $1-x^2=0$ or $-x^4 + 6x^2 - 1 = 0$

$$\therefore x = \pm 1$$

$$x^4 - 6x^2 + 1 = 0$$

$$x^4 - 6x^2 + 9 = 8$$

$$(x^2 - 3) = 8$$

$$x^2 = 3 \pm 2\sqrt{2}$$

$$= 1 \pm 2\sqrt{2} + 2$$

$$= (1 \pm \sqrt{2})^2$$

$$\therefore x = \pm (1 \pm \sqrt{2})$$

$\therefore$ The six solutions are $\pm 1, \pm 1 + \sqrt{2}, \pm 1 - \sqrt{2}$